Semester VI

PHYSICS-C XIII: ELECTROMAGNETIC THEORY
(Credits: Theory-04, Practicals-02)

Theory: 60 Lectures

Maxwell Equations: Review of Maxwell's equations. Displacement Current. Vector and Scalar Potentials. Gauge Transformations: Lorentz and Coulomb Gauge. Boundary Conditions at Interface between Different Media. Wave Equations. Plane Waves in Dielectric Media. Poynting Theorem and Poynting Vector. Electromagnetic (EM) Energy Density. Physical Concept of Electromagnetic Field Energy Density. Momentum Density and Angular Momentum Density. **(12 Lectures)**

EM Wave Propagation in Unbounded Media: Plane EM waves through vacuum and isotropic dielectric medium, transverse nature of plane EM waves, refractive index and dielectric constant, wave impedance. Propagation through conducting media, relaxation time, skin depth. Wave propagation through dilute plasma, electrical conductivity of ionized gases, plasma frequency, refractive index, skin depth, application to propagation through ionosphere. **(10 Lectures)**

EM Wave in Bounded Media: Boundary conditions at a plane interface between two media. Reflection & Refraction of plane waves at plane interface between two dielectric media-Laws of Reflection & Refraction. Fresnel's Formulae for perpendicular & parallel polarization cases, Brewster's law. Reflection & Transmission coefficients.Total internal reflection, evanescent waves. Metallic reflection (normal Incidence) **(10 Lectures)**

Polarization of Electromagnetic Waves: Description of Linear, Circular and Elliptical Polarization. Propagation of E.M. Waves in Anisotropic Media. Symmetric Nature of Dielectric Tensor. Fresnel's Formula. Uniaxial and Biaxial Crystals. Light Propagation in Uniaxial Crystal. Double Refraction. Polarization by Double Refraction. Nicol Prism. Ordinary & extraordinary refractive indices. Production & detection of Plane, Circularly and Elliptically Polarized Light. Phase Retardation Plates: Quarter-Wave and Half-Wave Plates. Babinet Compensator and its Uses. Analysis of Polarized Light **(12 Lectures)**

Rotatory Polarization: Optical Rotation. Biot's Laws for Rotatory Polarization.Fresnel's Theory of optical rotation. Calculation of angle of rotation. Experimental verification of Fresnel's theory. Specific rotation. Laurent's half-shade polarimeter. **(5 Lectures)**

Wave Guides: Planar optical wave guides. Planar dielectric wave guide. Condition of continuity at interface. Phase shift on total reflection. Eigenvalue equations. Phase and group velocity of guided waves. Field energy and Power transmission. **(8 Lectures)**

Optical Fibres: Numerical Aperture. Step and Graded Indices (Definitions Only). Single and Multiple Mode Fibres. **(3 Lectures)**

Chapter 1

Maxwell Equations: Review of Maxwell's equations. Displacement Current. Vector and Scalar Potentials. Gauge Transformations: Lorentz and Coulomb Gauge. Boundary Conditions at Interface between Different Media. Wave Equations. Plane Waves in Dielectric Media. Poynting Theorem and Poynting Vector. Electromagnetic (EM) Energy Density. Physical Concept of Electromagnetic Field Energy Density. Momentum Density and Angular Momentum Density. **(12 Lectures)**

Que1: Write the differential form of Maxwell's equation in vacuum.

Ans:

In the language of differential vector calculus

- Gauss's law:

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0} \tag{1}$$

- Gauss's law for magnetism:

$$\nabla \cdot \vec{\mathbf{B}} = 0 \tag{2}$$

- Maxwell-Faraday equation

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{3}$$

- Ampere's law, with Maxwell's correction

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \left(\vec{\mathbf{J}} + \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right) \tag{4}$$

Que2: Give physical interpretation of Maxwell's equation.

Ans:

Gauss's law: enclosed charges

$\nabla \cdot \vec{\mathbf{E}} = \rho/\epsilon_0$:

- Integrate over a closed volume:

$$\int_V (\nabla \cdot \vec{\mathbf{E}}) dV = \int_V \frac{\rho}{\epsilon_0} dV \tag{5}$$

- Use a mathematical identity (Gauss's theorem)

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \frac{Q_{\text{enclosed}}}{\epsilon_0} \tag{6}$$

- Relationship between electric field on a closed surface and the charge enclosed inside it
- The part in red: source of the electric field
- Leads to Coulomb's law if Q is a point charge at the centre of $\vec{\mathbf{S}}$, a sphere of radius r: $\boxed{E_r \cdot 4\pi r^2 = Q/\epsilon_0}$

Gauss's law: no magnetic monopoles

$\nabla \cdot \vec{\mathbf{B}} = 0$ **:**

- Integrate over a closed volume:

$$\int_V (\nabla \cdot \vec{\mathbf{B}}) dV = 0 \qquad (7)$$

- Use a mathematical identity (Gauss's theorem)

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0 \qquad (8)$$

- Relationship between magnetic field on a closed surface and the magnetic charge enclosed inside it
- The part in red: source of the magnetic field.
- Vanishing of the source $\Rightarrow$ no magnetic monopoles

Maxwell-Faraday equation: flux through a loop

$\nabla \times \vec{\mathbf{E}} = -\partial \vec{\mathbf{B}} / \partial t$:

- Integrate over a surface whose boundary is a loop:

$$\int_{\vec{S}} (\nabla \times \vec{\mathbf{E}}) \cdot d\vec{\mathbf{S}} = \int_{\vec{S}} -(\partial \vec{\mathbf{B}} / \partial t) \cdot d\vec{\mathbf{S}} \qquad (9)$$

- Use a mathematical identity (Stokes' theorem)

$$\oint \vec{\mathbf{E}} \cdot d\vec{\ell} = -\int_{\vec{S}} \frac{\partial}{\partial t} (\vec{\mathbf{B}} . d\vec{\mathbf{S}}) \qquad (10)$$

(If the loop does not change with time)

- The induced EMF is

$$\mathcal{E} \equiv \oint \vec{\mathbf{E}} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_{\vec{S}} (\vec{\mathbf{B}} . d\vec{\mathbf{S}}) = -\frac{\partial \Phi}{\partial t} \qquad (11)$$

$\Phi \equiv \int_{\vec{S}} \vec{\mathbf{B}} . d\vec{\mathbf{S}}$

- Relationship between electric field along a loop and the rate of change of magnetic flux through an open surface whose boundary is the loop
- No sources needed: it is a relationship between $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$

Ampere's law with Maxwell's corrections

$\nabla \times \vec{\mathbf{B}} = \mu_0(\vec{\mathbf{J}} + \epsilon_0 \partial \vec{\mathbf{E}}/\partial t)$:

- Integrate over a surface whose boundary is a loop:

$$\int_{\vec{S}} (\nabla \times \vec{\mathbf{B}}) \cdot d\vec{\mathbf{S}} = \mu_0 \int_{\vec{S}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{S}} + \mu_0 \epsilon_0 \int_{\vec{S}} (\partial \vec{\mathbf{E}}/\partial t) \cdot d\vec{\mathbf{S}} \qquad (12)$$

- Use a mathematical identity (Stokes' theorem)

$$\oint \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_0 I + \mu_0 \epsilon_0 \int_{\vec{S}} \frac{\partial}{\partial t} (\vec{\mathbf{E}}.d\vec{\mathbf{S}}) \qquad (13)$$

- Relationship between magnetic field along a loop and the rate of change of magnetic flux through an open surface whose boundary is the loop
- $I = \oint_{\vec{S}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{S}}$ is the conduction current
- $\mu_0 \int_{\vec{S}} \frac{\partial}{\partial t} (\vec{\mathbf{E}}.d\vec{\mathbf{S}})$ is often called "displacement current", this is the correction by Maxwell to Ampere's law

Que3: Write the macroscopic form of Maxwell's equations

Ans:

Inside a dielectric medium (static case)

- Gauss's law always valid, when ρ is the total charge: $\nabla \cdot \vec{\mathbf{E}} = \rho/\epsilon_0$
- Part of the charge is due to polarization induced in the medium, which gives rise to the "bound charge": $\rho_b = -\nabla \cdot \vec{\mathbf{P}}$, where $\vec{\mathbf{P}}$ is the polarization
- Then $\epsilon_0 \nabla \cdot \vec{\mathbf{E}} = (\rho_b + \rho_{fr}) = -\nabla \cdot P + \rho_{fr}$, where ρ_{fr} is the free charge density

- Defining $\vec{\mathbf{D}} = \epsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}$, we get Gauss's law in terms of the free charge density:

$$\nabla \cdot \vec{\mathbf{D}} = \rho_{\text{fr}} \tag{14}$$

- The relation $\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}}$ defines the dielectric permittivity of the medium, ϵ. This is in general not a number but a tensor, and may not be constant. Wherever it is constant, the dielectric is called "linear".

Inside a magnetic medium (static case)

- Maxwell-Faraday equation always valid, when $\vec{\mathbf{J}}$ is the total current: $\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$
- Part of the current is due to magnetization induced in the medium, which gives rise to the "surface current": $\vec{\mathbf{J}}_{\text{surface}} = \nabla \times \vec{\mathbf{M}}$, where $\vec{\mathbf{M}}$ is the magnetization
- Then $\nabla \times \vec{\mathbf{B}} = (\vec{\mathbf{J}}_{\text{surface}} + \vec{\mathbf{J}}_{\text{fr}}) = \mu_0 \nabla \times M + \mu_0 \vec{\mathbf{J}}_{\text{fr}}$, where $\vec{\mathbf{J}}_{\text{fr}}$ is the free current density

- Defining $\vec{\mathbf{H}} = \vec{\mathbf{B}}/\mu_0 - \vec{\mathbf{M}}$, we get Ampere's law in terms of the free charge density:

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_{\text{fr}} \tag{15}$$

- The relation $\vec{\mathbf{B}} = \mu \vec{\mathbf{H}}$ defines the magnetic permeability of the medium, μ. This is in general not a number but a tensor, and may not be constant. Wherever it is constant, the magnetic medium is called "linear".

Maxwell's equations: "macroscopic" form

$$\nabla \cdot \vec{D} = \rho_{fr} \quad (16)$$

$$\nabla \cdot \vec{B} = 0 \quad (17)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (18)$$

$$\nabla \times \vec{B} = \vec{J}_{fr} + \frac{\partial \vec{D}}{\partial t} \quad (19)$$

These are equivalent to the equations (1)–(4), with the substitutions

$$\rho = \rho_{fr} + \rho_b\,, \qquad \vec{J} = \vec{J}_{fr} + \vec{J}_{surface} \quad (20)$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}\,, \qquad \vec{B} = \mu_0(\vec{H} + \vec{M}) \quad (21)$$

$$\rho_b = -\nabla \cdot \vec{P}\,, \qquad \vec{J}_{surface} = \nabla \times \vec{M} + \frac{\partial \vec{D}}{\partial t}\,. \quad (22)$$

Que4: Define Scalar and vector potential in electromagnetism and write electric and magnetic fields in terms of potentials.

Ans:

A potential is a function whose derivative gives a field. Fields are associated with forces; potentials are associated with energy.

The magnetic vector potential $\vec{A}$ is defined so that the magnetic field $\vec{B}$ is given by:

$$\vec{B} = \nabla \times \vec{A} \quad (1)$$

The electric scalar potential ϕ is defined so that the electric field $\vec{E}$ is given by:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \qquad (2)$$

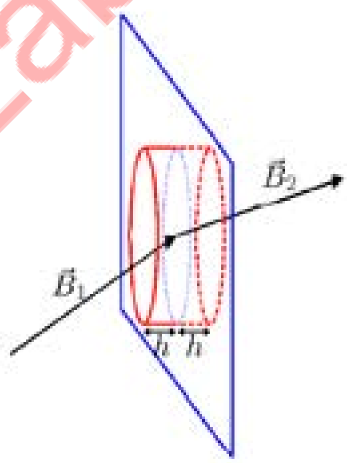
Note that in general, the scalar and vector potentials are functions of position and time.

Que5: Derive boundary conditions at interface between two different media.

Ans:

Boundary Conditions 1: Normal Component of $\vec{B}$

We can use Maxwell's equations to derive the boundary conditions on the magnetic field across a surface. Consider a "pillbox" across the surface.



Take Maxwell's equation:

$$\nabla \cdot \vec{B} = 0 \tag{1}$$

integrate over the volume of the pillbox, and apply Gauss' theorem:

$$\int_V \nabla \cdot \vec{B}\, dV = \oint_S \vec{B} \cdot d\vec{S} = 0 \tag{2}$$

where V is the volume of the pillbox, and S is its surface. We can break the integral over the surface into three parts: over the flat ends (S_1 and S_2) and over the curved wall (S_3):

$$\int_{S_1} \vec{B} \cdot d\vec{S} + \int_{S_2} \vec{B} \cdot d\vec{S} + \int_{S_3} \vec{B} \cdot d\vec{S} = 0 \tag{3}$$

In the limit that the length of the pillbox approaches zero, the integral over the curved surface also approaches zero. If each end has a *small* area A, then equation (3) becomes:

$$-B_{1n}A + B_{2n}A = 0 \tag{4}$$

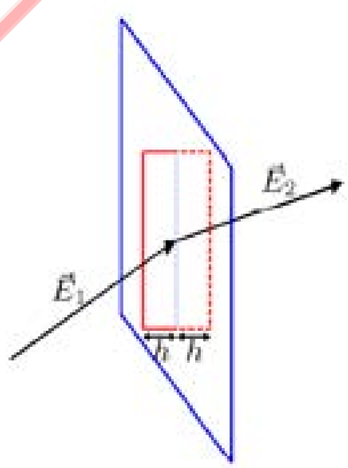
or:

$$B_{1n} = B_{2n} \tag{5}$$

In other words, the normal component of the magnetic field $\vec{B}$ must be continuous across the surface.

