

This equation represents the number of photons per unit volume lying in the frequency range  $\nu$  and  $\nu + d\nu$ .

The *energy density* of radiation of frequencies between  $\nu$  and  $\nu + d\nu$  can now be found by multiplying equation (8) with the energy of the photon  $h\nu$ . Therefore, if  $u(\nu) d\nu$  represents the energy density of radiation within the specified frequency range, then we get the energy distribution law

$$u(\nu) d\nu = \left( \frac{n(\nu) d\nu}{V} h\nu \right) = \frac{8\pi\nu^2 d\nu}{c^3} \cdot \left( \frac{h\nu}{e^{h\nu/kT} - 1} \right)$$

i.e.

$$u(\nu) d\nu = \frac{8\pi h \nu^3}{c^3} \frac{d\nu}{e^{h\nu/kT} - 1} \quad \dots(9)$$

This is well known *Planck's law of radiation in terms of frequency*.

Equation (9) in terms of wave-length  $\lambda$  becomes

$$u(\lambda) d\lambda = \frac{8\pi h}{\lambda^3} \frac{(-cd\lambda/\lambda^2)}{e^{hc/\lambda kT} - 1}$$

(Since  $\nu = \frac{c}{\lambda}$  and  $d\nu = -\frac{cd\lambda}{\lambda^2}$ )

negative sign occurs because when  $\nu$  increases,  $\lambda$  decreases

i.e. terms of modulus

$$|u(\lambda) d\lambda| = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda kT} - 1} \quad \dots(10)$$

This is well known *Planck's law of radiation in terms of wavelength*.

Hence the Bose-Einstein statistics while confirming the validity of Planck's radiation law, has the merit of using only photons and no other hypothetical resonators in the deduction of the law governing the black-body radiation.

**Q: what is Bose- Einstein condensation? Derive the expression for temperature at which it is achieved.**

**Ans:**

From Equation (25) the approximate value of  $A$  is

$$A = e^{-\alpha} = \frac{n}{V} \frac{h^3}{(2\pi mkT)^{3/2}}$$

If the density of particles is increased and/or the temperature is decreased, the value of  $A$  increases (or  $\alpha = -\mu/kT$  decreases). Then the behaviour of the perfect gas departs farther and farther from that of the classical perfect gas due to the fact that the velocities of the particles are subject to quantum statistics and not to classical statistics. The gas under this condition is said to be *degenerate* and the parameter  $A$  is called the *degeneracy parameter*.

As the expression for  $A$  contains three variables, *viz.*,  $m$  the mass of the particle,  $n/V$  the particle density, *i.e.*, the number of particles per unit volume and  $T$  the absolute temperature of

the gas. obviously the criterion of degeneracy will be based on the magnitude of  $\frac{(n/V)}{(mT)^{3/2}}$ . Hence, the degree of degeneracy will be large when temperature is low, particle density is large and the mass of each boson is small. On account of the above three factors, the gas can become degenerate in three different ways:

For low energy values the maximum admissible value of  $A$  is '1' [refer eqn. (5)] and consequently  $\alpha$  can never be negative. Thus for low energy values the limiting case of highest degeneracy in Bose-Einstein gas reaches when  $A = 1, \alpha = 0$ . Then the maximum value of  $f_1(\alpha)$  will be given by

$$[f_1(\alpha)]_{\max} = f_1(0) = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \dots = 2.612.$$

So the maximum value of particle density  $n/V$  will be given by

$$\left(\frac{n}{V}\right)_{\max} = \frac{(2\pi mkT)^{3/2}}{h^3} (2.612).$$

Since equation (32) corresponds to the limiting case of the Bose-Einstein degeneration, on solution of equation (15) can exist for

$$\frac{n}{V} > \frac{(2\pi mkT)^{3/2}}{h^3} \cdot 2.612 \quad \dots(33)$$

because this would involve  $A > 1$ .

The fact that no value of  $n/V$  can be greater than that given by equation (32) can be alternatively expressed in terms of the critical temperature  $T_0$ , defined as

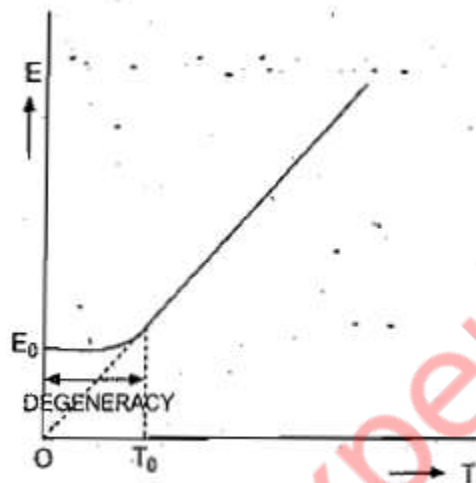
$$\frac{n}{V} = \frac{(2\pi mkT_0)^{3/2}}{h^3} \cdot 2.612, \quad \dots(34a)$$

$$\text{i.e.,} \quad T_0 = \frac{h^3}{2\pi mk} \left( \frac{1}{2.612} \cdot \frac{n}{V} \right)^{2/3} \quad \dots(34b)$$

Thus the critical temperature  $T_0$  is the lowest temperature for which a solution of equation (15) is possible, i.e. there is one solution of equation (15) for  $T < T_0$ .  $T_0$  is therefore the temperature at which the degeneracy of energy levels starts. If we plot a graph between the energy  $E$  and the temperature  $T$  of the gas a curve of the type shown in Fig. 8.2 is obtained. Below the critical temperature  $T_0$  full line shows the relation between  $E$  and  $T$  of a degenerate gas, while the dotted line through the origin that of a non-degenerate gas.  $E_0$  is termed as zero point energy which will be understood in the application of Fermi-Dirac statistics. The critical temperature  $T_0$  is 5K for helium gas. Now, the question arises why there is no solution of equation (15) for  $T < T_0$ . The reason is that while arriving

at Bose-Einstein distribution, because of the closeness of energy levels, we have assumed the continuous distribution in place of discrete distribution, and hence have replaced the summation by integration; while at low temperatures, the number of particles begin to crowd into lower energy levels and a large number of particles may occupy the ground state  $\epsilon_0 = 0$ . This means that at low temperatures we must be careful in replacing the summation into integration.

Now, from equation (12) the number of particles lying between energy range  $\epsilon$  and  $\epsilon + d\epsilon$  is given by



(Fig. 8.2)

$$n(\epsilon) d\epsilon = \frac{g(\epsilon) d\epsilon}{e^{\alpha + \epsilon/kT}}$$

$$g(\epsilon) = \frac{4\pi mV}{h^3} (2m\epsilon)^{1/2}$$

It may be now noted that for ground state  $\epsilon = \epsilon_0 = 0$ ,  $g(\epsilon) = 0$ , while actually it should be unity  $g(0) = 1$  as there is one state at  $\epsilon = 0$ . Therefore, the above distribution [eqn. (35)] gives incorrect result for ground state, while this state is very important at low temperature. We further note at  $\epsilon \neq 0$ ,  $g(\epsilon) \neq 0$  and therefore the above distribution holds good.

Consequently, the distribution (35) can still be applied for all states except ground state which should be treated separately.

