[This question paper contains 6 printed pages.]
Your Roll No.................
Sr. No. of Question Paper : 1116
Unique Paper Code : 32221601
Name of the Paper : Electromagnetic Theory
Name of the Course
Serrester
Duration: 3 Hours
: B.Sc. (Hons) Physics-CBCs

Instructions for Candidates VI

Maximum Marks : 75
2022

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Question No. $\mathbf{1}$ is compulsory.
3. Answer any four of the remaining six.
4. Use of non-programmable calculator is allowed.
5. Attempt any 5 parts of this question.
a) A conductor of square cross section and having conductivity $3.8 \times 10^{7} \mathrm{~S} / \mathrm{m}$ is 50 m long. It measures 0.5 cm on either side. Calculate the skin depth if conductor carries current at 150 kHz .
(b) The conduction current density in a material is given as $\overrightarrow{J_{C}}=0.02 \sin \left(10^{\circ} \mathrm{t}\right) \mathrm{A} / \mathrm{m}^{2}$. Find the displacement current density if $\sigma=10^{5} \mathrm{mho} / \mathrm{m}$, $\epsilon_{\mathrm{T}}=6.5$.
(c) What is plasma frequency and minimum penetration depth for collision free plasma having $10^{12}$ electrons $/ \mathrm{m}^{3}$.
(d) A 300 mm long tube containing $56 \mathrm{~cm}^{3}$ of sugar solution produces an optical rotation of $12^{\circ}$ when placed in a polarimeter. If the specific rotation of sugar solution is $66^{\circ}$, calculate the quantity of sugar contained in the tube.
(e) For glass air interface $\left(\mathrm{n}_{1}=1.5, \mathrm{n}_{2}=1.0\right)$, find the reflection and transmission coefficients for normal incidence.
(1) A left circularly polarized wave $(\lambda=5893 \mathrm{~A})$ is incident normally on calcite crystal of thickness 0.00514 mm . Find the state of polarization of the emergent beam. Take $n_{0}=1.65836$ and $\mathrm{n}_{\mathrm{e}}=1.48641$.
(g) A circularly polarized electromagnetic wave is propagating in the $z$-direction in free space and is described by the following equation

$$
\overrightarrow{\mathrm{E}}=5 \cos (\omega \mathrm{t}-\mathrm{kz}) \hat{\mathrm{x}}+5 \sin (\omega \mathrm{t}-\mathrm{kz}) \hat{\mathrm{y}} \mathrm{Vm}^{-1}
$$

The wavelength is $6 \times 10^{-7} \mathrm{~m}$. Find the corresponding magnetic field and the average of the Poynting vector.
2. (a) State and establish the Poynting theorem for conservation of energy for electromagnetic fields. Explain the physical significance of each term in the equation.
(b) Using electromagnetic scalar and vector potentials, show that the four Maxwell equations can be written as two coupled second order differertial equations.
(c) What is the ratio of amplitudes of conduction current density and displacement current densities density if applied field is $\overrightarrow{\mathrm{E}}=\overrightarrow{\mathrm{E}}_{0} \mathrm{e}^{-t / \tau}$, where $\tau$ is real.
3. (a) Discuss the propagation of high frequency electromagnetic wave in plasma. Show that the critical frequency for the propagation of electromagnetic wave in plasma is given by $\mathrm{f}_{\mathrm{c}}=9{\sqrt{n_{0}}}$, where $\mathrm{n}_{0}$ is the electron density.