Boundary Conditions 2: Tangential Component of $\vec{E}$

Consider a loop spanning the surface.



Take Maxwell's equation:

$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{6}$$

Integrate over the surface bounded by the loop and apply Stokes' theorem to get:

$$\int_S \nabla \times \vec{E} \cdot d\vec{S} = \oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{S} \tag{7}$$

Now take the limit in which the width of the loop becomes zero. The contributions to the integral around the loop C from the narrow ends become zero, as does the integral of the magnetic field across the area bounded by the loop. We are left with:

$$E_{1t} l - E_{2t} l = 0 \tag{8}$$

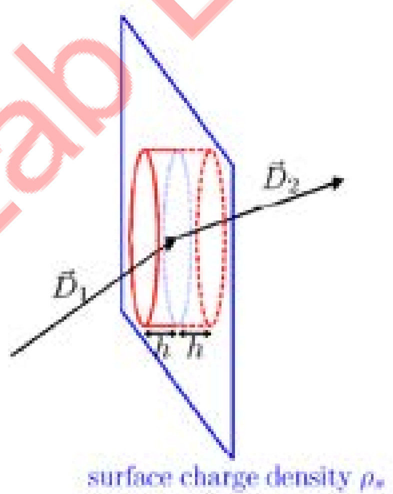
which means that:

$$E_{1t} = E_{2t} \tag{9}$$

Therefore, the tangential component of the electric field is continuous across the boundary.

Boundary Conditions 3: Normal Component of $\vec{D}$

Consider a pillbox crossing the boundary.



Take Maxwell's equation:

$$\nabla \cdot \vec{D} = \rho \tag{10}$$

Integrate over the volume of the pillbox, and apply Gauss' theorem:

$$\int_V \nabla \cdot \vec{D}\, dV = \oint_S \vec{D} \cdot d\vec{S} = \int_V \rho\, dV \tag{11}$$

Now we take the limit in which the height of the pillbox becomes zero. We assume that there is a surface charge density ρ_s. If the flat ends of the pillbox have (small) area A, then:

$$-D_{1n}A + D_{2n}A = \rho_s A \tag{12}$$

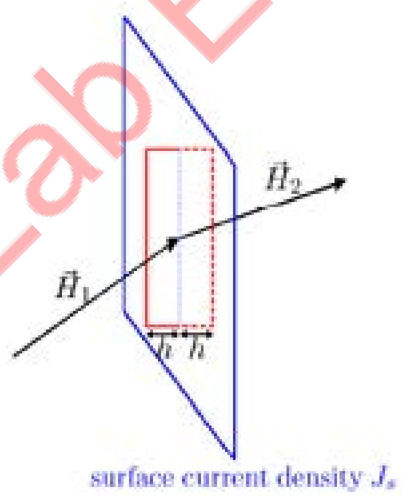
Dividing by the area A, we arrive at:

$$D_{2n} - D_{1n} = \rho_s \tag{13}$$

Note that if the surface charge density is zero, the normal component of $\vec{D}$ is continuous across the surface. However, this is not true for the normal component of $\vec{E}$, unless the two materials have identical permittivities.

Boundary Conditions 4: Tangential Component of $\vec{H}$

Consider a loop across the boundary.



Take Maxwell's equation:

$$\nabla \times \vec{H} = \vec{J} + \dot{\vec{D}} \tag{14}$$

Integrate over the surface bounded by the loop, and apply Stokes' theorem to obtain:

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{S} + \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\vec{S} \tag{15}$$

As before, take the limit where the lengths of the narrow edges of the loop become zero. Then we find that:

$$H_{1t} l - H_{2t} l = J_{s\perp} l \tag{16}$$

or:

$$H_{1t} - H_{2t} = J_{s\perp} \tag{17}$$

where $J_{s\perp}$ represents a surface current density perpendicular to the direction of the tangential component of $\vec{H}$ that is being matched.

The concept of surface current density is analogous to that of surface charge density: it represents a finite current in an infinitesimal layer of the material.

If the material has finite conductivity, then an infinitesimal layer of the material has infinite resistance, and no current can flow (if the electric field is finite).

Therefore, for a material with finite conductivity, we have:

$$H_{1t} = H_{2t} \tag{18}$$

That is, the tangential component of $\vec{H}$ is continuous across the boundary.

The general conditions on electric and magnetic fields at the boundary between two materials can be summarised as follows:

Boundary condition:	Derived from...	...applied to:
$B_{2n} = B_{1n}$	$\nabla \cdot \vec{B} = 0$	pillbox
$E_{2t} = E_{1t}$	$\nabla \times \vec{E} = -\dot{\vec{B}}$	loop
$D_{2n} - D_{1n} = \rho_s$	$\nabla \cdot \vec{D} = \rho$	pillbox
$H_{2t} - H_{1t} = -J_{s\perp}$	$\nabla \times \vec{H} = \vec{J} + \dot{\vec{D}}$	loop

Que6: What are Gauge transformations? Explain Coulomb and Lorentz gauge.

Ans:

Gauge transformations

In electro- and magentostatics, we showed that we could always choose our conventional reference points,

$$V \to 0 \text{ as } r \to \infty \qquad \nabla \cdot \mathbf{A} = 0$$

without placing any peculiar constraints on $\mathbf{E}$ or $\mathbf{B}$. Now we have two, more complicated equations to simplify, and a more general approach is more fruitful.

Consider performing a transformation on $\mathbf{A}$ and V: add a vector to $\mathbf{A}$ and a scalar to V, giving *new* potential functions:

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\alpha} \qquad V' = V + \beta$$

Now, we can't add just any old thing to the potentials; we need for the *fields* arising from the new potentials to be the same as those from the old:

$$\mathbf{B}' = \mathbf{B}$$
$$\nabla\times\mathbf{A}' = \nabla\times\mathbf{A} + \nabla\times\boldsymbol{\alpha}$$
$$\Downarrow$$
$$\nabla\times\boldsymbol{\alpha} = 0 \quad , \quad \text{or}$$
$$\boldsymbol{\alpha} = \nabla\lambda' \quad ,$$

$$\mathbf{E}' = \mathbf{E}$$
$$\nabla V' = \nabla V + \nabla\beta$$
$$-\mathbf{E}' - \frac{1}{c}\frac{\partial\mathbf{A}'}{\partial t} = -\mathbf{E} - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} + \nabla\beta$$
$$\nabla\beta = \frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{A}-\mathbf{A}'\right) = -\frac{1}{c}\frac{\partial\boldsymbol{\alpha}}{\partial t}$$

where λ' is a scalar function of $\mathbf{r}$ and t.

Combine those last two results:

$$\nabla\beta = -\frac{1}{c}\frac{\partial}{\partial t}\nabla\lambda'$$
$$\nabla\left(\beta + \frac{1}{c}\frac{\partial\lambda'}{\partial t}\right) = 0$$

and integrate the second one over volume, applying the fundamental theorem of calculus:

$$\beta + \frac{1}{c}\frac{\partial\lambda'}{\partial t} = f(t)$$

The integration "constant" f is not a function of position

We can combine the integration "constant" f with λ' by defining

$$\lambda = \lambda' - c\int_0^t f(t')\,dt$$

Then, $\beta = -\frac{1}{c}\frac{\partial\lambda'}{\partial t} + f(t) = -\frac{1}{c}\left(\frac{\partial\lambda}{\partial t} + cf(t)\right) + f(t) = -\frac{1}{c}\frac{\partial\lambda}{\partial t}$

$$\boldsymbol{\alpha} = \nabla\lambda' = \nabla\lambda \quad ,$$

Thus for any scalar function $\lambda = \lambda(\mathbf{r},t)$, the transformation

$$V' = V - \frac{1}{c}\frac{\partial\lambda}{\partial t} \qquad \mathbf{A}' = \mathbf{A} + \nabla\lambda$$

Gauge transformation

makes new potentials but leaves the fields $\mathbf{E}$ and $\mathbf{B}$ unchanged.

This sort of operation on potentials is called a gauge transformation, and a particular choice of λ is called a gauge condition.

- Clever choices of λ can simplify one or the other of the second-order differential equations for the potentials.
- The solution of these simpler equations for the transformed potentials gives the same fields as the solutions to the untransformed, complicated equations

For instance, to simplify $\nabla^2 V + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = -4\pi\rho$,
we could pick λ such that

$$\boxed{\nabla \cdot \boldsymbol{A} = 0} \qquad \text{Coulomb gauge}$$

- Coulomb gauge only does a lot of good in magnetoquasistatics, because otherwise the time derivative of $\boldsymbol{A}$ gets big enough that you have to remember that

$$\boldsymbol{E} = -\nabla V - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \quad .$$

Lorentz gauge

It's hard to compute $\boldsymbol{A}$ in Coulomb gauge. On the other hand, we could choose λ such that

$$\boxed{\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial V}{\partial t} = 0 \quad ,} \qquad \text{Lorentz gauge}$$

for which the second-order PDEs we saw several pages back become

$$\nabla^2 V + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = -4\pi\rho$$

$$\Rightarrow \boxed{\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -4\pi\rho}$$

and
$$\left(\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}\right) - \nabla\left(\nabla\cdot A + \frac{1}{c}\frac{\partial V}{\partial t}\right) = -\frac{4\pi}{c}J$$
$$\Rightarrow \boxed{\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi}{c}J}$$

- Not utterly simple, but at least V and A are separately determined, and the four equations are very similar to one another.

Que7: Explain the concept of Poynting vector and Poynting Theorem.

Ans: Poynting vector

When electromagnetic wave travels in space, it carries energy and energy density is always associated with electric fields and magnetic fields.

The rate of energy travelled through per unit area i.e. the amount of energy flowing through per unit area in the perpendicular direction to the incident energy per unit time is called poynting vector.

Mathematically poynting vector is represented as

$$\vec{P} = \vec{E}\times\vec{H}\left(=\frac{\vec{E}\times\vec{B}}{\mu}\right)$$

the direction of poynting vector is perpendicular to the plane containing $\vec{E}$ and $\vec{H}$. Poynting vector is also called as instantaneous energy flux density. Here rate of energy transfer $\vec{P}$ is perpendicular to both $\vec{E}$ and $\vec{H}$. Since it represents the rate of energy transfer per unit area, its unit is W/m^2.

Poynting Theorem

Poynting theorem states that the net power flowing out of a given volume V is equal to the time rate of decrease of stored electromagnetic energy in that volume decreased by the conduction losses.

i.e. total power leaving the volume = rate of decrease of stored electromagnetic energy − ohmic power dissipated due to motion of charge

Que8: Give Mathematical proof of Poynting theorem.

Ans:

Proof : The energy density carried by the electromagnetic wave can be calculated using Maxwell's equations

as $\text{div } \vec{D} = 0$ ...(i) $\text{div } \vec{B} = 0$ ...(ii) $\text{Curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ...(iii)

$$\text{and Curl } \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{...(iv)}$$

taking scalar product of (iii) with H and (iv) with $\vec{E}$

i.e.
$$\vec{H} \text{ curl } \vec{E} = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \quad \text{...(v)}$$

and
$$\vec{E}.\text{ curl } \vec{H} = \vec{E}.\vec{J} + \vec{E}.\frac{\partial \vec{D}}{\partial t} \quad \text{...(vi)}$$

doing (vi) – (v) i.e.
$$\vec{H}.\text{curl } \vec{E} - \vec{E}.\text{ curl } \vec{H} = -\vec{H}.\frac{\partial \vec{B}}{\partial t} - \vec{E}.\vec{J} - \vec{E}.\frac{\partial \vec{D}}{\partial t}$$

$$= -\left[\vec{H}.\frac{\partial \vec{B}}{\partial t} + \vec{E}.\frac{\partial \vec{D}}{\partial t}\right] - \vec{E}.\vec{J}$$

as
$$\text{div}\left(\vec{A} \times \vec{B}\right) = \vec{B}.\text{ curl } \vec{A} - \vec{A} \text{ curl B}$$

so $$\text{div}\left(\vec{E}\times\vec{H}\right) = -\left[\vec{H}.\frac{\partial\vec{B}}{\partial t}+\vec{E}.\frac{\partial\vec{D}}{\partial t}\right]-\vec{E}.\vec{J} \qquad \text{...(vii)}$$

But $$\vec{B} = \mu\vec{H} \quad \text{and} \quad \vec{D} = \varepsilon\vec{E}$$

so $$\vec{H}.\frac{\partial\vec{B}}{\partial t} = \vec{H}.\frac{\partial}{\partial t}\left(\mu\vec{H}\right)=\frac{1}{2}\mu\frac{\partial}{\partial t}\left(H^2\right)$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{2}\vec{H}.\vec{B}\right]$$

and $$\vec{E}.\frac{\partial\vec{D}}{\partial t} = \vec{E}.\frac{\partial}{\partial t}\left(\varepsilon\vec{E}\right)=\frac{1}{2}\varepsilon\frac{\partial}{\partial t}(E)^2=\frac{\partial}{\partial t}\left[\frac{1}{2}\vec{E}.\vec{D}\right]$$

so from equation (vii) $$\text{div}\left(\vec{E}\times\vec{H}\right) = -\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\vec{H}.\vec{B}+\vec{E}.\vec{D}\right)\right]-\vec{E}.\vec{J}$$

or $$\vec{E}.\vec{J} = -\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\vec{H}.\vec{B}+\vec{E}.\vec{D}\right)\right]-\text{div}\left(\vec{E}\times\vec{H}\right) \qquad \text{...(viii)}$$

Integrating equation (viii) over a volume V enclosed by a surface S

$$\int_V \vec{E}.\vec{J}\,dV = -\int_V\left|\frac{\partial}{\partial t}\left\{\frac{1}{2}\left(\vec{H}.\vec{B}+\vec{E}.\vec{D}\right)\right\}\right|dV-\int_V \text{div}\left(\vec{E}\times\vec{H}\right)dV$$

or $$\int_V \vec{E}.\vec{J}\,dV = -\int_V \left[\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right] dV - \int_S \left(\vec{E}\times\vec{H}\right).ds$$

as $\vec{B} = \mu\vec{H}$, $\vec{D} = \varepsilon\vec{E}$ and $\int_V div\left(\vec{E}\times\vec{H}\right)dV = \int_S \left(\vec{E}\times\vec{H}\right).ds$

or $$\int_V \left(\vec{E}\,.\,\vec{J}\right)dV = -\frac{\partial}{\partial t}\int_V \left[\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right] dV - \int_S \left(\vec{E}\times\vec{H}\right).ds$$

or $$\int_s \left(\vec{E}\times\vec{H}\right).ds = -\int_V \frac{\partial\, U_{em}}{\partial t}\,dV - \int_V \left(\vec{E}\,.\,\vec{J}\right)dV$$

or $$\boxed{\int_s \vec{P}.ds = -\int_V \frac{\partial\, U_{em}}{\partial t}\,dV - \int_V \left(\vec{E}\,.\,\vec{J}\right)dV} \quad \left(as\ \vec{P} = \vec{E}\times\vec{H}\right) \quad ...(ix)$$

This is also known as work-energy theorem. This is also called as the energy conservation law in electromagnetism.