For a single state, we have

$$n_i = \frac{g_i}{e^{\alpha + \beta\epsilon_i} - 1}$$

For ground state  $\epsilon_i = \epsilon_0 = 0$  and  $g_i = 1$ . Therefore, the number of particles in the ground state is given by

$$n \rightarrow n_0 = \frac{1}{e^{\alpha} - 1} \quad \dots(37)$$

Therefore, the total number  $n$  of particles for the degenerate case may be expressed as

$$n = n_0 + \int n(\epsilon) d\epsilon = n_0 + \int_0^{\infty} \frac{4\pi m V}{h^3} \cdot \frac{(2m\epsilon)^{1/2}}{e^{\alpha + \epsilon/kT} - 1} d\epsilon$$

$$= n_0 + n' \quad \dots(38)$$

are

$$n' = \frac{4\pi m V}{h^3} \cdot \sqrt{(2m)} \int_0^{\infty} \frac{\epsilon^{1/2} d\epsilon}{e^{\alpha + \epsilon/kT} - 1} \quad \dots(39)$$

$$= \frac{V}{h^3} (2\pi m k T)^{3/2} f_1(\alpha) \quad [\text{using (16)}]$$

$$= n \left( \frac{T}{T_0} \right)^{3/2} \frac{f_1(\alpha)}{f_1(0)} \quad \dots(40)$$

[using (34a) as  $f_1(0) = 2.612$ ]

As  $f_1(\alpha) < f_1(0)$ , therefore  $n'$  given by equation (40) acquires its maximum value when  $\alpha = 0$ . Thus the maximum number of particles ( $n'$ ) occupying states above the ground state is given by

$$n' = n \left( \frac{T}{T_0} \right)^{3/2} \quad (\text{for } T < T_0) \quad \dots(41)$$

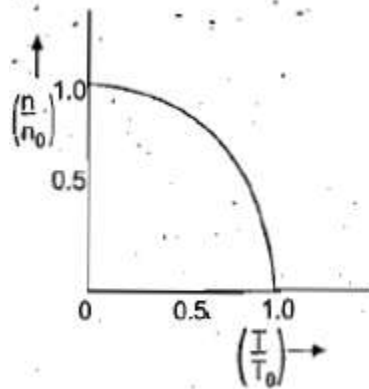
Therefore the rest of particles, given by

$$n_0 = n - n' = n \left[ 1 - \left( \frac{T}{T_0} \right)^{3/2} \right] \quad (\text{for } T < T_0) \quad \dots(42)$$

must condense into the ground state.

From equation (42), it is obvious that when the temperature of a Bose-Einstein gas is lowered below the critical temperature  $T_0$ , the number of particles in the ground state rapidly increases. This rapid increase in the population of the ground state below the critical temperature  $T_0$  for a Bose-Einstein gas is called the Bose-Einstein condensation. It is obvious from equation (34b) that the critical temperature  $T_0$  at which the Bose-Einstein condensation starts depends upon the particle density  $n/V$  of the gas.

Equation (42) is plotted in Fig. 8.3 which represents the fraction of particles condensed in the ground state for  $T \leq T_0$ . At the ground state  $\epsilon = 0$ , the particles of a Bose-Einstein degenerate gas condensed in the ground state do not contribute to the energy.



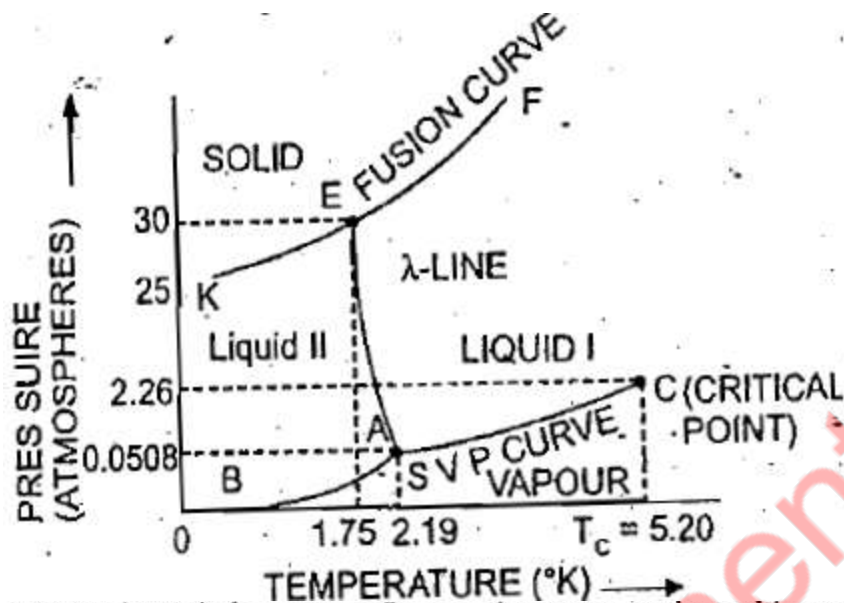
(Fig. 8.3)

Q: Draw and explain phase equilibrium diagram of helium.

Ans:

The phase equilibria of helium are represented diagrammatically in fig. 10.11 in which all portions are not on the same scale. A study of the phase diagram shows that it is entirely different from that of all other substances. The fusion curve (or the solid-liquid phase line) and the saturated vapour pressure curve (or the liquid-vapour phase line) do not meet in a point, as in the case of other substances and if we pursue the vapour pressure curve down to lower temperature it is found that the vapour and the liquid continue in equilibrium down to the absolute zero. Thus the three phases solid, liquid and vapour are never found to coexist

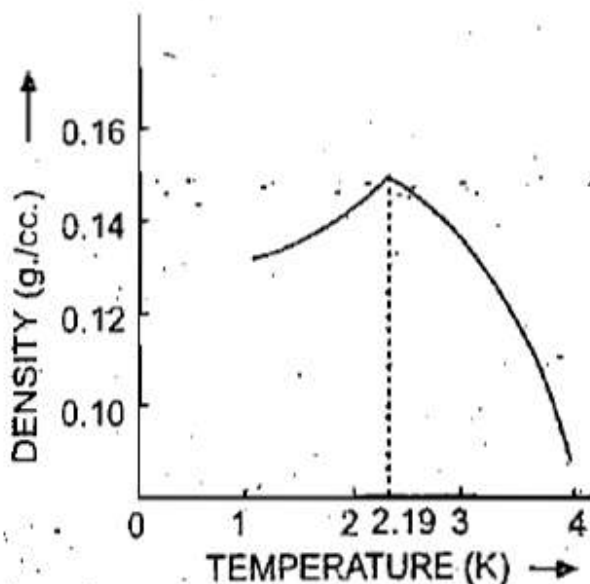
or in other words helium has no triple point in the conventional sense of terms. This is certainly an extraordinary thing and shows that helium is a unique liquid.



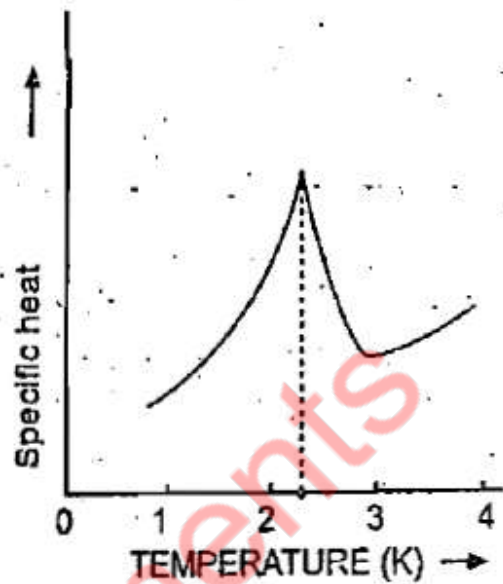
The curve FEK (Fig. 10.11) shows the influence of pressure on the melting point. It proceeds upwards to the right, takes a sharp bend at E and flattens out at low temperatures meeting the pressure axis at 25 atmospheres indicating that helium will not solidify even at 0 K if it is not subjected to pressures exceeding 25 atmospheres. The S.V.P. curve, on the other hand, appears to proceed normally to the left towards the origin ( $p=0, T=0$ ) but to the right it terminates at critical point C corresponding to a temperature of 5.2 K and a pressure of 2.26 atmospheres. The point A (2.19 K) is known as the  $\lambda$  point of liquid helium under its own pressure.

For helium in the liquid phase, there is a phase transition called the  $\lambda$ -transition, which divides the liquid state into two phases, helium I and helium II. The fusion curve and S.V.P. curve are joined by the  $\lambda$ -line running between the points E ( $T=1.75\text{ K}, p=30\text{ atm}$ ) and A ( $T=2.19\text{ K}$  and  $p=0.05\text{ atm}$ ) with helium I to its right and helium II to its left. Thus helium is present in the liquid form on either side of the  $\lambda$ -line. Kamerlingh Onnes, in the course of his investigations found that liquid helium shows an extremely interesting behaviour if it is cooled below its boiling point (4.2 K) to about 2.18 K. He found that the density passes through an abrupt maximum at 2.19 K decreasing slightly thereafter as shown in fig. 10.12. The density first rises with the fall of temperature from 4.2 K upto 2.19 K reaches a maximum value of 0.1462 at 2.19 K and then decreases with the decrease of temperature. Thus below 2.19 K, the liquid helium which was contracting when cooled now begins to expand.