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## 4

(b) A material has $\sigma=6.0 \times 10^{-2} \Omega^{-1} / \mathrm{m}, \mu=\mu_{0}$ and $\varepsilon_{\mathrm{r}}=7.0$. A plane wave of frequency $10^{9} \mathrm{~Hz}$ with amplitude $200 \mathrm{~V} / \mathrm{m}$ is propagating along positive $z$ direction. Find (a) $E_{x}$ at $(x=0 \mathrm{~cm}, y=0 \mathrm{~cm}$, $z=3 \mathrm{~cm}, \mathrm{t}=0.16 \mathrm{~ns})(\mathrm{b}) \mathrm{H}_{y}$ at $(\mathrm{x}=0 \mathrm{~cm}$, $y=0 \mathrm{~cm}, \mathrm{z}=3 \mathrm{~cm} . \mathrm{t}=0.16 \mathrm{~ns})$.
4. (a) Derive the Fresnel's equation for reflection and transmission of a plane electromagnetic wave at the boundary separating between two dielectric media when electric field vector is perpendicular to the plane of incidence.

An electromagnetic wave propagating in a dielectric medium with $\varepsilon=16 \varepsilon_{0}$ along the $z$ direction. It strikes another dielectric medium with $\varepsilon=4 \varepsilon_{0}$ at $\mathrm{z}=0$. If the incoming wave has a maximum value of $0.2 \mathrm{~V} / \mathrm{m}$ at the interface, and its angular frequency is $300 \mathrm{M} \mathrm{rad} / \mathrm{s}$, determine the power densities of the incident, reflected, and transmitted waves.
5. (a) Derive Fresnel's formula for the propagation of light in Anisotropic crystals. Also explain how this leads to the phenomenon of double refraction.
(b) Discuss the propagation of light in uniaxial crystals. Explain the difference between a positive uniaxial crystal and a negative uniaxial crystal. Also give one example of each.
6. (a) Explain the construction and working of a Babinet Compensator. How is it used to analyze the elliptically polarized light?
(b) A right-handed circularly polarized plane wave with electric field magnitude of $3 \mathrm{mV} / \mathrm{m}$ is travelling in the $+Y$ direction in a dielectric medium with $\varepsilon=4 \varepsilon_{0}$ and $\mu=\mu_{0}=0$. If the frequency is 100 MHz , obtain expressions for $\vec{E}(y, t)$ and $\vec{H}(y, t)$.
7. (a) Starting with Maxwell's equations, derive the wave equations for a symmetric planar dielectric waveguide with refractive index profile as:
$\mathrm{n}=\mathrm{n}_{1} \quad-\mathrm{d} / 2<\mathrm{x}<\mathrm{d} / 2$
$n=n_{2} \quad-d / 2>x>d / 2$
Using the boundary conditions, obtain the eigenvalue equation for symmetric TE modes.

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(b) Consider a symmetric planar waveguide with the follow ing parameters.

$$
\mathrm{n}_{1}=1.5 ; \mathrm{n}_{2}=1.48 ; \mathrm{d}=3.912 \mu \mathrm{~m}
$$

How many TE modes exist for $\lambda_{0}=1 \mu \mathrm{~m}$ ? Determine the corresponding propagation constants.

Qua


Q1b
b.) Aust- Maxwell displacement current for a plane wave is

$$
\begin{aligned}
\vec{J}_{D}=\varepsilon \frac{\partial \vec{E}}{\partial t} & =\varepsilon \frac{\partial\left[\vec{E}_{0} \operatorname{Cos}(\vec{k} \cdot \vec{r}-\omega t)\right]}{\partial t} \\
& =\varepsilon \omega \vec{E}
\end{aligned}
$$

4 By Ohm's law; $\vec{J}_{c}=\sigma \vec{E}$
On taking ratio

$$
\begin{aligned}
& \quad \frac{\overrightarrow{J_{D}}}{\overrightarrow{J_{C}}}=\frac{\varepsilon \omega}{\sigma} \\
& \vec{J}_{D}=\frac{\varepsilon \omega}{\sigma} \vec{J}_{c}
\end{aligned}
$$