Chapter 2

EM Wave Propagation in Unbounded Media: Plane EM waves through vacuum and isotropic dielectric medium, transverse nature of plane EM waves, refractive index and dielectric constant, wave impedance. Propagation through conducting media, relaxation time, skin depth. Wave propagation through dilute plasma, electrical conductivity of ionized gases, plasma frequency, refractive index, skin depth, application to propagation through ionosphere. **(10 Lectures)**

Que1: Derive solutions for plane waves travelling in (i) vacuum and (ii) isotropic linear dielectric medium and show the transverse nature of plane waves. Also calculate the energy density and flux.

Ans:

One of the most important consequences of the Maxwell equations is the equations for electromagnetic wave propagation in a linear medium. In the absence of free charge and current densities the Maxwell equations are

$$\nabla \cdot \mathbf{D} = 0 \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{7.1}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

The wave equations for $\mathbf{E}$ and $\mathbf{B}$ are derived by taking the curl of $\nabla \times \mathbf{H}$ and $\nabla \times \mathbf{E}$

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} \tag{7.2}$$

$$\nabla \times \nabla \times \mathbf{H} = \nabla \times \frac{\partial \mathbf{D}}{\partial t}$$

For uniform isotropic linear media we have $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$, where ε and μ are in general complex functions of frequency ω. Then we obtain

$$\nabla \times \nabla \times \mathbf{E} = -\varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad (7.3)$$
$$\nabla \times \nabla \times \mathbf{B} = -\varepsilon\mu \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Since $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$ and, similarly, $\nabla \times \nabla \times \mathbf{B} = -\nabla^2 \mathbf{B}$,

$$\nabla^2 \mathbf{E} = \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad (7.4)$$
$$\nabla^2 \mathbf{B} = \varepsilon\mu \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Monochromatic waves may be described as waves that are characterized by a single frequency. Assuming the fields with harmonic time dependence $e^{-i\omega t}$, so that $\mathbf{E}(\mathbf{x}, t) = \mathbf{E}(\mathbf{x})e^{-i\omega t}$ and $\mathbf{B}(\mathbf{x}, t) = \mathbf{B}(\mathbf{x})e^{-i\omega t}$ we get the Helmholtz wave equations

$$\nabla^2 \mathbf{E} + \varepsilon\mu\omega^2 \mathbf{E} = 0 \qquad (7.5)$$
$$\nabla^2 \mathbf{B} + \varepsilon\mu\omega^2 \mathbf{B} = 0$$

Plane waves in vacuum

Suppose that the medium is vacuum, so that $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$. Further, suppose $\mathbf{E}(\mathbf{x})$ varies in only one dimension, say the z-direction, and is independent of x and y.

$$\frac{d^2 \mathbf{E}(z)}{dz^2} + k^2 \mathbf{E}(z) = 0 \qquad (7.6)$$

where the wave number $k = \omega/c$. This equation is mathematically the same as the harmonic oscillator equation and has solutions

$$\mathbf{E}_k(z) = \mathcal{E} e^{\pm ikz} \qquad (7.7)$$

where $\mathcal{E}$ is a constant vector. Therefore, the full solution is

$$\mathbf{E}_k(z,t) = \mathcal{E}e^{\pm ikz-i\omega t} = \mathcal{E}e^{-i\omega\left(t\mp\frac{z}{c}\right)} \tag{7.8}$$

This represents a sinusoidal wave traveling to the right or left in the z-direciton with the speed of light c. Using the Fourier superposition theorem, we can construct a general solution of the form

$$\mathbf{E}(z,t) = \mathbf{F}(z-ct) + \mathbf{G}(z+ct) \tag{7.9}$$

Plane waves in a nonconducting, nonmagnetic dielectric

In a nonmagnetic dielectric, we have $\mu = \mu_0$ and the index of refraction

$$n(\omega) = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}} \tag{7.10}$$

We see that the results are the same as in vacuum, except that the velocity of wave propagation or the phase velocity is now $v = c/n$ instead of c. Then the wave number is

Transverse Nature of EM waves

$$k(\omega) = n(\omega)\frac{\omega}{c} \tag{7.11}$$

Electromagnetic plane wave of frequency ω and wave vector

Suppose an electromagnetic plane wave with direction of propagation $\mathbf{n}$ to be constructed, $\mathbf{n}$ where is a unit vector. Then the variable z in the exponent must be replaced by $\mathbf{n}\cdot\mathbf{x}$, the projection of $\mathbf{x}$ in the $\mathbf{n}$ direction. Thus an electromagnetic plane wave with direction of propagation $\mathbf{n}$ is described by

$$\begin{aligned} \mathbf{E}(\mathbf{x},t) &= \mathcal{E}e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} = \mathcal{E}e^{ik\mathbf{n}\cdot\mathbf{x}-i\omega t} \\ \mathbf{B}(\mathbf{x},t) &= \mathcal{B}e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} = \mathcal{B}e^{ik\mathbf{n}\cdot\mathbf{x}-i\omega t} \end{aligned} \tag{7.12}$$

where $\mathcal{E}$ and $\mathcal{B}$ are complex constant vector amplitudes of the plane wave. $\mathbf{E}$ and $\mathbf{B}$ satisfy the wave equations (Eq. 7.5), therefore the dispersion relation is given as

$$k^2 = \varepsilon\mu\omega^2 = \left(n\frac{\omega}{c}\right)^2 \quad \rightarrow \quad k = n\frac{\omega}{c} \tag{7.13}$$

Let us substitute the plane wave solutions (Eq. 7.12) into the Maxwell equations. This substitution will impose conditions on the constants, $\mathbf{k}$, $\mathcal{E}$ and $\mathcal{B}$, for the plane wave functions to be solutions of the Maxwell equations. For the plane waves, one sees that the operators

$$\frac{\partial}{\partial t} = -i\omega, \qquad \nabla = i\mathbf{k}$$

Thus the Maxwell equations become

$$\begin{matrix} \nabla\cdot\mathbf{D} = 0 & \nabla\times\mathbf{H} = \dfrac{\partial\mathbf{D}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0 & \nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t} \end{matrix} \quad \rightarrow \quad \begin{matrix} \mathbf{k}\cdot\mathcal{E} = 0 & \mathbf{k}\times\mathcal{B} = -\varepsilon\mu\omega\mathcal{E} \\ \mathbf{k}\cdot\mathcal{B} = 0 & \mathbf{k}\times\mathcal{E} = \omega\mathcal{B} \end{matrix} \tag{7.14}$$

where $\mathbf{k} = k\mathbf{n}$. The direction $\mathbf{n}$ and frequency ω are completely arbitrary. The divergence equations demand that

$$\mathbf{n}\cdot\mathcal{E} = 0 \quad \text{and} \quad \mathbf{n}\cdot\mathcal{B} = 0 \tag{7.15}$$

This means that $\mathbf{E}$ and $\mathbf{B}$ are both perpendicular to the direction of propagation $\mathbf{n}$. The magnitude of $\mathbf{k}$ is determined by the refractive index of the material

$$k = n\frac{\omega}{c} \tag{7.16}$$

Then $\mathcal{B}$ is completely determined in magnitude and direction

$$\mathcal{B} = \sqrt{\varepsilon\mu}\,\mathbf{n}\times\mathcal{E} = \frac{n}{c}\mathbf{n}\times\mathcal{E} \tag{7.17}$$

Energy density and flux

The time averaged energy density is

$$u = \frac{1}{4}(\mathbf{E}\cdot\mathbf{D}^* + \mathbf{B}\cdot\mathbf{H}^*) = \frac{1}{4}\left(\varepsilon\mathbf{E}\cdot\mathbf{E}^* + \frac{1}{\mu}\mathbf{B}\cdot\mathbf{B}^*\right)$$

This gives

$$u = \frac{\varepsilon}{2}|\mathcal{E}|^2 = \frac{1}{2}n^2|\mathcal{E}|^2$$

The time averaged energy flux is given by the real part of the complex Poynting vector

$$\mathbf{S} = \frac{1}{2}(\mathbf{E}\times\mathbf{H}^*)$$

Thus the energy flow is

$$\mathbf{S} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathcal{E}|^2\mathbf{n} = \frac{1}{2}n^2|\mathcal{E}|^2\cdot v\mathbf{n} = u\mathbf{v}$$

Que2 : Derive solutions for plane waves travelling in conducting medium. Also derive the expression of (i)Propagation constant (ii) wave impedance (iii) skin depth for a conductor.

Ans:

We will consider a plane electromagnetic wave travelling in a linear dielectric medium such as air along the z direction and being incident at a conducting interface. The medium will be taken to be a linear medium. So that one can describe the electrodynamics using only the E and H vectors

As the medium is linear and the propagation takes place in the infinite medium, the vectors $\vec{E}, \vec{H}$ and $\vec{k}$ are still mutually perpendicular. We take the electric field along the x direction, the magnetic field along the y- dirrection and the propagation to take place in the z direction. Further, we will take the conductivity to be finite and the conductor to obey Ohm's law, $\vec{J} = \sigma\vec{E}$. Consider the pair of curl equations of Maxwell.

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$\nabla\times\vec{H} = \vec{J} + \epsilon\frac{\partial\vec{E}}{\partial t} = \sigma\vec{E} + \epsilon\frac{\partial\vec{E}}{\partial t}$$

Let us take $\vec{E}, \vec{H}$ and $\vec{k}$ to be respectively in x, y and z direction. We the nhave,

$$(\nabla \times \vec{E})_y = \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$

i.e.,

$$\frac{\partial E_x}{\partial z} + \mu \frac{\partial H_y}{\partial t} = 0 \qquad (1)$$

and

$$(\nabla \times \vec{H})_x = -\frac{\partial H_y}{\partial z} = \sigma E_x + \epsilon \frac{\partial E_x}{\partial t}$$

i.e.

$$\frac{\partial H_y}{\partial z} + \sigma E_x + \epsilon \frac{\partial E_x}{\partial t} = 0 \quad (2)$$

We take the time variation to be harmonic ($\sim e^{i\omega t}$) so that the time derivative is equivalent to a multiplication by $i\omega$. The pair of equations (1) and (2) can then be written as

$$\frac{\partial E_x}{\partial z} + i\mu\omega H_y = 0$$
$$\frac{\partial H_y}{\partial z} + \sigma E_x + i\omega\epsilon E_x = 0$$

We can solve this pair of coupled equations by taking a derivative of either of the equations with respect to z and substituting the other into it,

$$\frac{\partial^2 E_x}{\partial z^2} + i\mu\omega \frac{\partial H_y}{\partial z} = \frac{\partial^2 E_x}{\partial z^2} - i\mu\omega(\sigma + i\omega\epsilon)E_x = 0$$

Define, a complex constant γ through

$$\gamma^2 = i\mu\omega(\sigma + i\omega\epsilon)$$

in terms of which we have,

$$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0 \quad (3)$$

In an identical fashion, we get

$$\frac{\partial^2 H_y}{\partial z^2} - \gamma^2 H_y = 0 \quad (4)$$

Solutions of (3) and (4) are well known and are expressed in terms of hyperbolic functions,

$$E_x = A\cosh(\gamma z) + B\sinh(\gamma z)$$
$$H_y = C\cosh(\gamma z) + D\sinh(\gamma z)$$

where A, B, C and D are constants to be determined. If the values of the electric field at z=0 is E_0 and that of the magnetic field at z=0 is H_0, we have $A = E_0$ and $C = H_0$.