The specific heat of liquid helium at constant volume  $C_v$  has been plotted in fig. 10.13. The specific heat increases, upto 2.19 K, and at this temperature there is a sudden and abnormal increase in its value. Beyond 2.19 K the specific heat first decreases and then increases. The dielectric constant also behaves in a similar manner. The specific heat temperature graph at 2.19 K looks like the Greek letter lambda ( $\lambda$ ) and hence this temperature (2.19 K) at which specific heat



(Fig. 10.12 (a))



(Fig. 10.12 (b))

changes abruptly is called the  $\lambda$ -point. There is a fundamental difference between the natures of the liquid above and below the  $\lambda$ -point (2.19 K). Liquid helium above 2.19 K which behaves in a normal way is called liquid helium I and that existing below this temperature is called the liquid helium II because of its abnormal properties. No heat is evolved or absorbed during the transition from one form of helium to another. In other words, no latent heat is involved in the transition  $He I \rightarrow He II$  which suggests that :

- (a) the entropy is continuous across the curve i.e., the entropy of  $He II$  is practically the same as that of the  $He I$  and
- (b) there is no change of density during transitions, i.e., the density of both types of liquid is about the same.

While the viscosity of liquids increases with decrease in temperature, that of liquid helium I decreases, in this respect  $He I$  resembles a gas. The viscosity of  $He II$  is almost zero and it can flow rapidly through a narrow capillaries. Liquid helium-I is a normal liquid while helium II presents a very anomalous behaviour in the sense that its thermal conductivity is abnormally high, its internal friction is practically zero and that when it is forced through a capillary, the emerging liquid cools while that which remains behind warms up. The densities of the two liquids are about the same, somewhere in the neighbourhood of 0.1462 g/cc which is far less than that of the lightest of other liquids like ether and gasoline.

**Q: write down some properties of helium II. Explain fountain effect.**

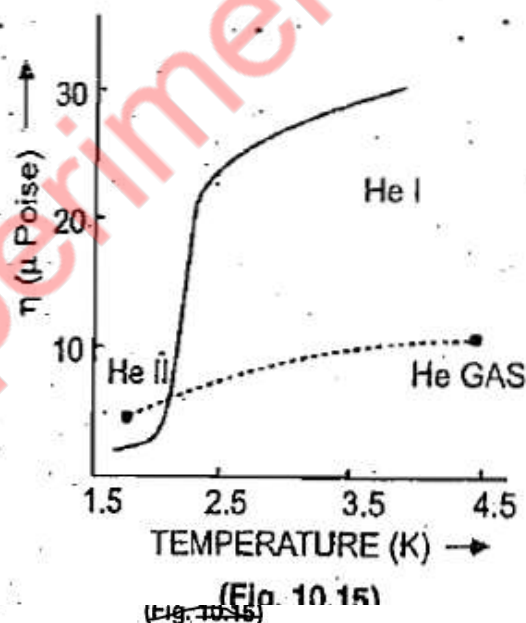
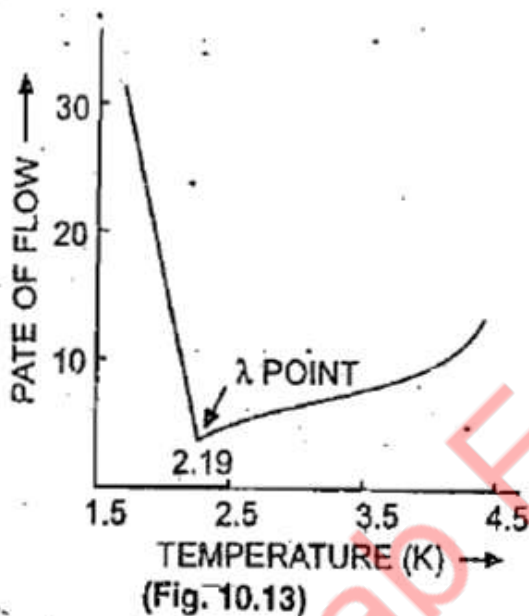
**Ans:**

In many of its properties, the behaviour of liquid helium II is quite unlike that of any other substance. It behaves strongly and shows some very unique properties. Some of the more interesting ones are given below :

(1) **Extreme fluidity.** One of the most significant properties of liquid helium II is that its viscosity is abnormally low even less than that of hydrogen gas and that its rate of flow through capillaries is independent of the pressure difference across them. According to Poiseuille's law the

rate of flow of liquid through a capillary tube or an annular space is inversely proportional to the viscosity. If the rate of flow of liquid helium through a fine annulus is measured as a function of temperature, the curve plotted in Fig. 10.14 (a) is obtained. At the  $\lambda$ -point the rate of flow increases abruptly and below it the flow is found to be extra-ordinary large thus proving the experimental evidence of a very low viscosity of liquid helium II. The values of the viscosity coefficient are plotted in Fig. 10.14 (b) as obtained by the oscillating disc method. It may be noted that there is a sharp discontinuity at the  $\lambda$ -point and in course of the curve on the two sides of it is different. The viscosity falls by a factor of about ten on passing through the  $\lambda$ -point. In fact Kapitza found that

$$\frac{\eta \text{ for liquid He II}}{\eta \text{ for liquid He I}} \approx 10^{-3}$$



Thus liquid *He II* has practically zero viscosity and can flow rapidly, almost without resistance, through narrow tubes. that is why the liquid *He II* is called a *superfluid*, a new frictionless state of matter. The properties associated with it have been characterised as *superfluidity*.

(ii) **High heat conductivity.** Heat conductivity of liquid helium I is quite normal of the order of  $10^4$ , which in turn is of the same order as for gases at ordinary temperatures. But *He II* is found to have an extra ordinary high coefficient of thermal conductivity. The heat transported per unit temperature gradient is several 10 times as great as that in copper at room temperature, *He II* is said to be about 800 times more conducting than copper and about  $13.5 \times 10^6$  times more than liquid *He I*, the absolute value being  $820 \text{ cal cm}^{-1} \text{ deg}^{-1} \text{ second}^{-1}$ . It is also found that the heat flow is not proportional to the temperature gradient.



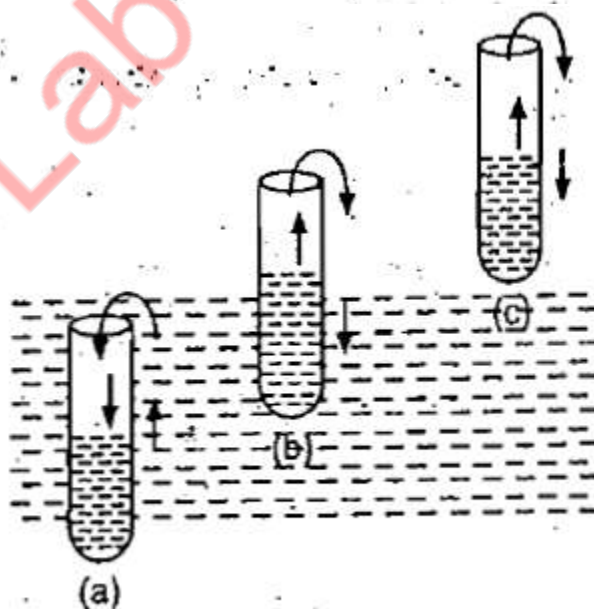
(iii) Formation of films over solid surfaces. Liquid helium II can creep along solid surfaces in the form of a highly mobile film generally known as the Rollin (sometimes also called Rollin-Simon) film which is of the order of several hundred atoms thick. The properties of the film were investigated by Daunt & Mendelssohn who observed the following remarkable behaviour of helium.

If a tube containing helium II be placed in a helium II bath, it is observed that :

(a) If the liquid level inside the tube is lower than that outside, the liquid helium from outside starts creeping into the tube along its surface in the direction of arrows shown in Fig. 10.15 (a) The process continues till the levels, inside and outside the tube, are equalized.