Given that $\vec{J}_{c}=0.02 \sin \left(10^{9} t\right) \mathrm{A} / \mathrm{m}^{2}$

$$
\begin{aligned}
\sigma & =10^{5} \mathrm{mh} / / \mathrm{m} \\
\varepsilon_{r} & =6.5 \\
\overrightarrow{J_{D}} & =\frac{6.5 \times 8.85 \times 10^{-12} \times 10^{9}}{10^{5}}(0.02) \sin \left(10^{9} \mathrm{t}\right) \\
& =6.5 \times 8.85 \times 10^{-12} \\
& =1.15 \times 10^{-3} \times 10^{-5} \sin \left(10^{9} t\right) \\
\overrightarrow{J_{D}} & =1.15 .10^{-8} \sin \left(10^{9} t\right) \mathrm{A} / \mathrm{m}^{2}
\end{aligned}
$$

Q1c
(.) Ans- Plasma frequency $\left(f_{p}\right)=\frac{\omega_{p}}{2 \pi}$

$$
\begin{aligned}
& \omega_{p}=\left(\frac{n_{e} e^{2}}{\varepsilon_{0} m}\right)^{1 / 2} \\
& f_{p}=\frac{\omega_{p}}{2 \pi} \approx \sqrt[9]{n_{e}}
\end{aligned}
$$

Given

$$
n_{c}=10^{12} e^{-1} \mathrm{~m}^{3}
$$

Penetration depth $=\delta=\frac{C}{\sqrt{\omega_{p}^{2}-\omega^{2}}}$
$\delta$ will be minimum if $\omega=0$ \& $\omega=0$ for $d . C$.

$$
\text { So } \begin{aligned}
\delta_{\text {min. }}=\frac{C}{w_{p}} & =\frac{3 \times 10^{8}}{2 \times 3.14 \times 9 \times 10^{6}} \\
& =0.5 \mathrm{~m} .
\end{aligned}
$$

Q1d
di) Dine $l=$ path length $=300 \mathrm{~mm}=300 \times \frac{100}{100} \mathrm{dm}$

$$
=3 d m
$$

Specific rotation $=[x]_{\lambda}^{\top}=60^{\circ}$
and $\alpha=12^{\circ}$
Now we know $[\alpha]_{\lambda}^{\top}=\frac{\alpha}{l \times c}$
Here $C$ is the concentration

$$
\begin{aligned}
& \Rightarrow 60^{\circ}=\frac{12}{3 \times c} \\
& \Rightarrow \quad C=\frac{12}{60 \times 3}=\frac{1}{15} \mathrm{~g} / \mathrm{mL} \\
& V=56 \mathrm{~cm}^{3}=56 \mathrm{~mL}
\end{aligned}
$$

Q1e
e) Ans: we know

$$
R=\left[\frac{n_{1}-n_{2}}{n_{1}+n_{2}}\right]^{2} \text { and } T=\frac{4 n_{1} n_{2}}{\left(n_{1}+n_{2}\right)^{2}}
$$

Here $n_{1}=1.5$ and $n_{2}=1.0$ (Give

$$
\begin{aligned}
\text { so } & R=\left(\frac{1.5-1}{1.5+1}\right)^{2}=\frac{1}{25}=0.04 \\
T & =\frac{4(1.5)(1)}{(1.5+1)^{2}}=\frac{6}{(2.5)^{2}} \\
& =\frac{6}{6.25}=0.96
\end{aligned}
$$

f) Ans ir Given

$$
\begin{aligned}
& \lambda=5893 \AA \\
& t=0.00514 \mathrm{~mm} \\
& n_{0}=1.65836 \\
& n_{e}=1.48641
\end{aligned}
$$

baths of
Path difference between ordinary and extraordinary ray =

$$
\begin{aligned}
& =(1.65836-1.48641) t \\
& =(1.65836-1.48641) \times(0.00514 \\
& =8838.23 \times 10^{-10} \mathrm{~m} \\
\text { Phase diff } & =\frac{2 \pi}{\lambda} \times 8838.83 \times 10^{-10}=2 \pi \times(1.499)
\end{aligned}
$$

Gina
Now the thickaress of the plate is such that the phase diff $(\phi)=3 \pi$. So the emergent light will be planepolarised light.