In order to determine the constants B and D, let us return back to the original first order equations (1) and (2)

$$\frac{\partial E_x}{\partial z} + i\mu\omega H_y = 0$$
$$\frac{\partial H_y}{\partial z} + \sigma E_x + i\omega\epsilon E_x = 0$$

Substituting the solutions for E and H

$$\gamma E_0\sinh(\gamma z) + B\gamma\cosh(\gamma z) + i\omega\mu(H_0\cosh(\gamma z) + D\sinh(\gamma z)) = 0$$

This equation must remain valid for all values of z, which is possible if the coefficients of *sinh* and cosh terms are separately equated to zero,

$$E_0\gamma + i\omega\mu D = 0$$
$$B\gamma + i\omega\mu H_0 = 0$$

The former gives,

$$D = -\frac{\gamma}{i\omega\mu}E_0$$
$$= -\frac{\sqrt{i\mu\omega(\sigma + i\omega\epsilon)}}{i\omega\mu}$$
$$= -\sqrt{\frac{\sigma + i\omega\epsilon}{i\omega\mu}}E_0$$
$$= -\frac{E_0}{\eta}$$

where

$$\eta = \sqrt{\frac{i\omega\mu}{\sigma + i\omega\epsilon}}$$

Likewise, we get,

$$B = -\eta H_0$$

Substituting these, our solutions for the E and H become,

$$E_x = E_0 \cosh(\gamma z) - \eta H_0 \sinh(\gamma z)$$
$$H_y = H_0 \cosh(\gamma z) - \frac{E_0}{\eta}\sinh(\gamma z)$$

The wave is propagating in the z direction. Let us evaluate the fields when the wave has reached $z = l$,

$$E_x = E_0 \cosh(\gamma l) - \eta H_0 \sinh(\gamma l)$$
$$H_y = H_0 \cosh(\gamma l) - \frac{E_0}{\eta}\sinh(\gamma l)$$

If l is large, we can approximate

$$\sinh(\gamma l) \approx \cosh(\gamma l) = \frac{e^{\gamma l}}{2}$$

we then have,

$$E_x = (E_0 - \eta H_0)\frac{e^{\gamma l}}{2}$$
$$H_y = (H_0 - \frac{E_0}{\eta})\frac{e^{\gamma l}}{2}$$

The ratio of the magnitudes of the electric field to magnetic field is defined as the "characteristic impedance" of the wave

$$\left|\frac{E_x}{H_y}\right| = \eta = \sqrt{\frac{i\omega\mu}{\sigma + i\omega\epsilon}}$$

Let us look at the full three dimensional version of the propagation in a conductor. Once again, we start with the two curl equations,

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \sigma\vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}$$

Take a curl of both sides of the first equation,

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial (\nabla \times \vec{H})}{\partial t}$$

As there are no charges or currents, we ignore the divergence term and substitute for the curl of H from the second equation,

$$\nabla^2 \vec{E} = \mu \frac{\partial}{\partial t}\left(\sigma\vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}\right)$$

$$= \sigma\mu \frac{\partial \vec{E}}{\partial t} + \sigma\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

We take the propagating solutions to be

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

so that the above equation becomes,

$$k^2\vec{E} = (i\omega\mu\sigma + \omega^2\mu\epsilon)\vec{E}$$

so that we have, the complex propagation constant to be given by

$$k^2 = i\omega\mu\sigma + \omega^2\mu\epsilon$$

so that

$$k = \sqrt{\omega\mu(\omega\epsilon + i\sigma)}$$

k is complex and its real and imaginary parts can be separated by standard algebra,

we have

$$k = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\left(1 + \sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}}\right)^{1/2} + i\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} - 1\right)^{1/2}\right]$$

Thus the propagation vector β and the attenuation factor α are given by

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\sqrt{\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} + 1\right)}$$

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\sqrt{\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} - 1\right)}$$

The ration $\frac{\sigma}{\omega\epsilon}$ determines whether a material is a good conductor or otherwise. Consider a good conductor for which $\sigma \gg \omega\epsilon$. For this case, we have,

$$\beta = \alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$$

The speed of electromagnetic wave is given by

$$v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\sigma\mu}}$$

The electric field amplitude diminishes with distance as $e^{-\alpha z}$. The distance to which the field penetrates before its amplitude diminishes be a factor e^{-1} is known as the "**skin depth**" , which is given by

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

The wave does not penetrate much inside a conductor. Consider electromagnetic wave of frequency 1 MHz for copper which has a conductivity of approximately $6 \times 10^7 \Omega^{-1}\mathrm{m}^{-1}$. Substituting these values, one gets the skin depth in Cu to be about 0.067 mm. For comparison, the skin depth in sea water which is conducting because of salinity, is about 25 cm while that for fresh water is nearly 7m. Because of small skin depth in conductors, any current that arises in the metal because of the electromagnetic wave is confined within a thin layer of the surface.

Que3: Explain EM wave propagation through plasma and derive characteristic plasma frequency.

Ans:

Consider a point particle of mass m and electric charge q interacting with a linearly polarized, sinusoidal, electromagnetic plane wave that propagates in the z -direction. Provided that the wave amplitude is not sufficiently large to cause the particle to move at relativistic speeds, the electric component of the wave exerts a much greater force on the particle than the magnetic component. [This follows, from standard electrodynamics, because the ratio of the magnetic to the electric force is of order $B_0 v/E_0$, where E_0 is the amplitude of the wave electric field-strength, $B_0 = E_0/c$ the amplitude of the wave magnetic field-strength, v the particle velocity, and c the velocity of light in vacuum. Hence, the ratio of the forces is approximately v/c

Suppose that the electric component of the wave oscillates in the x -direction, and takes the form

$$E_x(z,t) = E_0 \cos(\omega t - k z), \tag{789}$$

where k is the wavenumber, and ω the angular frequency. The equation of motion of the particle is thus

$$m \frac{d^2 x}{dt^2} = q E_x, \tag{790}$$

where x measures its wave-induced displacement in the x -direction. The previous equation can be solved to give

$$x = -\frac{q E_0}{m \omega^2} \cos(\omega t - k z). \tag{791}$$

Thus, the wave causes the particle to execute sympathetic simple harmonic oscillations, in the x -direction, with an amplitude that is directly proportional to its charge, and inversely proportional to its mass.

Suppose that the wave is actually propagating through an unmagnetized electrically neutral plasma consisting of free electrons, of mass m_e and charge $-e$, and free ions, of mass m_i and charge $+e$. Since the plasma is assumed to be electrically neutral, each species must have the same equilibrium number density, n_e. Given that the electrons are much less massive than the ions (i.e., $m_e \ll m_i$), but have the same charge (modulo a sign), it follows from Equation (791) that the wave-induced oscillations of the electrons are of much higher amplitude than those of the ions. In fact, to a first approximation, we can say that the electrons oscillate while the ions remain stationary. Assuming that the electrons and ions are evenly distributed throughout the plasma, the wave-induced displacement of an individual electron generates an effective electric dipole moment in the x -direction of the form $p_x = -e\, x$ (the other component of the dipole is a stationary ion of charge $+e$ located at $x = 0$).

Hence, the x -directed electric dipole moment per unit volume is

$$P_x = n_e\, p_x = -n_e\, e\, x. \tag{792}$$

Given that all of the electrons oscillate according to Equation (791) (with $q = -e$ and $m = m_e$), we obtain

$$P_x(z,t) = -\frac{n_e\, e^2 E_0}{m_e\, \omega^2} \cos(\omega t - k z). \tag{793}$$

And,

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\epsilon_0}\left(\frac{\partial P_x}{\partial t} + \frac{\partial H_y}{\partial z}\right), \tag{794}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0}\frac{\partial E_x}{\partial z}. \tag{795}$$

Thus, writing E_x in the form (789), H_y in the form

$$H_y(z,t) = Z^{-1}\,E_0\,\cos(\omega\,t - k\,z), \tag{796}$$

where Z is the effective impedance of the plasma, and P_x in the form (793). Equations (794) and (795) yield

the nonlinear dispersion relation

$$\omega^2 = k^2\,c^2 + \omega_p^2, \tag{797}$$

where $c = 1/\sqrt{\epsilon_0\,\mu_0}$ is the velocity of light in vacuum, and the so-called *plasma frequency*,

$$\omega_p = \left(\frac{n_e\,e^2}{\epsilon_0\,m_e}\right)^{1/2}, \tag{798}$$

is the characteristic frequency of collective electron oscillations in the plasma (Stix 1962). Equations (794) and (795) also yield

$$Z = \frac{Z_0}{n}, \tag{799}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space, and

$$n = \frac{k\,c}{\omega} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$$

the effective refractive index of the plasma. We, thus, conclude that sinusoidal electromagnetic waves propagating through an unmagnetized plasma have a nonlinear dispersion relation. Moreover, this nonlinearity arises because the effective refractive index of the plasma is frequency dependent.

Substituting values of permittivity, electron mass and electronic charge, plasma frequency becomes, $f_c = 9\sqrt{N_0}$

Chapter 3

EM Wave in Bounded Media: Boundary conditions at a plane interface between two media. Reflection & Refraction of plane waves at plane interface between two dielectric media-Laws of Reflection & Refraction. Fresnel's Formulae for perpendicular & parallel polarization cases, Brewster's law. Reflection & Transmission coefficients.Total internal reflection, evanescent waves. Metallic reflection (normal Incidence) **(10 Lectures)**

Que1: Derive laws of reflection and refraction for plane waves incident at normal interface between two dielectric media.

Ans:

We begin with the simplest possible case: a plane wave normally incident on a plane dielectric interface. We will see that the boundary conditions are satisfied only if reflected and transmitted waves are present.

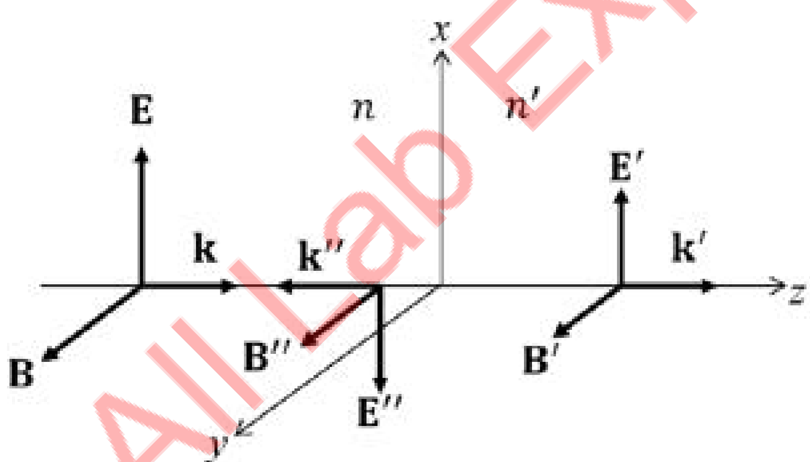


Fig 7.5 Reflection and transmission at normal incidence

Fig. 7.5 describes the incident wave ($\mathbf{E}, \mathbf{B}$) travelling in the z-direction, the reflected wave ($\mathbf{E}'', \mathbf{B}''$) travelling in the minus z-direction, and the transmitted wave ($\mathbf{E}', \mathbf{B}'$) travelling in the z-direction. The interface is taken as coincident with the xy-plane at $z = 0$, with two dielectric media with the indices of refraction, n for $z < 0$ and n' for $z > 0$. The electric fields, which are assumed to be linearly polarized in the x-direction, are described by

$$\begin{cases} \mathbf{E} = \mathbf{e}_x E e^{i(kz-\omega t)} \\ \mathbf{E}' = \mathbf{e}_x E' e^{i(k'z-\omega t)} \\ \mathbf{E}'' = -\mathbf{e}_x E'' e^{i(-kz-\omega t)} \end{cases} \tag{7.49}$$

where

$$k = n\frac{\omega}{c}, \qquad k' = n'\frac{\omega}{c} \tag{7.50}$$

From Eq. 7.17,

$$\mathcal{B} = \frac{n}{ck}\mathbf{k} \times \mathcal{E}$$

Therefore, the magnetic fields associated with the electric fields are given by eq 7.49

$$\begin{cases} c\mathbf{B} = \mathbf{e}_y n E e^{i(kz-\omega t)} \\ c\mathbf{B}' = \mathbf{e}_y n' E' e^{i(k'z-\omega t)} \\ c\mathbf{B}'' = \mathbf{e}_y n E'' e^{i(-kz-\omega t)} \end{cases} \tag{7.51}$$

Clearly the reflected and transmitted waves must have the same frequency ω as the incident wave if boundary conditions at $z = 0$ are to be satisfied for all t. The E-field must be continuous at the boundary,

$$E - E'' = E' \tag{7.52}$$

The H-field must also be continuous, and for nonmagnetic media ($\mu = \mu' = \mu_0$), so must be the B-field:

$$n(E + E'') = n'E' \tag{7.53}$$

Eqs. 7.52 and 7.53 can be solved simultaneously for the amplitudes E' and E'' in terms of the incident amplitude E:

$$E'' = \frac{n' - n}{n' + n} E, \qquad E' = \frac{2n}{n' + n} E \tag{7.54}$$

The Fresnel coefficients for normal incidence reflection and transmission are defined as

$$r = \frac{E''}{E} = \frac{n' - n}{n' + n}, \qquad t = \frac{E'}{E} = \frac{2n}{n' + n} \tag{7.55}$$

For $n' > n$, there is a phase reversion for the reflected wave.

What is usually measureable is the reflected and transmitted average energy fluxes per unit area (a.k.a., the intensity of EM wave) given by the magnitude of the Poynting vector

$$S = \frac{1}{2}|\mathbf{E} \times \mathbf{H}^*| = \frac{1}{2} n c \varepsilon_0 |E|^2 \tag{7.56}$$

We define the reflectance R and the transmittance T for normal incidence by the ratios of the intensities

$$R = \frac{S''}{S} = |r|^2 = \left(\frac{n' - n}{n' + n}\right)^2, \qquad T = \frac{S'}{S} = \frac{n'}{n}|t|^2 = \frac{4nn'}{(n' + n)^2} \tag{7.57}$$

With the Fresnel coefficients given by Eq. 7.55, R and T satisfy

$$R + T = 1 \tag{7.58}$$

for any pair of nonconducting media. This is an expression of energy conservation at the interface.

Que2: Derive laws of reflection and refraction for plane waves incident at oblique interface between two dielectric media and calculate Fresnel coefficients.

Ans:

Oblique incidence

We consider reflection and refraction at the boundary of two dielectric media at oblique incidence. The discussion will lead to three well-known optical laws: Snell's law, the law of reflection, and Brewster's law governing polarization by reflection. Fig. 7.6 depicts the situation that the wave vectors, $\mathbf{k}$, $\mathbf{k}'$, and $\mathbf{k}''$, are coplanar and lie in the xz-plane. The media for $z < 0$ and $z > 0$ have the indices of refraction, n and n', respectively. The unit normal to the boundary is $\mathbf{n}$. The plane defined by $\mathbf{k}$ and $\mathbf{n}$ is called the *plane of incidence*, and its normal is in the direction of $\mathbf{k} \times \mathbf{n}$.