(b) If the tube be raised up so that the level of helium inside it is higher than that outside it, the liquid from inside the tube starts creeping out of it along its surface in the direction of arrow [Fig. 10.15 (b)] till the two levels again become the same.

(c) If the tube is lifted entirely out of the *He* bath, the liquid inside the tube now creeps out along the surface of the tube, collects at its bottom in the form of drops and falls into the liquid below till the whole liquid inside the tube is drained out [Fig. 10.15 (c)] and the tube is empty.

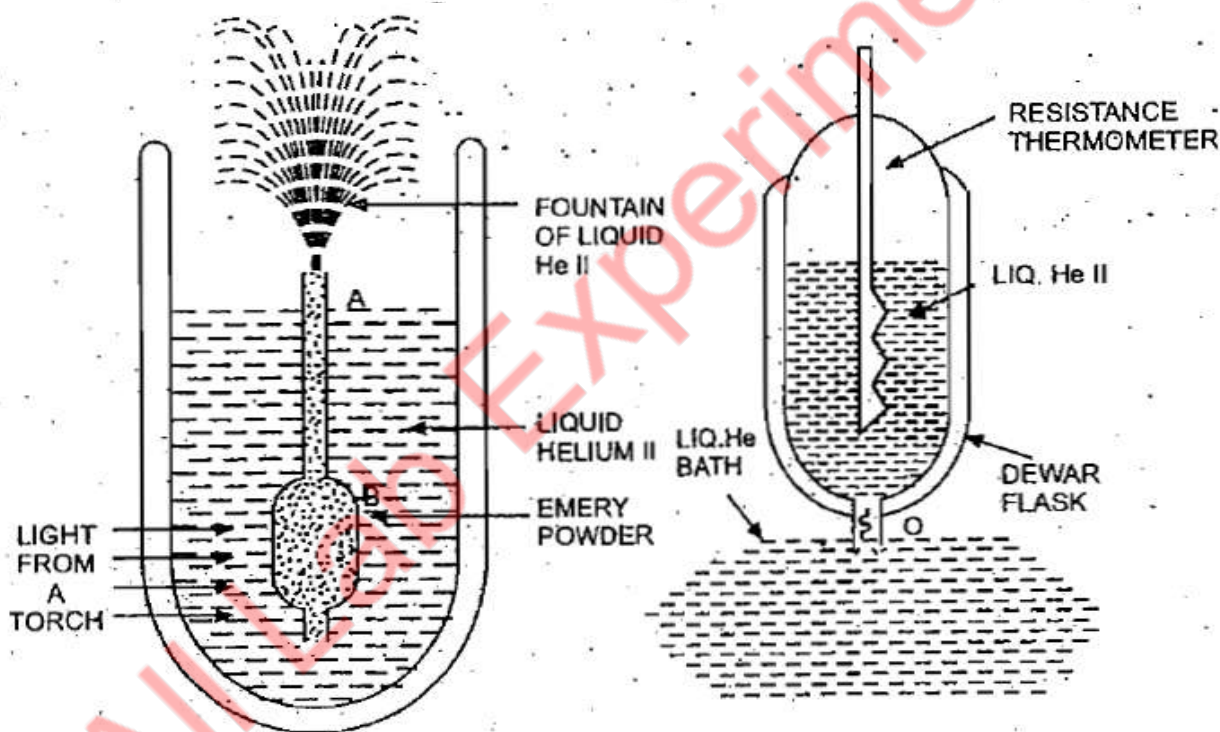


(Fig. 10.15)

All these experiments indicate that the liquid helium II seeks the lowest level, but the rate of transfer of the liquid is independent of the difference in level and the nature of the surface. It, however, depends upon the temperature and the perimeter of the surface over which it has to pass.

Thus liquid, *He II* seems to defy gravity by creeping out of the containing vessel by coating the walls with a thin film of the liquid.

(iv) The 'Fountain' and 'Mechano-caloric' Effect. Another set of peculiar properties of liquid *He II* is manifested in the 'Fountain' and 'Mechano-caloric' effects. The fountain effect, which is illustrated in Fig. 10.16 shows that heat locally supplied to the system produces a pressure difference. The phenomenon was discovered by Allen and Jones, in 1938 but the apparatus shown in Fig. 10.16 is one due to Allen and Misener. In the apparatus, *He II* is taken in a tube *AB* open at both the ends and kept inside a Dewar flask also containing liquid helium II. The upper part *A* of the tube is in the form of a capillary and kept projecting out of the liquid bath while the lower part *B* with a small opening *O* at the bottom is packed with fine *emery powder*. The interspaces between the powder particles provide fine *capillaries* for the liquid *He II* to flow into the tube. When heat radiations from an ordinary torch are allowed to fall on the powder, liquid *He* is found to spurt



(Fig. 10.16)

(Fig. 10.17)

out of the capillary tube in the form of a fountain which has been observed as high as 30 cm. This is a consequence of absorption of energy by emery powder.

The mechano-caloric effect is illustrated in Fig. 10.17 which is opposite to the fountain effect and shows that the flow of liquid gives rise to a temperature difference. A Dewar flask is filled partly with the liquid helium II and provided with a small hole *O* at the bottom which is plugged by fine powder. A resistance thermometer is introduced in the Dewar flask and then its top is closed. If now the flask is above the level of the liquid *He* bath, the liquid from the flask drains out through the power and its temperature is found to increase by about  $0.1^{\circ}\text{C}$ . If on the other hand, the Dewar flask is dipped into the bath and the liquid helium is allowed to flow into it through the opening *O* via the powder, the temperature inside the flask is observed to decrease.

## Chapter 5: Fermi Dirac statistics

**Q: derive an expression for Fermi-Dirac distribution law.**

**Ans:**

In accordance with the postulate of equal a priori probability of states, the probability  $\Omega$  of the system for occurring with the specified distribution is proportional to the total number of eigen states *i.e.*,

$$\Omega = \prod_i \frac{g_i!}{n_i! (g_i - n_i)!} \times \text{constant} \quad \dots(3)$$

The Fermi-Dirac distribution law can now be obtained by determining the most probable distribution.

Taking log of equation (3), we get

$$\begin{aligned} \log \Omega &= \log \left\{ \prod_i \frac{g_i!}{n_i! (g_i - n_i)!} \times \text{constant} \right\} \\ &= \sum_i [\log g_i! - \log n_i! - \log (g_i - n_i)!] + \text{constant} \quad \dots(4) \end{aligned}$$

As  $n_i$  and  $g_i$  are large numbers, therefore using Stirling approximation, eqn. (4) reduces to

$$\begin{aligned} \log \Omega &= \sum_i [g_i - \log g_i - \log n_i + n_i - (g_i - n_i) \log (g_i - n_i) + (g_i - n_i)] + \text{constant} \\ &= \sum_i [(n_i - g_i) \log (g_i - n_i) + g_i \log g_i - n_i \log n_i] + \text{constant} \quad \dots(5) \end{aligned}$$

Remembering that  $g_i$  is not subject to variation and  $n_i$  varies continuously, the differentiation of above equation gives

$$\begin{aligned} -\delta (\log \Omega) &= \sum_i \{ \log (g_i - n_i) - \log n_i \} \delta n_i \\ &= - \sum_i \left\{ \log \frac{n_i}{g_i - n_i} \right\} \delta n_i \quad \dots(6) \end{aligned}$$

For most probable distribution  $\Omega = \Omega_{\max}$  so  $\delta (\log \Omega_{\max}) = 0$ . Thus the conditions of most probable distribution gives

$$\sum_i \left\{ \log \frac{n_i}{g_i - n_i} \right\} \delta n_i = 0. \quad \dots(7)$$