Q1g
1.). g.) Ans $\quad \vec{E}=5 \cos (\omega t-k z) \hat{x}+5 \sin (\omega t-x) y$

$$
\begin{aligned}
& \dot{\lambda}=6 \times 10^{-7} \mathrm{~m} \\
& k=\frac{2 \pi}{\lambda}=\frac{2 \pi}{6 \times 10^{-7}}
\end{aligned}
$$

It can be seen from expression of $\vec{E}$ that Direction of propagation of wave (i.e, wave vector) will be $z$-direction.

$$
\begin{aligned}
& \vec{B}=\frac{\vec{k} \times \vec{E}}{\omega}=\frac{k y \sin a d n}{\omega t} \\
& \left.=\frac{2 \pi \times 5}{6 \times 10^{-7}} \sqrt{\cos (\omega t-k z)(z \times \hat{x})+} \quad \sin (\omega t-k z)(z \times \hat{y})\right] \\
& =\frac{5 \pi \times 10^{7}}{3 \omega}[\cos (\omega t-k z) \hat{y}-\sin (\omega t-k z) \hat{x}]
\end{aligned}
$$

* Here medium is free space.


Q2a

Q2b

Q2c
(2.).) Ansi- Conduction current density $\left(\overrightarrow{v_{c}}\right)=$
\& Displacement current density $\left(F_{D}\right)$

$$
\begin{aligned}
& =\varepsilon \frac{d \vec{E}}{d t} \\
\text { Given } \vec{E} & =\overrightarrow{E_{0}} e^{-t / \tau}, \tau \text { is real. } \\
\overrightarrow{J_{C}} & =\sigma e^{-t / \tau} \overrightarrow{E_{0}} \\
\overrightarrow{J_{D}} & =\varepsilon\left(\frac{-1}{\tau}\right) e^{-t / \tau} \overrightarrow{E_{0}} \\
\overrightarrow{J_{C}} & =\frac{\sigma e^{-t / \tau} \overrightarrow{E_{0}}}{-\varepsilon / \tau e^{-t / \tau} \vec{t}_{0}}=\frac{-\sigma \tau}{\varepsilon}
\end{aligned}
$$

$$
\frac{\overrightarrow{J_{C}}}{\overrightarrow{J_{D}}}=\frac{-\sigma \tau}{\varepsilon}
$$



Q3b


## Q4a

Refer to chapter 2, page 12 for the solution to this question
Q4b
Refer to chapter 2, page 12 for the solution to this question
Q5a

### 2.2 Optics of (simple) anisotropic media

In order to understand the principal behavior of anisotropic materials, it is best to first ignore any dispersive property (frequency dependence) of the material. Owing to the Kramers-Kronig relation the material will then be lossless. This is what we mean by $a$ simple anisotropic material.

For a given material, the values $\epsilon_{1}, \epsilon_{2}, \epsilon_{3}$ and the corresponding principal axes are determined by the microscopic arrangements and symmetries of the material's constituents (atoms, molecules, etc). It is one of the tasks of solid-state physics to determine these quantities. In optics we take these quantities as given input.

We consider time-harmonic plane waves, i.e., E, D, B, H depend on $\mathbf{r}$ and t through the common factor $\mathrm{e}^{\mathrm{i}(k \hat{k} \cdot \hat{r}-\omega t)}$ as in $\mathbf{E}(\mathbf{r}, t)=\mathbf{E}_{0} \mathrm{e}^{\mathrm{i}(k \hat{k} \cdot \mathbf{r}-\omega t)}$ with a constant vector $\mathbf{E}_{0}$ (same goes for $\mathbf{D}, \mathbf{B}$, and $\mathbf{H}$ ). We also ignore (free) charges and (free) currents, hence we set $\rho=0$ and $\mathbf{j}=0$. Then we can deduce from the Maxwell equations (1.45) and (1.47):