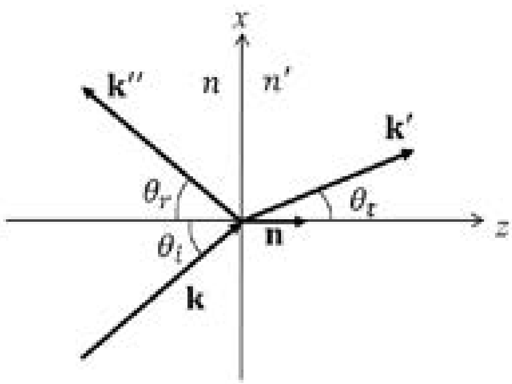


Fig 7.6 Reflection and transmission at oblique incidence. Incident wave $\mathbf{k}$ strikes plane interface between different media, giving rise to a reflected wave $\mathbf{k}''$ and refracted wave $\mathbf{k}'$.

The three plane waves are:

Incident

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \qquad (7.59)$$
$$\mathbf{B} = \frac{n}{ck}\,\mathbf{k} \times \mathbf{E}$$

Refracted

$$\mathbf{E}' = \mathbf{E}_0' e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega t)} \qquad (7.60)$$
$$\mathbf{B}' = \frac{n'}{ck'}\,\mathbf{k}' \times \mathbf{E}'$$

Reflected

$$\mathbf{E} = \mathbf{E}_0'' e^{i(\mathbf{k}''\cdot\mathbf{x}-\omega t)} \qquad (7.61)$$
$$\mathbf{B} = \frac{n}{ck}\,\mathbf{k}'' \times \mathbf{E}''$$

where

$$k = n\frac{\omega}{c}, \qquad k' = n'\frac{\omega}{c} \qquad (7.62)$$

Phase matching on the boundary

Not only must the refracted and reflected waves have the same frequency as the incident wave, but also the phases must match everywhere on the boundary to satisfy boundary conditions at all points on the plane at all times:

$$(\mathbf{k}\cdot\mathbf{x})_{z=0} = (\mathbf{k}'\cdot\mathbf{x})_{z=0} = (\mathbf{k}''\cdot\mathbf{x})_{z=0} \tag{7.63}$$

This condition has three interesting consequences. Using the vector identity

$$\mathbf{n}\times(\mathbf{n}\times\mathbf{x}) = (\mathbf{n}\cdot\mathbf{x})\mathbf{n} - \mathbf{x} \tag{7.64}$$

and $\mathbf{n}\cdot\mathbf{x} = 0$ on the boundary, we obtain

$$\mathbf{x} = -\mathbf{n}\times(\mathbf{n}\times\mathbf{x}) \tag{7.65}$$

We substitute this into Eq. 7.63,

$$\mathbf{k}\cdot\mathbf{x} = -\mathbf{k}\cdot[\mathbf{n}\times(\mathbf{n}\times\mathbf{x})] = -(\mathbf{k}\times\mathbf{n})\cdot(\mathbf{n}\times\mathbf{x}) \tag{7.66}$$

and similarly for the other members of Eq. 7.63. Since $\mathbf{x}$ is an arbitrary vector on the boundary, Eq. 7.63 can hold if and only if

$$\mathbf{k}\times\mathbf{n} = \mathbf{k}'\times\mathbf{n} = \mathbf{k}''\times\mathbf{n} \tag{7.67}$$

This implies that

(i) All three vectors, $\mathbf{k}$, $\mathbf{k}'$ and $\mathbf{k}''$, lie in a plane, i.e., $\mathbf{k}'$ and $\mathbf{k}''$ lie in the plane of incidence;

(ii) Law of reflection: $|\mathbf{k}\times\mathbf{n}| = |\mathbf{k}''\times\mathbf{n}| \rightarrow k\sin\theta_i = k\sin\theta_r$, thus

$$\theta_i = \theta_r \tag{7.68}$$

(iii) Snell's Law: $|\mathbf{k}\times\mathbf{n}| = |\mathbf{k}'\times\mathbf{n}| \rightarrow k\sin\theta_i = k'\sin\theta_t$, thus

$$n\sin\theta_i = n'\sin\theta_t \tag{7.69}$$

Boundary conditions and Fresnel coefficients

At all points on the boundary, normal components of **D** and **B** and tangential components of **E** and **H** are continuous. The boundary conditions at $z = 0$ are

$$\begin{aligned} &\text{(i)} & [\varepsilon(\mathbf{E}_0 + \mathbf{E}_0'') - \varepsilon'\mathbf{E}_0']\cdot\mathbf{n} &= 0 \\ &\text{(ii)} & [\mathbf{k}\times\mathbf{E}_0 + \mathbf{k}''\times\mathbf{E}_0'' - \mathbf{k}'\times\mathbf{E}_0']\cdot\mathbf{n} &= 0 \\ &\text{(iii)} & [\mathbf{E}_0 + \mathbf{E}_0'' - \mathbf{E}_0']\times\mathbf{n} &= 0 \\ &\text{(iv)} & \left[\frac{1}{\mu}(\mathbf{k}\times\mathbf{E}_0 + \mathbf{k}''\times\mathbf{E}_0'') - \frac{1}{\mu'}\mathbf{k}'\times\mathbf{E}_0'\right]\times\mathbf{n} &= 0 \end{aligned} \tag{7.70}$$

In applying the boundary conditions it is convenient to consider two separate situations: the incident plane wave is linearly polarized with its polarization vector (a) perpendicular (s-polarization) and (b) parallel (p-polarization) to the plane of incidence (see Fig. 7.7). For simplicity, we assume the dielectrics are nonmagnetic ($\mu = \mu' = \mu_0$).

(a) s-polarization

The E-fields are normal to $\mathbf{n}$, therefore (i) in Eq. 7.70 is automatically satisfied. (iii) and (iv) give

$$E_0 + E_0'' - E_0' = 0 \tag{7.71}$$

and

$$n(E_0 - E_0'')\cos\theta_i - n'E_0'\cos\theta_t = 0 \tag{7.72}$$

while (ii), using Snell's law, duplicates (iii). With Eqs. 7.71 and 7.72, we obtain the s-pol Fresnel coefficients,

$$t_s = \frac{E_0'}{E_0} = \frac{2n\cos\theta_i}{n\cos\theta_i + n'\cos\theta_t} = \frac{2n\cos\theta_i}{n\cos\theta_i + \sqrt{n'^2 - n^2\sin^2\theta_i}} \tag{7.73}$$

and

$$r_s = \frac{E_0''}{E_0} = \frac{n\cos\theta_i - n'\cos\theta_t}{n\cos\theta_i + n'\cos\theta_t} = \frac{n\cos\theta_i - \sqrt{n'^2 - n^2\sin^2\theta_i}}{n\cos\theta_i + \sqrt{n'^2 - n^2\sin^2\theta_i}} \tag{7.74}$$

where, using Snell's law, we could write

$$\cos\theta_t = \sqrt{1 - (n/n')^2\sin^2\theta_i} \tag{7.75}$$

(b) p-polarization

The B-fields are normal to $\mathbf{n}$, therefore (ii) in Eq. 7.70 is automatically satisfied. (iii) and (iv) give

$$\cos\theta_i\,(E_0 - E_0'') - \cos\theta_t\,E_0' = 0 \tag{7.76}$$

and

$$n(E_0 + E_0'') - n'E_0' = 0 \tag{7.77}$$

while (i), using Snell's law, duplicates (iv). With Eqs. 7.76 and 7.76, we obtain the p-pol Fresnel coefficients,

$$t_p = \frac{E_0'}{E_0} = \frac{2n\cos\theta_i}{n'\cos\theta_i + n\cos\theta_t} = \frac{2nn'\cos\theta_i}{n'^2\cos\theta_i + n\sqrt{n'^2 - n^2\sin^2\theta_i}} \tag{7.78}$$

and

$$r_p = \frac{E_0''}{E_0} = \frac{n' \cos\theta_i - n\cos\theta_t}{n'\cos\theta_i + n\cos\theta_t} = \frac{n'^2\cos\theta_i - n\sqrt{n'^2 - n^2\sin^2\theta_i}}{n'^2\cos\theta_i + n\sqrt{n'^2 - n^2\sin^2\theta_i}} \qquad (7.79)$$

For normal incidence, $r_p = -r_s = -(n-n')/(n+n')$, because we assign opposite directions for $\mathbf{E}$ and $\mathbf{E}''$ for p-polarization.

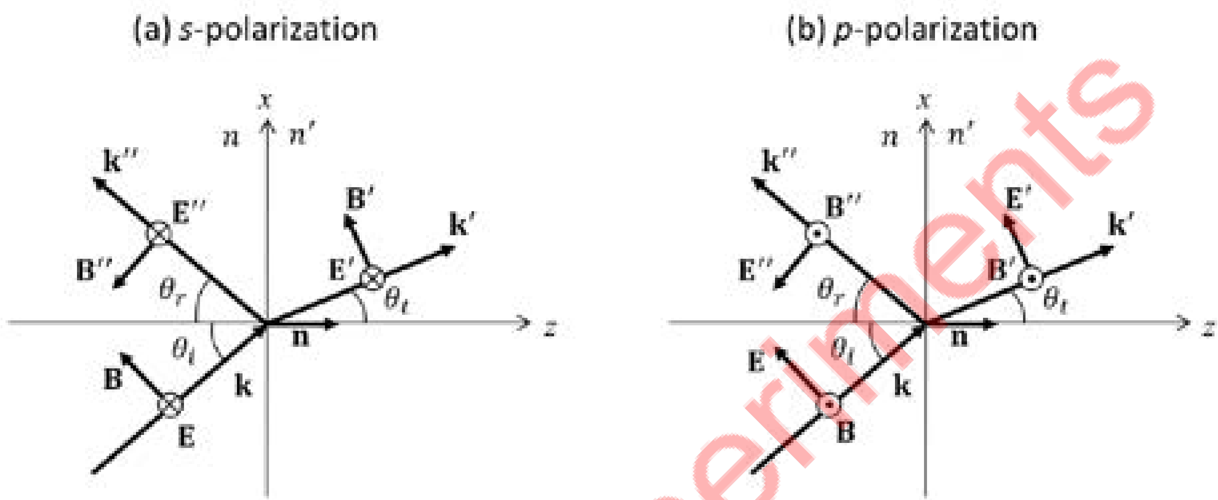


Fig 7.7 Reflection and refraction with polarization (a) perpendicular (s-polarization) and (b) parallel (p-polarization) to the plane of incidence

For certain purposes, it is more convenient to express the Fresnel coefficients in terms of the incident and refraction angles, θ_i and θ_t only. Using the Snell's law, $n\sin\theta_i = n'\sin\theta_t$, we can write

$$t_s = \frac{2n\cos\theta_i}{n\cos\theta_i + n'\cos\theta_t} = \frac{2\cos\theta_i}{\cos\theta_i + \dfrac{n'}{n}\cos\theta_t} = \frac{2\cos\theta_i\sin\theta_t}{\sin\theta_t\cos\theta_i + \cos\theta_t\sin\theta_i}$$

then

$$t_s = \frac{2 \sin\theta_t \cos\theta_i}{\sin(\theta_t + \theta_i)} \tag{7.80}$$

Similarly,

$$r_s = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \tag{7.81}$$

$$t_p = \frac{2 \sin\theta_t \cos\theta_i}{\sin(\theta_t + \theta_i)\cos(\theta_t - \theta_i)} \tag{7.82}$$

and

$$r_p = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)} \tag{7.83}$$

Que3: Derive Brewster's angle and explain total internal reflection.

Brewster's angle and total internal reflection

We next consider the dependence of R and T on the angle of incidence, using the Fresnel coefficients.

Brewster angle

We see that r_p in Eq. 7.88 vanishes when $\theta_t + \theta_i = \pi/2$. Using Snell's law, we can determine *Brewster's angle* $\theta_B = \theta_i$ at which the p-polarized reflected wave is zero:

$$n \sin\theta_B = n' \sin\left(\frac{\pi}{2} - \theta_B\right) = n' \cos\theta_B$$

or

$$\tan\theta_B = \frac{n'}{n} \tag{7.84}$$

Polarization at the Brewster angle is a practical means of producing polarized radiation. If a plane wave of mixed polarization is incident on a plane interface at the Brewster angle, the reflected radiation is completely s-polarized. The generally lower reflectance for p-polarized lights accounts for the usefulness of polarized sunglasses. Since most outdoor reflecting surfaces are horizontal, the plane of incidence for most reflected glare reaching the eyes is vertical. The polarized lenses are oriented to eliminate the strongly reflected s-component. Fig. 7.8 shows R_s and R_p as a function of θ_i with $n = 1$ and $n' = 1.5$, as for an air-glass interface. The Brewster angle is $\theta_B = 56^\circ$ for this case.