The two subsidiary conditions are

(i) The total number of particles of the system is constant

*i.e.*  $n = \sum_i n_i = \text{constant}$

*i.e.*  $\delta n = \sum_i \delta n_i = 0 \quad \dots(8)$

(ii) Total energy of the system is constant

*i.e.*  $E = \sum_i n_i \epsilon_i = \text{constant}$

*i.e.*  $\delta E = \sum_i \epsilon_i \delta n_i = 0. \quad \dots(9)$

Now to apply the Lagrangian method of undetermined multipliers, we multiply equation (8) by  $\alpha$  and equation (9) by  $\beta$  and adding the resulting expressions to equation (7); so that we get

$$\sum_i \left\{ \log \left( \frac{n_i}{g_i - n_i} \right) + \alpha + \beta \epsilon_i \right\} \delta n_i = 0. \quad \dots(10)$$

As the variation  $\delta n_i$  are independent of each other, we get

$$\log \frac{n_i}{g_i - n_i} + \alpha + \beta \epsilon_i = 0.$$

or

$$\frac{n_i}{g_i - n_i} = e^{-(\alpha + \beta \epsilon_i)}$$

i.e.

$$\frac{g_i - n_i}{n_i} = e^{\alpha + \beta \epsilon_i}$$

i.e.

$$\frac{g_i}{n_i} - 1 = e^{\alpha + \beta \epsilon_i}$$

i.e.

$$\frac{g_i}{n_i} = e^{\alpha + \beta \epsilon_i} + 1$$

i.e.

$$n_i = \frac{g_i}{e^{\alpha + \beta \epsilon_i} + 1} \quad \dots(11)$$

This equation represents the most probable distribution of the particles among various energy levels for a system obeying Fermi-Dirac statistics and is therefore known as *Fermi-Dirac Distribution law*.

**Q:** derive an expression of entropy using Fermi-Dirac distribution law.

**Ans:** Refer chap 4 ques 2

**Q:** calculate internal energy and pressure for completely degenerate Fermi Dirac gas.

**Ans:**

At absolute zero i.e., when  $T=0$ ,  $A \rightarrow \infty$ . In this case the Fermi-Dirac gas is completely degenerate.

At  $T=0$ , we have

$$f(\epsilon) = \frac{1}{\frac{1}{A} e^{\epsilon/kT} + 1} = \frac{1}{e^{(\epsilon - \epsilon_F)/kT} + 1} \quad \left\{ \begin{array}{l} = 1 \text{ for } 0 \leq \epsilon \leq \epsilon_F(0) \\ = 0 \text{ for } \epsilon > \epsilon_F(0) \end{array} \right.$$

So that

$$n(\epsilon) d\epsilon \begin{cases} = g_s \left( \frac{4\pi mV}{h^3} \right) (2m)^{1/2} \epsilon^{1/2} d\epsilon & \text{for } \epsilon \leq \epsilon_F(0) \\ = 0 & \text{for } \epsilon > \epsilon_F(0) \end{cases}$$

where  $\epsilon_F(0)$  is given by eqn. (18).

Now the total internal energy of perfect Fermi-Dirac gas at  $T=0$  i.e., zero point energy of Fermi gas is

$$\begin{aligned} E_0 &= \int_0^{\epsilon_F(0)} \epsilon n(\epsilon) d\epsilon = g_s \left( \frac{4\pi mV}{h^3} \right) (2m)^{1/2} \int_0^{\epsilon_F(0)} \epsilon^{3/2} d\epsilon \\ &= g_s \left( \frac{4\pi mV}{h^3} \right) (2m)^{1/2} \left[ \frac{\epsilon^{5/2}}{5/2} \right]_0^{\epsilon_F(0)} \\ &= \frac{2}{5} g_s \left( \frac{4\pi mV}{h^3} \right) (2m)^{1/2} [\epsilon_F(0)]^{5/2} \end{aligned}$$

Using equation (18), above equation gives

$$E_0 = \frac{3nh^2}{10m} \left[ \frac{3n}{4\pi V g_s} \right]^{2/3} = \frac{3}{5} n \epsilon_F(0)$$

Now the pressure at  $T=0$  (i.e., zero point pressure) is given by

$$\begin{aligned} P_0 &= \frac{2}{3} \frac{E_0}{V} \left[ \text{since } P = - \left( \frac{\partial E}{\partial T} \right)_{T,S} = \frac{2}{3} \frac{E}{V} \right] \\ &= \frac{1}{5} \frac{nh^2}{Vm} \left( \frac{3n}{4\pi g_s V} \right)^{2/3} \end{aligned}$$

From above two equations,

From equations (28) and (29) it is obvious that a strongly degenerate Fermi-Dirac gas possesses energy and exerts a pressure even at 0 K, quite unlike a Bose-Einstein and classical gases where the energy and pressure at absolute zero are zero.

**Q:** calculate internal energy, pressure and specific heat for strongly degenerate Fermi Dirac gas.

**Sol:**

In this case the Fermi-gas is strongly degenerate at low temperature and  $\epsilon$  is still positive.

From equation (17) the number of particles lying in the energy range between  $\epsilon$  and  $\epsilon + d\epsilon$  is

$$\begin{aligned} n(\epsilon) d\epsilon &= f(\epsilon) g(\epsilon) d\epsilon = g_s \frac{4\pi mV}{h^3} (2m)^{1/2} \frac{\epsilon^{1/2} d\epsilon}{e^{(\epsilon - \epsilon_F)/kT} + 1} \\ &= \frac{3}{2} \frac{n}{[\epsilon_F(0)]^{3/2}} \frac{\epsilon^{1/2} d\epsilon}{e^{(\epsilon - \epsilon_F)/kT} + 1} \end{aligned}$$

Therefore, the total number of particles is

$$n = \int_0^{\infty} n(\epsilon) d\epsilon = \frac{3}{2} \frac{n}{[\epsilon_F(0)]^{3/2}} \int_0^{\infty} \frac{\epsilon^{3/2} d\epsilon}{e^{(\epsilon - \epsilon_F)/kT} + 1}$$

and the total integral energy is

$$E = \int_0^{\infty} \epsilon n(\epsilon) d\epsilon = \frac{3}{2} \frac{n}{[\epsilon_F(0)]^{3/2}} \int_0^{\infty} \frac{\epsilon^{5/2} d\epsilon}{e^{(\epsilon - \epsilon_F)/kT} + 1}$$

To solve the integrals of above two equations, let us consider

$$I \equiv \int_0^{\infty} \frac{\phi(\epsilon) d\epsilon}{e^{(\epsilon - \epsilon_F)/kT} + 1}$$

where  $\phi(\epsilon)$  is a simple function of  $\epsilon$  such that  $\phi(\epsilon) = 0$  when  $\epsilon = 0$ .

The integral of equation (33) can be expanded using the method of Taylor's series expansion,

$$I = \int_0^{\infty} \frac{\phi(\epsilon) d\epsilon}{e^{(\epsilon - \epsilon_F)/kT} + 1} = \int_0^{\epsilon_F} \phi(\epsilon) d\epsilon + \frac{\pi^2}{6} (kT)^2 [\phi']_{\epsilon = \epsilon_F} + \frac{7\pi^4}{360} (kT)^4 [\phi''']_{\epsilon = \epsilon_F} + \dots \quad \dots(34)$$

where  $\phi'$ ,  $\phi'''$  etc. denote the first, third etc. differentials of the function  $\phi$ .