$$
\begin{align*}
& k \hat{\mathbf{k}} \times \mathbf{H}+\omega \mathbf{D}=0 \Rightarrow \mathbf{D} \perp \hat{\mathbf{k}}, \mathbf{H}  \tag{2.13}\\
& k \hat{\mathbf{k}} \times \mathbf{E}-\mu_{0} \omega \mathbf{H}=0 \Rightarrow \mathbf{H} \perp \hat{\mathbf{k}}, \mathbf{E}  \tag{2.14}\\
& \Rightarrow \mathbf{H} \perp \hat{\mathbf{k}}, \mathbf{E}, \mathbf{D}  \tag{2.15}\\
& \hat{\mathbf{k}} \times \mathbf{H}=\underbrace{\frac{k}{\mu_{0} \omega} \underbrace{\hat{\mathbf{k}} \times(\hat{\mathbf{k}} \times \mathbf{E})}_{=-\mathbf{E}+(\mathbf{E} \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}}}=-\frac{\omega}{k} \mathbf{D}}_{\text {coplanar }}  \tag{2.16}\\
& \underbrace{\underbrace{\frac{k^{2}}{\mu_{0} \omega^{2}}[\mathbf{E}-(\mathbf{E} \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}}]}}
\end{align*}
$$

Eq. (2.17) implies $\hat{\mathbf{k}} \cdot \mathbf{D}=0$, i.e., the divergence condition (2.13) for the displacement field is consistent with (2.17).

Furthermore, $\mathbf{E} \cdot \hat{\mathbf{k}}$ can be non-zero now. If the dispersion relation is known, this allows to determine the polarization (i.e., the direction) of $\mathbf{D}$.

## Note:

$$
\begin{equation*}
\mathbf{H}=\frac{k}{\mu_{0} \omega} \hat{\mathbf{k}} \times \mathbf{E} \tag{2.18}
\end{equation*}
$$

$$
\begin{equation*}
\Rightarrow \mathbf{S}=\frac{1}{2} \operatorname{Re}\left(\mathbf{E} \times \mathbf{H}^{*}\right) \tag{2.19}
\end{equation*}
$$

$$
\begin{equation*}
=\frac{1}{2} \frac{k}{\mu_{0} \omega} \operatorname{Re}\left[\left(\mathbf{E} \cdot \mathbf{E}^{*}\right) \hat{\mathbf{k}}-(\mathbf{E} \cdot \hat{\mathbf{k}}) \mathbf{E}^{*}\right] \tag{2.20}
\end{equation*}
$$

$\Rightarrow$ The Poynting-vector $\mathbf{S}$ is not necessarily parallel to the wave vector $\hat{\mathbf{k}}$.
We now insert the constitutive relation $\mathbf{D}=\epsilon_{0} \in \mathbf{E}$ into Eq. (2.17). This gives

$$
\begin{equation*}
\underbrace{\left({ }_{=}^{\epsilon}-\frac{c^{2} k^{2}}{\omega^{2}} \mathbf{1}+\frac{c^{2} k^{2}}{\omega^{2}} \hat{\mathbf{k}}: \hat{\mathbf{k}}\right)}_{\text {det }=0} \mathbf{E}=0 \text { non trivial solutions } \quad \text { independent of Coordinate system } \tag{2.21}
\end{equation*}
$$

Without loss of generality, we can assume that we are in a coordinate system that coincides with the material's principal axes:

$$
\stackrel{\epsilon}{=}=\left(\begin{array}{ccc}
\epsilon_{1} & &  \tag{2.22}\\
& \epsilon_{2} & \\
& & \epsilon_{3}
\end{array}\right), \quad \hat{k}=\left(\begin{array}{c}
k_{1} \\
k_{2} \\
k_{3}
\end{array}\right) .
$$