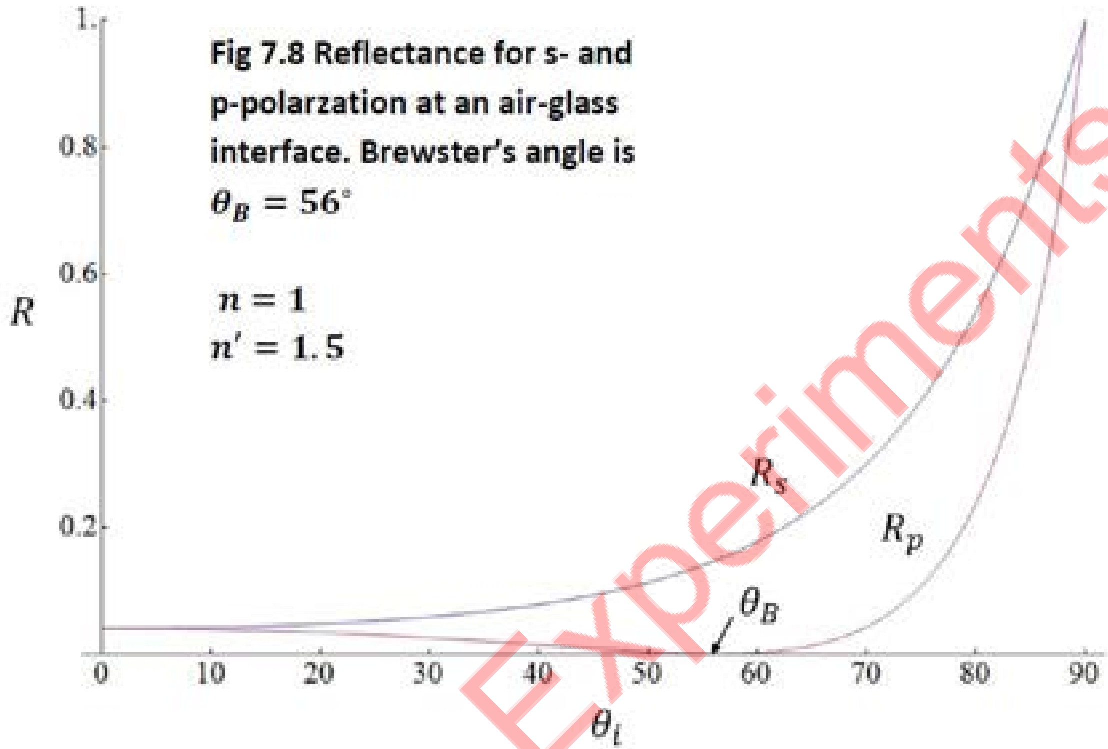


Fig 7.8 Reflectance for s- and p-polarzation at an air-glass interface. Brewster's angle is $\theta_B = 56^\circ$

<u>Total internal reflection</u>

There is another case in which $R_s = R_p = 1$. Eqs. 7.74 and 7.79 indicates that perfect reflection occurs for $\theta_t = \pi/2$. The incident angle for which $\theta_t = \pi/2$ is called the *critical angle*, $\theta_i = \theta_c$. From Snell's law

$$\sin\theta_c = \frac{n'}{n} \tag{7.85}$$

θ_c can exist only if $n > n'$, i.e., the incident and reflected waves are in a medium of larger index of refraction than the refracted wave.

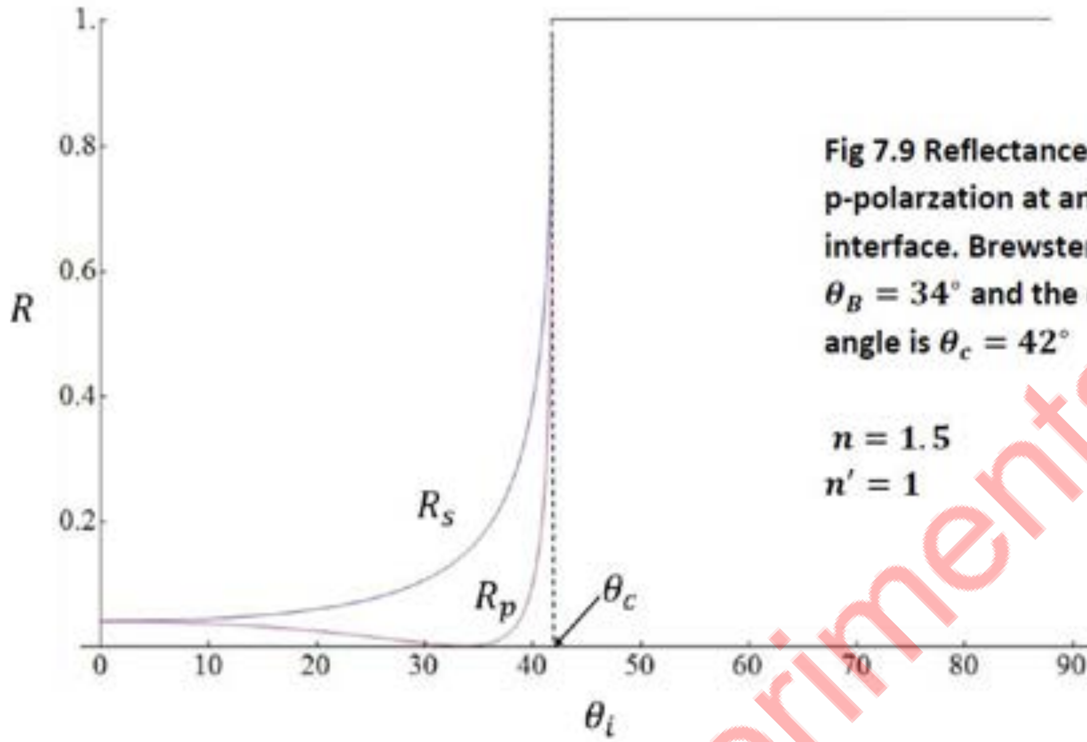


Fig 7.9 Reflectance for s- and p-polarzation at an air-glass interface. Brewster's angle is $\theta_B = 34^\circ$ and the critical angle is $\theta_c = 42^\circ$

$n = 1.5$
$n' = 1$

For waves incident at θ_c, the refracted wave is propagated parallel to the surface. There can be no energy flow across the surface. Hence at that angle of incidence there must be total reflection. For incident angles greater than the critical angle $\theta_i > \theta_c$, Snell's law gives

$$\sin\theta_t = \frac{n}{n'}\sin\theta_i > \frac{n}{n'}\sin\theta_c = 1$$

This means that θ_t is a complex angle with a purely imaginary cosine.

$$\cos\theta_t = i\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1} \tag{7.86}$$

Then Eqs. 7.74 and 7.79 indicates that r_s and r_p both take the form

$$r = \frac{a - ib}{a + ib}$$

where a and b are real, therefore,

$$R = |r|^2 = \left|\frac{a - ib}{a + ib}\right|^2 = 1$$

The result is that $R_s = R_p = 1$ for all $\theta_i > \theta_c$. This perfect reflection is called *total internal reflection*. The meaning of this total internal reflection becomes clear when we consider the propagation factor for the refracted wave:

$$e^{i\mathbf{k}'\cdot\mathbf{x}} = e^{ik'(x\sin\theta_t + z\cos\theta_t)} = e^{-\frac{z}{\delta}}e^{ik'\left(\frac{\sin\theta_i}{\sin\theta_c}\right)x} \qquad (7.87)$$

where

$$\frac{1}{\delta} = -ik'\cos\theta_t = k\sqrt{\sin^2\theta_i - \sin^2\theta_c} = \frac{2\pi}{\lambda}\sqrt{\sin^2\theta_i - \sin^2\theta_c} \qquad (7.88)$$

With the wavelength of the radiation, λ. This shows that, for $\theta_i > \theta_c$, the refracted wave is propagating only parallel to the surface and is attenuated exponentially beyond the interface. The attenuation occurs within a few wavelengths of the boundary except for $\theta_i \approx \theta_c$.

Que4: Explain reflection from metallic surface (normal incidence)

Ans:

REFLECTION FROM A METALLIC SURFACE

For a metallic medium the dielectric function and the index of refraction complex valued functions. This is also the case for semiconductors : insulators in certain frequency ranges near and at absorption bands. Fresn equations are still valid but the angles in the equations are now complex valı and do no longer have the obvious geometrical interpretation.

For normal incidence we have

$$R = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 \rightarrow \left|\frac{\tilde{n}_2 - \tilde{n}_1}{\tilde{n}_2 + \tilde{n}_1}\right|^2 = \frac{(n_2 - n_1)^2 + (k_2 - k_1)^2}{(n_2 + n_1)^2 + (k_2 + k_1)^2}$$

where

$$\tilde{n} = n + ik = n(1 + i\kappa)$$

where κ is the so called extinction coefficient.

For metallic systems

$$\tilde{n} = \sqrt{\tilde{\varepsilon}} = \sqrt{\varepsilon + 4\pi i \sigma/\omega}$$

For normal incidence the refracted, or rather transmitted, wave will vary as

$$\mathbf{E}_2 \sim e^{i\left(\tilde{n}k_0 z - \omega t\right)} = e^{-kk_0 z} e^{i\left(nk_0 z - \omega t\right)}$$

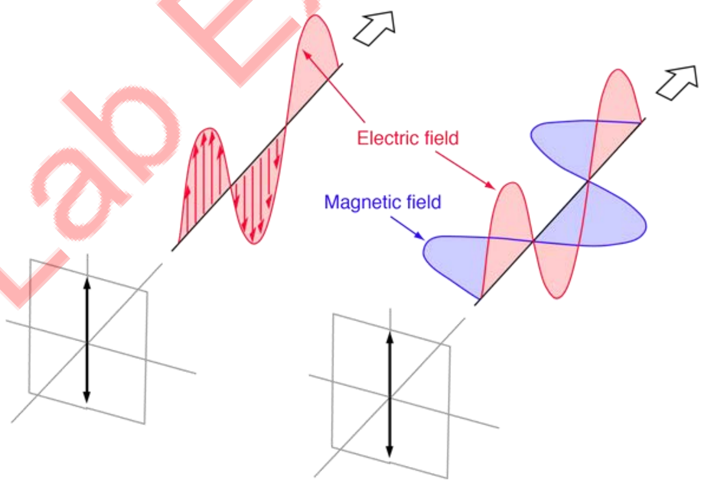
Chapter 4

Polarization of Electromagnetic Waves: Description of Linear, Circular and Elliptical Polarization. Propagation of E.M. Waves in Anisotropic Media. Symmetric Nature of Dielectric Tensor. Fresnel's Formula. Uniaxial and Biaxial Crystals. Light Propagation in Uniaxial Crystal. Double Refraction. Polarization by Double Refraction. Nicol Prism. Ordinary & extraordinary refractive indices. Production & detection of Plane, Circularly and Elliptically Polarized Light. Phase Retardation Plates: Quarter-Wave and Half-Wave Plates. Babinet Compensator and its Uses. Analysis of Polarized Light **(12 Lectures)**

Que1: Explain linear, circular and elliptical polarization of light.

Ans:
Linear Polarization

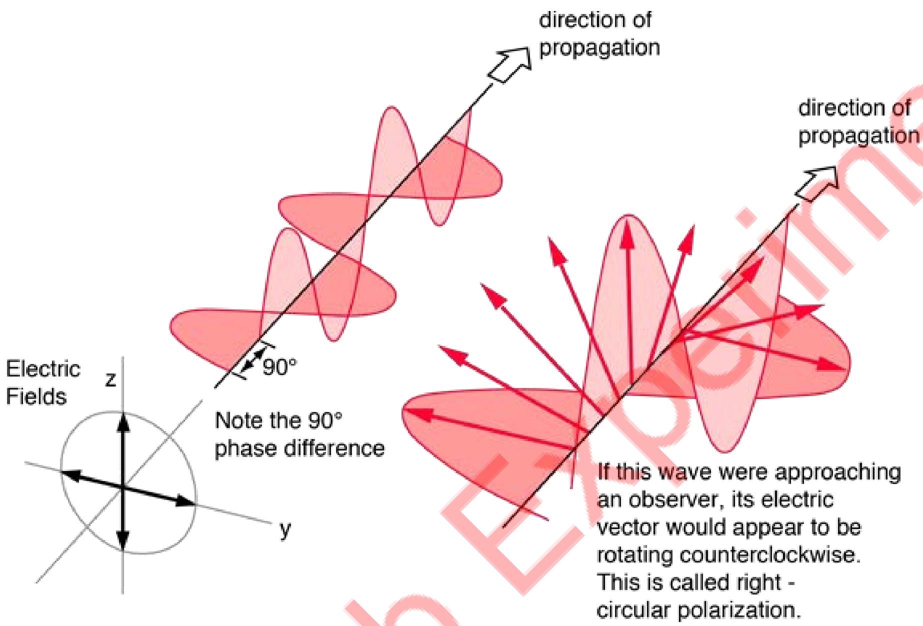
A plane electromagnetic wave is said to be linearly polarized. The transverse electric field wave is accompanied by a magnetic field wave as illustrated**.**



Circularly polarized light consists of two perpendicular electromagnetic plane waves of equal amplitude and 90° difference in phase. The light illustrated is right- circularly polarized.

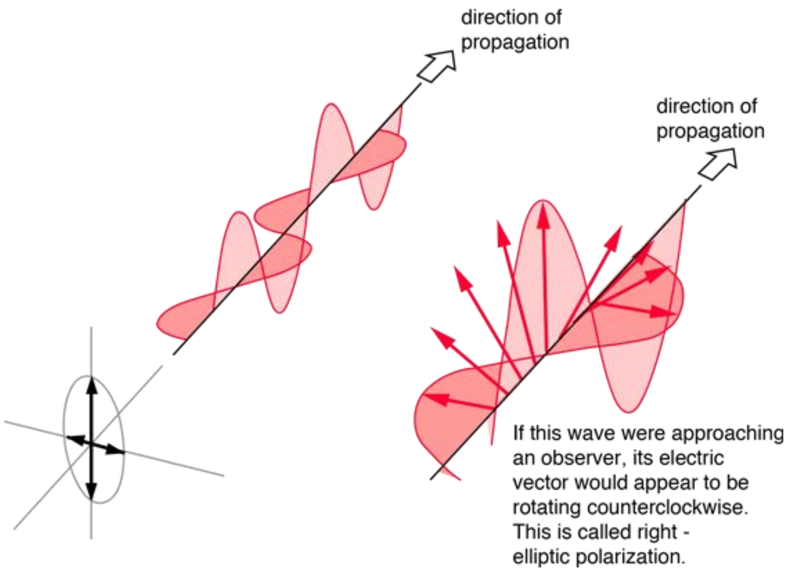
Circular Polarization

If light is composed of two plane waves of equal amplitude but differing in phase by 90°, then the light is said to be circularly polarized. If you could see the tip of the electric field vector, it would appear to be moving in a circle as it approached you. If while looking at the source, the electric vector of the light coming toward you appears to be rotating counterclockwise, the light is said to be right-circularly polarized. If clockwise, then left-circularly polarized light. The electric field vector makes one complete revolution as the light advances one wavelength toward you. Another way of saying it is that if the thumb of your right hand were pointing in the direction of propagation of the light, the electric vector would be rotating in the direction of your fingers.