Now, for

$$\phi(\epsilon) = \epsilon^{1/2}$$

$$\left. \begin{aligned} \int_0^{\epsilon_F} \phi(\epsilon) d\epsilon &= \frac{2}{3} (\epsilon_F)^{3/2} \\ \phi'(\epsilon) &= \left( \frac{\partial \phi}{\partial \epsilon} \right)_{\epsilon = \epsilon_F} = \frac{1}{2} \epsilon_F^{-1/2} \\ \phi''(\epsilon) &= -\frac{1}{4} \epsilon_F^{-3/2}, \quad \phi'''(\epsilon) = \frac{3}{8} \epsilon_F^{-5/2} \end{aligned} \right\}$$

$$\begin{aligned} n &= \frac{3}{2} \frac{n}{[\epsilon_F(0)]^{3/2}} \int_0^{\infty} \frac{\epsilon^{1/2} d\epsilon}{e^{(\epsilon - \epsilon_F)/kT} + 1} \\ &= \frac{3}{2} \frac{n}{[\epsilon_F(0)]^{3/2}} \left[ \frac{2}{3} \epsilon_F^{3/2} + \frac{\pi^2}{6} (kT)^2 \cdot \frac{1}{2} \epsilon_F^{-1/2} + \frac{7\pi^4}{360} (kT)^4 \cdot \frac{3}{8} \epsilon_F^{-5/2} + \dots \right] \end{aligned}$$

$$1 = \frac{\epsilon_F^{3/2}}{[\epsilon_F(0)]^{3/2}} \left[ 1 + \frac{1}{8} \left( \frac{\pi kT}{\epsilon_F} \right)^2 + \frac{7}{640} \left( \frac{\pi kT}{\epsilon_F} \right)^4 + \dots \right]$$

$$\frac{\epsilon_F}{\epsilon_F(0)} = \left[ 1 + \frac{1}{8} \left( \frac{\pi kT}{\epsilon_F} \right)^2 + \frac{7}{640} \left( \frac{\pi kT}{\epsilon_F} \right)^4 + \dots \right]^{-2/3}$$

Remembering that  $kT \ll \epsilon_F$ , we can take into account only the first two terms in the bracketed expression and write

$$\frac{\epsilon_F}{\epsilon_F(0)} = \left[ 1 + \frac{1}{8} \left( \frac{\pi kT}{\epsilon_F} \right)^2 \right]^{-2/3} \approx 1 - \frac{1}{12} \left( \frac{\pi kT}{\epsilon_F} \right)^2 \quad \dots(37)$$

This gives

$$\epsilon_F^2 = [\epsilon_F(0)]^2 \left[ 1 - \frac{1}{12} \left( \frac{\pi kT}{\epsilon_F} \right)^2 \right]^2 \approx [\epsilon_F(0)]^2 \left[ 1 - \frac{1}{6} \left( \frac{\pi kT}{\epsilon_F} \right)^2 \right]$$

$$\frac{1}{\epsilon_F^2} = \frac{1}{[\epsilon_F(0)]^2} \left[ 1 + \frac{1}{6} \left( \frac{\pi kT}{\epsilon_F} \right)^2 \right]$$

Now we make the crude approximation by putting  $\epsilon_F = \epsilon_F(0)$  in the second term of above expression, we get

$$\frac{1}{\epsilon_F^2} = \frac{1}{[\epsilon_F(0)]^2} \left[ 1 + \frac{1}{6} \left( \frac{\pi kT}{\epsilon_F(0)} \right)^2 \right] \quad \dots(39)$$

Now using equation (39), equation (37) gives

$$\epsilon_F = \epsilon_F(0) \left[ 1 - \frac{1}{12} (\pi kT)^2 \cdot \frac{1}{\epsilon_F^2(0)} \left[ 1 + \frac{1}{6} \left( \frac{\pi kT}{\epsilon_F(0)} \right)^2 \right] \right]$$

$$\epsilon_F(T) \approx \epsilon_F(0) \left[ 1 - \frac{1}{12} \left( \frac{\pi kT}{\epsilon_F(0)} \right)^2 \right]$$

$$\phi(\epsilon) = \epsilon^{3/2}$$

$$\int_0^{\epsilon_F} \phi(\epsilon) d\epsilon = \frac{2}{5} \epsilon_F^{5/2}$$

$$[\phi'(\epsilon)]_{\epsilon=\epsilon_F} = \frac{3}{2} \epsilon_F^{1/2}, \quad [\phi''(\epsilon)]_{\epsilon=\epsilon_F} = \frac{3}{4} \epsilon_F^{-1/2}$$

$$[\phi'''(\epsilon)]_{\epsilon=\epsilon_F} = -\frac{3}{8} \epsilon_F^{-3/2}$$

$$E = \frac{3}{2} \cdot \frac{n}{[\epsilon_F(0)]^{3/2}} \int_0^{\infty} \frac{\epsilon^{3/2} d\epsilon}{e^{(\epsilon-\epsilon_F)/kT} + 1}$$

$$= \frac{3}{2} \cdot \frac{n}{\epsilon_F^{3/2}(0)} \left[ \frac{2}{5} \epsilon_F^{5/2} + \frac{1}{6} (\pi kT)^2 \cdot \frac{3}{2} \epsilon_F^{1/2} - \frac{7}{360} (\pi kT)^4 \cdot \frac{3}{8} \epsilon_F^{-3/2} + \dots \right]$$

$$= \frac{3}{5} n \left[ \frac{\epsilon_F}{\epsilon_F(0)} \right]^{3/2} \cdot \epsilon_F \left[ 1 + \frac{5}{8} \left( \frac{\pi kT}{\epsilon_F} \right)^2 - \frac{7}{384} \left( \frac{\pi kT}{\epsilon_F} \right)^4 + \dots \right]$$

$$\approx \frac{3}{5} n \epsilon_F(0) \left( \frac{\epsilon_F}{\epsilon_F(0)} \right)^{5/2} \left[ 1 + \frac{5}{8} \left( \frac{\pi kT}{\epsilon_F} \right)^2 \right]$$

Now using equations (39) and (40), we get

$$\begin{aligned}
 E &\approx \frac{3}{5} n \epsilon_F(0) \left[ 1 - \frac{1}{12} \left( \frac{\pi kT}{\epsilon_F} \right)^2 \right]^{5/2} \left[ 1 + \frac{5}{8} (\pi kT)^2 \frac{1}{\epsilon_F^2(0)} \times \left[ 1 + \frac{1}{6} \left( \frac{\pi kT}{\epsilon_F(0)} \right)^2 \right] \right] \\
 &\approx \frac{3}{5} n \epsilon_F(0) \left[ 1 - \frac{5}{24} \left( \frac{\pi kT}{\epsilon_F(0)} \right)^2 \right] \left[ 1 + \frac{5}{8} \left( \frac{\pi kT}{\epsilon_F(0)} \right)^2 \right] \\
 &\approx \frac{3}{5} n \epsilon_F(0) \left[ 1 + \frac{5}{12} \left( \frac{\pi kT}{\epsilon_F(0)} \right)^2 \right]
 \end{aligned}$$

The corresponding pressure is

$$p = \frac{2}{3} \frac{E}{V} \approx \frac{2}{5} \left( \frac{n \epsilon_F(0)}{V} \right) \left[ 1 + \frac{5}{12} \left( \frac{\pi kT}{\epsilon_F(0)} \right)^2 \right]$$

$$\begin{aligned}
 C_V &= \left( \frac{\partial E}{\partial T} \right)_V = \frac{\partial}{\partial T} \left[ \frac{3}{5} n \epsilon_F(0) \left[ 1 + \frac{5}{12} \left( \frac{\pi kT}{\epsilon_F(0)} \right)^2 \right] \right]_V \\
 &= \frac{3}{5} n \epsilon_F(0) \left[ 0 + \frac{5}{6} \left( \frac{\pi k}{\epsilon_F(0)} \right)^2 \cdot 2T \right] \\
 &= \frac{1}{2} \frac{n k^2 \pi^2}{\epsilon_F(0)} T \\
 &= \lambda T,
 \end{aligned}$$

where  $\lambda = \left( \frac{n k^2 \pi^2}{2 \epsilon_F(0)} \right)$  is a constant quantity independent of temperature, i.e.,

$$C_V \propto T.$$

Thus, the specific heat of ideal strongly degenerate Fermi-Dirac gas is proportional to the absolute temperature. We have already seen that the Fermi gas is strongly degenerate at low temperature, therefore, this result holds at very low temperatures.

**Q:** derive an expression for Fermi temperature, zero point energy and zero point pressure of electron gas.

**Ans:**

For electrons  $s = \frac{1}{2}$  so that  $g_s = 2s + 1 = 2$ .

The Fermi energy at 0K is:



$$\begin{aligned}\epsilon_F(0) &= \frac{h^2}{2m} \left[ \frac{3n}{4\pi V g_s} \right]^{2/3} = \frac{h^2}{2m} \left[ \frac{3n}{4\pi V \cdot 2} \right]^{2/3} \\ &= \frac{h^2}{8m} \left( \frac{3n}{\pi V} \right)^{2/3} = 0.625 \times 10^{-17} \rho^{2/3} \text{ joule or } 39 \rho^{2/3} \text{ eV},\end{aligned}$$

where  $\rho = \left( \frac{mn}{V} \right)^{2/3} \text{ kg/m}^3$  is the density of the electron gas. For conduction electrons in metals  $\rho = 0.1 \text{ kg/m}^3$ .