Then, the requirement of the determinant being zero gives the celebrated Fresnel Equation:

$$
\begin{array}{r}
\left(\frac{c^{2} k^{2}}{\omega^{2}}\right)\left(\frac{\epsilon_{1} c^{2} k_{1}^{2}}{\omega^{2}}+\frac{\epsilon_{2} c^{2} k_{2}^{2}}{\omega^{2}}+\frac{\epsilon_{3} c^{2} k_{3}^{2}}{\omega^{2}}\right) \\
-\left(\frac{c^{2} k_{1}^{2}}{\omega^{2}} \epsilon_{1}\left(\epsilon_{2}+\epsilon_{3}\right)+\frac{c^{2} k_{2}^{2}}{\omega^{2}} \epsilon_{2}\left(\epsilon_{1}+\epsilon_{3}\right)+\frac{c^{2} k_{3}^{2}}{\omega^{2}} \epsilon_{3}\left(\epsilon_{1}+\epsilon_{2}\right)\right) \\
+\epsilon_{1} \epsilon_{2} \epsilon_{3}=0 \tag{2.25}
\end{array}
$$

where, $k^{2}=k_{1}^{2}+k_{2}^{2}+k_{3}^{2}$.
Q5b

Ordinary Ray ( 0-ray):
As per ray diagram, plane of polarization of 0 -ray is $I^{r}$ to the plane of figure: It follows snell's law of refraction.

fight: Double Refraction.
Exfra-Ordinaly Ray (E-ray):
Here, plane of polarization of e-gray is in the plane of figure. It does not follow shells law of refraction.

O-ray gravels with same velocity in all
directions within the crystal, whereas E-ray travels with different velocity in different directions within the crystal. But, the velocity of 0 -ray and $E$-rays is same along the direction of the optic axis.

$$
\begin{aligned}
& \mu_{0}=\frac{c}{v_{0}} \quad \text { and } \quad \mu_{e}=\frac{c}{v_{e}} \\
& \text { ( } \mu_{0}=6 \text { Constant } \\
& \text { as } V_{0} \text { is same in } \\
& \text { al directions) } \\
& \text { ( } \mu_{e}=\text { changes as } v_{e} \text { changes from } \\
& \text { direction to direction) }
\end{aligned}
$$



Fig:(a) ©re corystal (Uniasial)
(b) Ore corystal (Uniavial)

Babinet Compensator: [Jacques Babinet]
Retarders have been constructed of movable elements in order to produce variable retardance. Two of the most common designs based on movable wedges are the Babinet and Soleil, known as Babinet-Soleil Compensa-- tors. The term compensator is used for these elements because they are often used to allow adjustable comber. - Sation of retardance originating in a sample under test.
The Babinet Compensator, shown in fig. Consists of two cuedges of a (uniaxial) bire-fringent material (quartz). The bottom cuedge is fixed while the top wedge slides over the bottom by means of a micrometer. The optic axes of both the wedges are II to the outer faces of the wedge pair, but are $1^{r}$ to one another.
At any particular location across the face of the Babinet compensator, the net retardation is

$$
\left.\phi=\frac{2 \pi}{\lambda}\left(d_{1}-d_{2}\right)\left(\mu_{e}-\mu_{0}\right)\right]-(1)
$$

where, $d_{1} \& d_{2}$ are the thickness at that location.
If monochromatic polarized light oriented at $45^{\circ}$ to one of the optic apes is incident on the Babined compensator, one component of the light becomes
the expra-ordinaly component and the other is the ardinaly component in the $1^{\text {st }}$ wedge.

fig.2: Babinet compensator, where $O A$ is the optic axis.
When the light enters the $2^{\text {nd }}$ wedge, the components exchanges places, i.e., the extra-ordinaly becomes the ordinary y and vice-verga. An analyzer whose azimuth is $f^{r}$ to the original polarization can be placed behind the compensator to show the effect of the retardations.
Everywhere there is zero or a multiple of $2 \pi$ phase difference there will be a dark band. When the upper wedge is translated, the bands shift. These bands indicate the disadvantage of the Babinet compensator - a desired retardatice only occurs along these II bands.

By using Babinet compensator, o-rays and e-rays of light are produced. These: rays interfere and the interference fringes.


Qb
Please refer to chapter 5 for the solution to this question
Q7a
Refer chapter 6 for its solution
Q7b Refer chapter 6 for its solution