Elliptical Polarization

Elliptically polarized light consists of two perpendicular waves of unequal amplitude which differ in phase by 90°. The illustration shows right- elliptically polarized light.



If the thumb of your right hand were pointing in the direction of propagation of the light, the electric vector would be rotating in the direction of your fingers.

Que2: What is dilectric tensor and why is it symmetric in nature? Explain the plan wave propgation in anisotropic media and derive Fresnel formula.

Ans:

- In an isotropic medium, the induced polarization $\boldsymbol{P}$ is always parallel to the electric field $\boldsymbol{E}$ and is related by a scalar quantity (the susceptibility) that is independent of the field direction.

- In anisotropic media,

$$P_x = \epsilon_0(\chi_{11}E_x + \chi_{12}E_y + \chi_{13}E_z)$$
$$P_y = \epsilon_0(\chi_{21}E_x + \chi_{22}E_y + \chi_{23}E_z)$$
$$P_z = \epsilon_0(\chi_{31}E_x + \chi_{32}E_y + \chi_{33}E_z)$$

Along the principal axes of the crystal (vanishing off-diagonal elements),

$$P_x = \epsilon_0\chi_{11}E_x, \quad P_y = \epsilon_0\chi_{22}E_y, \quad P_z = \epsilon_0\chi_{33}E_z.$$

- In terms of the dielectric permittivity tensor ϵ_{ij}, $D_i = \epsilon_{ij} E_j$ where $\epsilon_{ij} = \epsilon_0(1+\chi_{ij})$

- For a homogeneous, nonabsorbing, and magnetically isotropic medium, the energy density of the stored electric field in the anisotropic medium

$$U_e = \frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D} = \frac{1}{2}E_i\epsilon_{ij}D_j$$

Differentiating the above eqn, $\dot{U}_e = \frac{1}{2}\epsilon_{ij}(\dot{E}_iE_j + E_i\dot{E}_j)$.

Using the Poynting theorem, the net power flow into a unit vol in a lossless medium is

$$-\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) = \boldsymbol{E}\cdot\dot{\boldsymbol{D}} + \boldsymbol{H}\cdot\dot{\boldsymbol{B}} \;\rightarrow\; -\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) = E_i\epsilon_{ij}\dot{E}_j + \boldsymbol{H}\cdot\dot{\boldsymbol{B}}$$

The first term must be equal to $\dot{U}_e$ (the Poynting vector corresponds to the energy flux).

$$\frac{1}{2}\epsilon_{ij}(\dot{E}_iE_j + E_i\dot{E}_j) = \epsilon_{ij}E_i\dot{E}_j \;\rightarrow\; \epsilon_{ij} = \epsilon_{ji} \text{ (symmetric)}$$

* For a lossless medium, $\epsilon_{ij} = \epsilon_{ji}^*$: the conservation of electromagnetic field energy requires that the dielectric tensor be "Hermitian".

Plane Wave Propagation in Anisotropic Media

- In an anisotropic medium such as a crystal, the phase velocity of light dependes on its state of *polarization* as well as its *direction* of propagation.

- For given direction of propagation in the medium, there exist, in general, *two* eigenwaves with well-defined eigen-phase velocities and polarization directions.

- Consider a monochromatic plane wave of angular frequency ω propagating in the anisotropic medium with an electric field $\boldsymbol{E}\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ and a magnetic field $\boldsymbol{H}\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ where $\boldsymbol{k}$ is the wavevector $\boldsymbol{k} = (\omega/c)n\boldsymbol{s}$ with $\boldsymbol{s}$ is a unit vector along the propagation, and n is the refraction index to be determined.

Maxwell's eqn: $\boldsymbol{k}\times\boldsymbol{E} = -\omega\mu\boldsymbol{H}$, $\boldsymbol{k}\times\boldsymbol{H} = -\omega\epsilon\boldsymbol{E}$

$$\rightarrow\quad \boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}) + \omega^2\mu\epsilon\boldsymbol{E} = 0$$

- In the principal coordinate system, $\epsilon = \begin{pmatrix}\epsilon_x & 0 & 0\\ 0 & \epsilon_y & 0\\ 0 & 0 & \epsilon_z\end{pmatrix}$

Then, the wave eqn is given by $\begin{pmatrix} \omega^2\mu\epsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2\mu\epsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2\mu\epsilon_z - k_x^2 - k_y^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$

For nontrival solutions, det $\begin{vmatrix} \omega^2\mu\epsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2\mu\epsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2\mu\epsilon_z - k_x^2 - k_y^2 \end{vmatrix} = 0$, representing

a 3-dim. surface of $\boldsymbol{k}$ space (momentum space). This surface is known as *the normal surface* and consists of two shells, which, in general, have 4 points in common

→ The *two lines* going through the origin and these points are known as *the optic axes*.

- Given a direction of propagation, there are in general two $\boldsymbol{k}$ values which are the intersections of the direction of propagation and the normal surface.
  These two $\boldsymbol{k}$ values correspond to two different phase velocities (ω/k) of the waves propagating along the chosen direction.
  The two phase velocities always correspond to *two mutually orthogonal* polarizations.

- The direction of the electric field vector associated with these propagation:

$$\begin{pmatrix} \dfrac{k_x}{k^2 - \omega^2\mu\epsilon_x} \\ \dfrac{k_y}{k^2 - \omega^2\mu\epsilon_y} \\ \dfrac{k_z}{k^2 - \omega^2\mu\epsilon_z} \end{pmatrix}$$

- For propagation in the direction of the optic axes, there is only one value of $\boldsymbol{k}$ and thus only one phase velocity. *There are two independent directions of polarization.*

- In terms of the direction cosines of the wavevector, using $\mathbf{k} = (\omega/c)n\mathbf{s}$ for the plane wave. (Fresnel's equation of wave normals)

$$\frac{s_x^2}{n^2 - \epsilon_x/\epsilon_0} + \frac{s_y^2}{n^2 - \epsilon_y/\epsilon_0} \frac{+s_z^2}{n^2 - \epsilon_z/\epsilon_0} = \frac{1}{n^2} \quad \text{and} \quad \begin{pmatrix} \dfrac{s_x}{n^2 - \epsilon_x/\epsilon_0} \\ \dfrac{s_y}{n^2 - \epsilon_y/\epsilon_0} \\ \dfrac{s_z}{n^2 - \epsilon_z/\epsilon_0} \end{pmatrix}$$

Que: 3 What are uniaxial and biaxial crystals?

Ans: For a given propagation direction, in general there are two perpendicular polarizations for which the medium behaves as if it had a single effective refractive index. In a uniaxial material, these polarizations are called the extraordinary and the ordinary ray (e and o rays), with the ordinary ray having the effective refractive index n_o and n_e. On the other hand, a biaxial crystal is characterized by three refractive indices α, β, and γ applying to its principal axes.

Que4: Explain the phenomenon of double refraction or birefrigence.

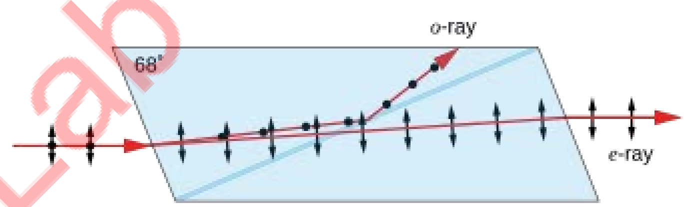
Ans:Double refraction, also called birefringence is an optical property in which a single ray of unpolarized light entering an anisotropic medium is split into two rays, each traveling in a different direction. One ray (called the extraordinary ray) is bent, or refracted, at an angle as it travels through the medium; while the other ray (called the ordinary ray) passes through the medium unchanged.

In case of double refraction, the ordinary ray and the extraordinary ray are polarized in planes vibrating at right angles to each other. Additionally, the refractive index (a number that determines the angle of bending specific for each medium) of the ordinary ray is observed to be constant in all directions; on the other hand, the refractive index of the extraordinary ray varies according to the direction taken because it has components that are both parallel and perpendicular to the crystal's optic axis. Because the speed of light waves in a medium is equal to their speed in a vacuum divided by the index of refraction for that particular wavelength, an extraordinary ray can move faster or slower than an ordinary ray.

Electromagnetic radiation propagates through space with oscillating electric and magnetic field vectors alternating in sinusoidal patterns which are perpendicular to one another and to the direction of wave propagation. As visible light is composed of both electrical and magnetic components, the velocity of light through a substance is significantly determined by the electrical conductivity of the material. Light waves travelling through a transparent crystal must interact with localized electrical fields during their journey. The relative speed at which electrical signals travel through a material varies with the type of signal and its interaction with the electronic structure, and is calculated by a property defined as the dielectric constant of the material. The relationship defining the interaction between a light wave and a crystal through which it passes is governed by the inherent orientation of lattice electrical vectors and the direction of the wave's electric vector component. Therefore, a meticulous consideration of the electrical properties of an anisotropic material is fundamental to the understanding of light wave interaction with the material as it propagates through.

Que5: Explain the principle of Nicol prism with a schematic.

Ans: A Nicol prism is a type of polarizer, an optical device made from calcite crystal used to produce and analyse plane polarized light. It is made in such a way that it eliminates one of the rays by total internal reflection, i.e. the ordinary ray is eliminated and only the extraordinary ray is transmitted through the prism. It was the first type of polarizing prism, invented in 1828 by William Nicol (1770–1851) of Edinburgh. It consists of a rhombohedral crystal of Iceland spar (a variety of calcite) that has been cut at an angle of 68° with respect to the crystal axis, cut again diagonally, and then rejoined as shown, using a layer of transparent Canada balsam as a glue.



Unpolarized light ray enters through the left face of the crystal, as shown in the diagram, and is split into two orthogonally polarized, differently directed rays by the birefringence property of the calcite. The ordinary ray, or o-ray, experiences a refractive index of no = 1.658 in the calcite and undergoes total internal reflection at the calcite–glue interface because its angle of incidence at the glue layer (refractive index n = 1.55) exceeds the critical angle for the interface. It passes out the top side of the upper half of the prism with some refraction, as shown. The extraordinary ray, or e-ray, experiences a lower refractive index (ne = 1.486) in the calcite and is not totally reflected at the interface because it strikes the interface at a sub-critical angle. The e-ray merely undergoes a slight refraction, or bending, as it passes through the interface into the lower half of the prism. It finally leaves the prism as a ray of plane-polarized light, undergoing another refraction, as it exits the far right side of the prism. The two exiting rays have polarizations orthogonal (at right angles) to each other, but the lower, or e-ray, is the more commonly used for further experimentation because it is again traveling in the original horizontal direction, assuming that the calcite prism angles have been properly cut. The direction of the upper ray, or o-ray, is quite different from its original direction because it alone suffers total internal reflection at the glue interface, as well as a final refraction on exit from the upper side of the prism

Que6: What is a wave plate or retarder? Explain its working principle in detail.

Ans: A waveplate or retarder is an optical device that alters the polarization state of a light wave travelling through it. Two common types of waveplates are the half-wave plate, which shifts the polarization direction of linearly polarized light, and the quarter-wave plate, which converts linearly polarized light into circularly polarized light and vice versa. A quarter-wave plate can be used to produce elliptical polarization as well.

Waveplates are constructed out of a birefringent material (such as quartz or mica), for which the index of refraction is different for different orientations of light passing through it. The behavior of a waveplate (that is, whether it is a half-wave plate, a quarter-wave plate, etc.) depends on the thickness of the crystal, the wavelength of light, and the variation of the index of refraction. By appropriate choice of the relationship between these parameters, it is possible to introduce a controlled phase shift between the two polarization components of a light wave, thereby altering its polarization.

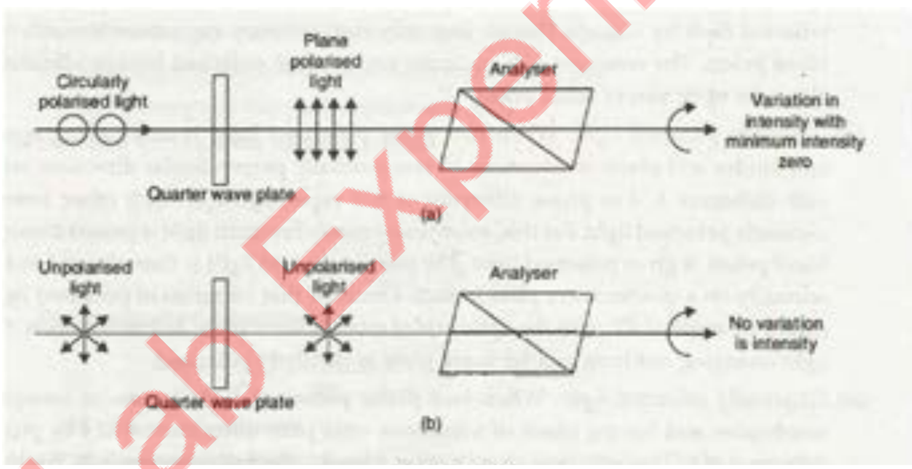
Principles of operation

A waveplate works by shifting the phase between two perpendicular polarization components of the light wave. A typical waveplate is simply a birefringent crystal with a carefully chosen orientation and thickness. The crystal is cut into a plate, with the orientation of the cut chosen so that the optic axis of the crystal is parallel to the surfaces of the plate. This results in two axes in the plane of the cut: the *ordinary axis*, with index of refraction n_o, and the *extraordinary axis*, with index of refraction n_e. The ordinary axis is perpendicular to the optic axis. The extraordinary axis is parallel to the optic axis. For a light wave normally incident upon the plate, the polarization component along the ordinary axis travels through the crystal with a speed $v_o = c/n_o$, while the polarization component along the extraordinary axis travels with a speed $v_e = c/n_e$. This leads to a phase difference between the two components as they exit the crystal. When $n_e < n_o$, as in calcite, the extraordinary axis is called the *fast axis*and the ordinary axis is called the *slow axis*. For $n_e > n_o$ the situation is reversed.