The Fermi temperature  $T_F$  for electron gas is

$$T_F = \frac{\epsilon_F(0)}{k} = \frac{h^2}{8mk} \left( \frac{3n}{\pi V} \right)^{2/3} = (4.52 \times 10^5 \rho^{2/3}) \text{ K.} \quad \dots(2)$$

For  $\rho = 0.1 \text{ kg/m}^3$ ,  $T_F = 10^5 \text{ K}$ .

Thus electron gas below  $10^5 \text{ K}$  temperature is degenerate.

The degeneracy factor of an electron gas, from eqn. (27) of section 8.20 is

$$A = \frac{1}{g_s} \cdot \frac{n}{V} \frac{h^3}{(2\pi mkT)^{3/2}} = \frac{1}{2} \frac{n}{V} \frac{h^3}{(2\pi mkT)^{3/2}}$$

As  $T \rightarrow 0$ ,  $A \rightarrow \alpha$ , therefore  $\frac{1}{e^{\alpha+x} + 1} = \frac{1}{\frac{1}{A} e^x + 1} \rightarrow 1$ ,

so that for low temperatures

$$\begin{aligned}n &= g_s \frac{V}{h^3} (2\pi mkT)^{3/2} \cdot \frac{2}{\sqrt{\pi}} \int_0^\infty x^{1/2} dx \text{ from (10)} \\ &= \frac{4V}{\sqrt{\pi}} \cdot \frac{(2\pi mkT)^{3/2}}{h^3} \int_0^A x^{1/2} dx.\end{aligned}$$

Here we have put  $g_s = 2$  and have replaced the upper limit by  $A$  at low temperatures near absolute zero since  $A \rightarrow \infty$ , when  $T \rightarrow 0$

$$n = \frac{4V}{\sqrt{\pi}} \frac{(2\pi mkT)^{3/2}}{h^3} \cdot \frac{A^{3/2}}{\frac{3}{2}}$$

$$A = \frac{h^2}{2mkT} \left( \frac{3n}{8\pi V} \right)^{2/3} = \frac{h^2}{2m^{5/2} \cdot kT} \left( \frac{3\rho}{8\pi} \right)^{2/3}$$

This equation represents the degeneracy factor of electron gas at low temperature near absolute zero.

Now substituting  $h = 6.6 \times 10^{-34}$  joule sec,  $m = 9 \times 10^{-31}$  kg,  $k = 1.38 \times 10^{-23}$  joule/deg and  $\rho = 0.1 \text{ kg/m}^3$ ,

$$A = \frac{4.66 \times 10^5}{T} \quad \dots(4)$$

This mean that at low temperatures the electron gas is strongly degenerate.

Zero point energy of the electron gas, from eqn. (28) is

$$\begin{aligned} E_0 &= \frac{3 n h^2}{10 m} \left( \frac{3 n}{4 \pi V g_s} \right)^{2/3} = \frac{3 n h^2}{10 m} \left( \frac{3 n}{4 \pi V \cdot 2} \right)^{2/3} \\ &= \frac{3 n h^2}{40 m} \left( \frac{3 n}{\pi V} \right)^{2/3} = \frac{3}{5} n \epsilon_F(0). \end{aligned}$$

Zero point pressure of the electron gas, from eqn. (29), is

$$\begin{aligned} P_0 &= \frac{1}{5} \cdot \frac{n h^2}{V m} \left( \frac{3 n}{4 \pi g_s V} \right)^{2/3} = \frac{1}{5} \cdot \frac{n h^2}{V m} \left( \frac{3 n}{8 \pi V} \right)^{2/3} \\ &= \frac{n h^2}{20 m V} \left( \frac{3 n}{\pi V} \right)^{2/3} \end{aligned}$$

**Q:** calculate the expression for electronic contribution to the specific heat of metals at low temperatures.

**Ans:**

The electronic contribution to the specific heat of metals at low temperatures is given by

$$\begin{aligned} C_V &= \left( \frac{\partial E}{\partial T} \right)_V = \frac{\partial}{\partial T} \left[ \frac{3}{5} n \epsilon_F(0) \left( 1 + \frac{5}{12} \left( \frac{\pi k T}{\epsilon_F(0)} \right)^2 \right) \right]_V \\ &= \frac{1}{2} n k \pi^2 \left( \frac{k T}{\epsilon_F(0)} \right) \\ &= \lambda T \end{aligned} \quad \dots(7)$$

where  $\lambda = \frac{n k^2 \pi^2}{2 \epsilon_F(0)}$  is constant quantity, independent of temperature.

i.e.,  $C_V \propto T$ .

Thus the electronic contribution to the specific heat is proportional to the absolute temperature and vanishes at the absolute zero.

**Q: show that mean energy of free-electron at absolute zero of 3/5 times of Fermi energy at absolute zero.**

**Ans:**

**Ans. Mean Free-electron Energy at Absolute Zero**

The number of electrons in an electron-gas having energies between  $\epsilon$  and  $\epsilon + d\epsilon$  is given by

$$n(\epsilon) d\epsilon = \frac{8\sqrt{2} \pi V m^{3/2}}{h^3} \frac{\epsilon^{1/2} d\epsilon}{e^{(\epsilon - \epsilon_F)/kT} + 1} \quad \dots(i)$$

where  $m$  is the mass of electron and  $V$  is the volume of the electron-gas.

If  $N$  is the total number of electrons, then the Fermi energy  $\epsilon_F$  is given by

$$\begin{aligned} \epsilon_F &= \frac{h^2}{2m} \left( \frac{3N}{8\pi V} \right)^{2/3} \\ (\epsilon_F)^{3/2} &= \frac{h^3}{2\sqrt{2} m^{3/2}} \frac{3N}{8\pi V} \\ \frac{8\sqrt{2} \pi V m^{3/2}}{h^3} &= \frac{3N}{2} (\epsilon_F)^{-3/2} \end{aligned}$$

Making this substitution in eq. (i), we get

$$n(\epsilon) d\epsilon = \frac{3N}{2} (\epsilon_F)^{-3/2} \frac{\epsilon^{1/2} d\epsilon}{e^{(\epsilon - \epsilon_F)/kT} + 1}$$

At  $T = 0$ , all of the electrons have energies less than or equal to  $\epsilon_F$  (i.e.  $\epsilon \leq \epsilon_F$ ), so that at  $T = 0$ , we have

$$e^{(\epsilon - \epsilon_F)/kT} = e^{-\infty} = 0.$$

Therefore, at absolute zero

$$n(\epsilon) d\epsilon = \frac{3N}{2} (\epsilon_F)^{-3/2} \epsilon^{1/2} d\epsilon.$$

Now, let us first find the total energy  $E_0$  at absolute zero, which is

$$\begin{aligned} E_0 &= \int_0^{\epsilon_F} \epsilon n(\epsilon) d\epsilon \\ &= \frac{3N}{2} (\epsilon_F)^{-3/2} \int_0^{\epsilon_F} \epsilon^{3/2} d\epsilon \\ &= \frac{3N}{2} (\epsilon_F)^{-3/2} \left( \frac{2}{5} \epsilon_F^{5/2} \right) \end{aligned}$$

$$= \frac{3}{5} N \epsilon_F.$$

The average energy  $\bar{\epsilon}_0$  is this total energy divided by the total number  $N$  of electrons. Thus

$$\bar{\epsilon}_0 = \frac{E_0}{N} = \frac{3}{5} \epsilon_F.$$

**Q: Give characteristics of white dwarf stars. Obtain the relation between mass and radius of white dwarfs.**

**Ans:**

*White dwarfs are stars which are much fainter, possess smaller diameter and are very dense as compared to other stars of the same mass.*

An ideal white dwarf has the following specifications :

Content  $\rightarrow$  helium

Mass  $\sim 10^{30}$  kg  $\sim$  mass of sun

Density  $\sim 10^{10}$  kg m<sup>-3</sup> = 10  $\times$  density of sun

Temperature  $10^7$  K  $\sim$  temperature of sun

At this extremely high temperature, the helium gas gets completely ionised and so the white dwarf may be regarded as a gas composed of helium nuclei and electrons. The gas of electrons behaves as an ideal Fermi-Dirac gas of density  $10^{36}$  electrons per metre<sup>3</sup>, this corresponds to Fermi energy of

$$\epsilon_F = \frac{h^2}{2m} \left( \frac{3N}{8\pi V} \right)^{2/3} = 20 \text{ MeV} \quad \dots(1)$$

So that the Fermi temperature is

$$T_F = \frac{\epsilon_F}{k} = \frac{20 \times 1.6 \times 10^{-13}}{1.38 \times 10^{-23}} = 10^{11} \text{ K} \quad \dots(2)$$

As  $\frac{T_F}{T} \gg 1$ , the electron gas is highly degenerate. We can regard it as an ideal Fermi gas at  $T=0$  (i.e. in the ground state) The effect of high electron density is to provide  $T_F \gg T$  and also to make the electrons relativistic energies due to increase of mean energy. The Fermi Dirac gas possesses zero point energy and enormous pressure  $P_0$  which is balanced by the gravitational attraction.

We shall first calculate the pressure  $P_0$  exerted by Fermi Dirac gas of relativistic electron in the ground state.

The single particle energy levels are given by

$$\epsilon_{ps} = [p^2 c^2 + m^2 c^4]^{1/2} \quad \dots(3)$$

where  $p$  is momentum,  $m$  is rest mass of electron and  $c$  is speed of light. The states of electrons are specified by momentum  $p$  and spins 's'. The ground state energy of Fermi gas is

$$E_0 = g_s \sum_{|p| < p_F} (p^2 c^2 + m^2 c^4)^{1/2} \quad \dots(4)$$

where  $p_F$  is Fermi momentum given by

$$p_F = \sqrt{2m \epsilon_F} = \sqrt{2m \times \frac{h^2}{2m} \left( \frac{3N}{8\pi V} \right)^{2/3}} \Rightarrow p_F = h \left( \frac{3N}{8\pi V} \right)^{1/3} \quad \dots(5)$$

As energy levels are quite close to each other, so the summation may be replaced by integration; we have-

$$\begin{aligned} g_s &= g(p) dp = (2s + 1) \frac{V 4\pi p^2 dp}{h^3} \\ &= \frac{2 \cdot V 4\pi p^2 dp}{h^3} \quad (\text{as } s = \frac{1}{2} \text{ for electron}) \\ E_0 &= \frac{2V}{h^3} \int_0^{p_F} (p^2 c^2 + m^2 c^4)^{1/2} 4\pi p^2 dp \quad \dots(6) \end{aligned}$$

Now putting

$$\begin{aligned} \frac{p}{mc} &= x \Rightarrow \frac{dp}{mc} = dx, \text{ we get} \\ E_0 &= \frac{8\pi V}{h^3} \int_0^{x_F} mc^2 [1 + x^2]^{1/2} (mcx)^2 \cdot mc dx \\ &= \frac{8\pi V m^4 c^5}{h^3} \int_0^{x_F} (1 + x^2)^{1/2} x^2 dx \\ &= \frac{8\pi V m^4 c^5}{h^3} f(x_F) \quad \dots(7) \end{aligned}$$

where  $f(x_F) = \int_0^{x_F} x^2 (1 + x^2)^{1/2} dx$

$$= \begin{cases} \frac{1}{2} x_F^3 \left( 1 + \frac{3}{10} x_F^2 + \dots \right); & x_F \leq 1 \\ \frac{1}{4} x_F^4 (1 + x_F^{-2} + \dots); & x_F \gg 1 \end{cases} \quad \dots(8)$$

where  $x_F \ll 1$  corresponds to non relativistic case and  $x_F \gg 1$  corresponds to the relativistic case,

with  $x_F = \frac{p_F}{mc}$ .

$$\begin{aligned} \text{Pressure } P_0 &= - \left( \frac{\partial E_0}{\partial V} \right) = - \frac{8\pi m^4 c^5}{h^3} \left[ f(x_F) + \frac{V \partial f(x_F)}{\partial x_F} \frac{\partial x_F}{\partial V} \right] \\ &= \frac{8\pi m^4 c^5}{h^3} \left[ \frac{1}{3} x_F^3 (1 + x_F^2)^{1/2} - f(x_F) \right] \end{aligned}$$

$$= \begin{cases} \frac{8\pi m^4 c^5}{15h^3} x_F^5 & x_F \ll 1 \\ \frac{2\pi m^4 c^5}{3h^3} (x_F^4 - x_F^2), & x_F \gg 1 \end{cases} \quad \dots(9)$$

If  $m$  is mass of electron,  $m_p$  the mass of proton and  $N$  is the number of helium atoms, then mass of star,

$$M = (m + 2m_p) N = 2m_p N \quad (\text{because } m \ll m_p) \quad \dots(10)$$

Radius of star is given by  $V = \frac{4}{3} \pi R^3 \Rightarrow R = \left(\frac{3V}{4\pi}\right)^{1/3} \quad \dots(11)$

$$\begin{aligned} \frac{V}{N} &= \frac{\frac{4}{3} \pi R^3}{M/2m_p} = \frac{8}{3} \pi \cdot \left(\frac{m_p R^3}{M}\right) \\ \text{Then } p_F &= h \cdot \left(\frac{3N}{8\pi V}\right)^{1/3} = h \left(\frac{3}{8\pi} \cdot \frac{3}{8\pi} \frac{M}{m_p R^3}\right)^{1/3} = \frac{h}{2\pi R} \cdot \left(\frac{9\pi M}{8 m_p}\right)^{1/3} \\ \Rightarrow x_F &= \frac{p_F}{mc} = \frac{h}{2\pi mc} \left(\frac{9\pi M}{8 m_p}\right)^{1/3} = \frac{\bar{M}^{1/3}}{\bar{R}} \quad \dots(12) \end{aligned}$$

where  $\bar{M} = \frac{9\pi M}{8 m_p}$  and  $\bar{R} = \frac{R}{(h/2\pi mc)}$

The enormous zero point pressure is balanced by the gravitational attraction to bind the star. The work done to form the star of radius  $R$  from the state of infinite diluteness is given by

$$W = \int_{\infty}^R P_0 \cdot 4\pi r^2 dr$$

where  $P_0$  is pressure of uniform  $F \cdot D$  gas

This work is calculated by gravitational self energy which is given by

$$U_s = -\frac{3}{5} \frac{GM^2}{R} \quad \text{where } G \text{ is gravitational constant}$$

$$\int_0^R P_0 4\pi r^2 dr = -\frac{3}{5} \frac{GM^2}{R} \quad \dots(13)$$

Differentiating both sides with respect to  $R$ , we get equilibrium condition

$$\begin{aligned} P_0 4\pi R^2 &= \frac{3}{5} \frac{GM^2}{R^2} \\ \Rightarrow P_0 &= \frac{3}{20\pi} \frac{Gm^2}{R^4} \quad \dots(14) \end{aligned}$$

In terms of  $\bar{M}$  and  $\bar{R}$  it is expressed in

$$P_0 = \frac{3}{20\pi} G \left(\frac{8 m_p}{9\pi} \bar{M}\right)^2 \times \left(\frac{2\pi mc}{h}\right)^4 \cdot \frac{1}{R^4}$$

This is the value of equilibrium pressure in terms of  $\bar{M}$  and  $\bar{R}$

Using equation (9) for extreme relativistic case  $x_F \gg 1$ , no white dwarf can have a mass larger than


$$M_0 = \left( \frac{8}{9\pi} m_p \right) \left( \frac{45}{64} \pi \right)^{3/2} \left( \frac{hc}{2\pi G m_p^2} \right)^{3/2} = 10^{30} \text{ kg}$$

= mass of sun

The refined estimate gives

$$M_0 = 1.4 \times \text{mass of sun}$$

This mass is called the Chandrasekhar Limit.

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