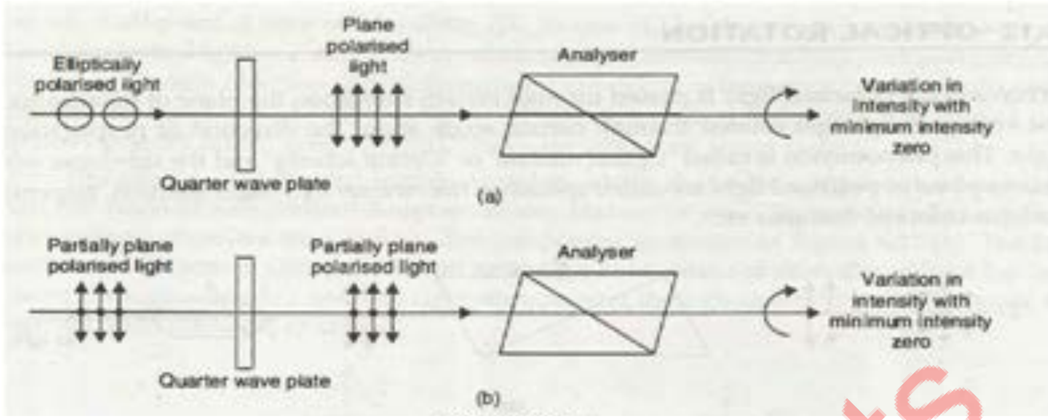
Que7: How will you differentiate among plane polarised, circularly polarised and elliptically polarised light?

Ans: **Plane Polarised Light**: The light beam is allowed to fall on Nicol prism. If on rotation of Nicol prism, intensity of emitted light can be completely extinguished at two places in each rotation, then light is plane polarised.

Circularly Polarised Light: The light beam is allowed to fall on a Nicol prism. If on rotation of Nicol prism the intensity of emitted light remains same, then light is either circularly polarised or unpolarised. To differentiate between unpolarised and circularly polarised light, the light is first passed through quarter wave plate and then through Nicol prism. Because if beam is circularly polarised then after passing through quarter wave-plate an extra difference of λ/ 4 is introduced between ordinary and extraordinary component and gets converted into plane polarised. Thus on rotating the Nicol, the light can.be extinguished at two plates. If, on the other hand, the beam is unpolarised, it remains unpolarised after passing through quarter wave plate and on rotating the Nicol, there is no change in intensity of emitted light (Figure 6.18).



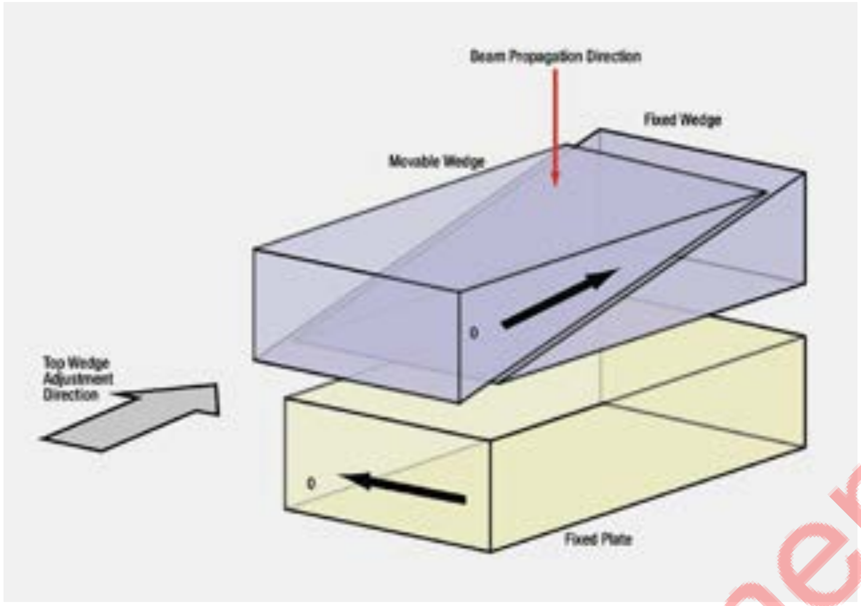
(i) **Elliptically Polarised Light**. The light beam is allowed to fall on Nicol prism. If on rotation of Nicol prism, the intensity of emitted light varies from maximum to minimum, then light is either elliptically polarised or a mixture of plane polarized and unpolarised. To differentiate between the two, the light is first passed through quarter wave plate and then through Nicol prism. Because, if beam is elliptically polarised, then after passing through quarter wave plate, an extra path difference of λ/ 4 is introduced between 0-ray and E-ray and get converted into plane polarized Thus, on rotating the Nicol, the light can be extinguished l’lt two places. If, on the other hand, beam is mixture of polarised and unpolarised it remains mixture after passing through quarter wave plate and on rotating the Nicol intensity of emitted light varies from maximum to minimum.



Que8: Explain briefly the working of Babinet-Soleil Compensator.

Ans: A Babinet-Soleil Compensator is a continuously variable zero-order retarder (wave plate) that can be used over a broad spectral range. The variable retardance is attained by adjusting the position of a long birefringent wedge with respect to a short fixed birefringent wedge. The wedge angle and fast axis orientation is the same for both wedges so as to hve uniform retardance across the entire clear aperture of the Babinet compensator.

A compensator plate is connected to the fixed wedge, with its fast axis orthogonal to both the fast axis of the wedges and the propagation direction of the light. When the long birefringent wedge is arranged such that the total thickness of the two stacked wedges is equal to the thickness of the compensator plate, the net retardance of light passing through the Soleil-Babinet compensator is zero. The position of the long wedge can then be adjusted with a micrometer in order to create a retardance transmitted beam of light.



Que 9: Differentiate Positive and negative uniaxial crystals.

Ans: Uniaxial birefringence is classified as positive when the extraordinary index of refraction n_e is greater than the ordinary index n_o. Negative birefringence means that $\Delta n = n_e - n_o$ is negative. In other terms , the polarization of the fast (or slow) wave is perpendicular to the optical axis when the birefringence of the crystal is positive (or negative, respectively).

Chapter 5

Rotatory Polarization: Optical Rotation. Biot's Laws for Rotatory Polarization.Fresnel's Theory of optical rotation. Calculation of angle of rotation. Experimental verification of Fresnel's theory. Specific rotation. Laurent's half-shade polarimeter. **(5 Lectures)**

Que 1: Explain rotary polarization of light.

Ans:
Rotary Polarization

When a beam of plane polarized light propagates through certain substances or crystals, the plane of polarization of the emergent beam rotated through a certain angle.

This phenomenon is called rotatory polarization and this property of the crystal and other substances is called optical activity or optical rotation and substances which show this property are called optically active substances.

Que 2: Discuss Biot's law of optical rotation?

Ans: He showed (Biot's Law) that the amount of rotation of the plane of polarization of light passing through an optically active medium is proportional to the length of its path, and to the concentration, if the medium is a solution of an active solute in an inactive solvent, and that the rotation is roughly inversely proportional to the square of the wavelength of the light

$$\theta \alpha C$$
$$\theta \alpha 1/\lambda^2$$

Que 3: Explain Fresnel's theory of optical rotation and give mathematical proof.

Ans:
Fresnel Theory

- This explanation was based on the following **assumptions**:

1. A plane polarized light falling on an optically active medium along its optic axis splits up into two circularly polarized vibrations of equal amplitudes and rotating in opposite directions –one clockwise and other anticlockwise.

2. In an **optically inactive** substance these two circular components travel with the same speed along the optic axis. Hence at emergence they give rise to a plane polarized light without any rotation of the plane of polarization.

3. In an **optically active crystal**, like quartz, two circular components travel with different speeds so that relative phase difference is developed between them.

4. In dextro-rotatory substance $v_r > v_l$ and in leavo rotatory substance $v_l > v_r$..

5. On emergence from an optically active substance the two circular vibrations recombine to give plane polarized light whose plane of vibration has been rotated w.r.t that of incident light through a certain angle depends on the phase diff between the two vibrations.

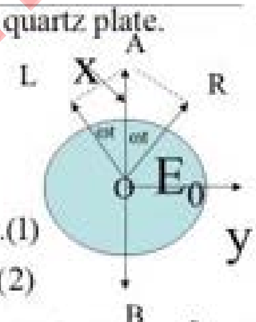
Mathematical Treatment

Let a beam of plane polarized light be incident normally on a quartz plate.

Let the vibrations in the incident polarised beam be

$$E_x = 2E_o \cos \omega t \text{ and } E_y = 0$$

where $2E_o$ is the amplitude of the incident vibrations.



the above eq^n can be rewritten as $E_x = E_o \cos \omega t + E_o \cos \omega t$...(1)

and $E_y = E_o \sin \omega t - E_o \sin \omega t$...(2)

From the Huygen's principle of superposition, $E_x = E_x^R + E_x^L$ and $E_y = E_y^R + E_y^L$

Therefore eq^{n}s (1)and (2)may be considered to be the resultant of the two circular vibrations represented

by the eq^{n}s $E_x^R = E_o \cos \omega t$ and $E_y^R = E_o \sin \omega t$...(3)

components of clockwise circular motion in two mutually $\perp^r$ directions.

$E_x^L = E_o \cos \omega t$ and $E_y^L = -E_o \sin \omega t$...(4)

components of anticlockwise circular motion in two mutually $\perp^r$ directions

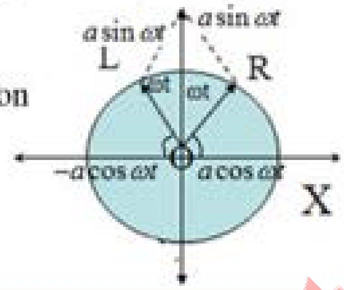
(for optically inactive substance- the angular speeds of L and R components are same)

If the resultant vibrations for the emergent beam

along the x axis: $E_x = E_x^R + E_x^L = 2E_o \cos \omega t$

along the y axis: $E_y = E_y^R + E_y^L = 0$

Plane of vibration is along original direction



The result shows that two oppositely directed circular motions of equal velocity combine to give linear motion along the direction of motion (optically inactive material)

For optically active substances

* According to Fresnel the two circular components are propagated through the plate with different angular speeds. So when they emerges out of the crystal there is a phase difference δ between them.

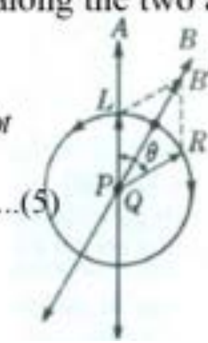
* Suppose clockwise component advances in front of the other.

$E_x^R = E_o \cos(\omega t + \delta)$ $\quad E_y^R = E_o \sin(\omega t + \delta)$ $\quad$ [clockwise]

$E_x^L = E_o \cos \omega t$ $\quad E_y^L = -E_o \sin \omega t$ $\quad$ [anti clockwise]

The resultant displacement along the two axes are

$$E_x = E_x^R + E_x^L$$
$$= E_o \cos(\omega t + \delta) + E_o \cos \omega t$$
$$= 2E_o \cos\frac{\delta}{2} \cos\left(\omega t + \frac{\delta}{2}\right) \quad ....(5)$$



$$E_y = E_y^R + E_y^L$$
$$= E_o \sin(\omega t + \delta) - E_o \sin \omega t$$
$$= 2E_o \sin\frac{\delta}{2} \cos\left(\omega t + \frac{\delta}{2}\right) \quad ....(6)$$

$$E_x = 2E_o \cos\frac{\delta}{2}\cos\left(\omega t + \frac{\delta}{2}\right) \;;\; E_y = 2E_o \sin\frac{\delta}{2}\cos\left(\omega t + \frac{\delta}{2}\right)$$

These resultant vibrations along the x and y axes are $\perp^r$ to each other and are in the same period and phase.

Dividing eq^n(5) by (6) we get

$$\frac{E_y}{E_x} = \frac{\sin\frac{\delta}{2}}{\cos\frac{\delta}{2}} = \tan\frac{\delta}{2}$$

This is equation of straight line inclined at δ/2 with x-axis.
That is with the vibrations of incident light.

Que4 : Define specific rotation of an optically active substance.

Ans: The specific rotation of an optically active substance at a given temperature for a given wavelength of light is defined as the rotation (in degrees) produced by the path of one decimeter length in a substance of unit density (concentration)

$$\alpha_\lambda^T = \frac{\theta}{\ell C} \quad \text{or} \quad \alpha_\lambda^T = \frac{10\theta}{\ell C} \text{ (If } \ell \text{ is in cm)}$$

The unit of specific rotation is deg.(decimeter)-1(gm/cc)-1

The molecular rotation is given by the product of the specific rotation and molecular weight of the substance

Specific rotation for solids

$$\theta \propto l \text{ or } \theta = \alpha l$$

The specific rotation of an optically active solid substance at a given temperature for a given wavelength of light is defined as the rotation (in degrees) produced by the path of 1 mm length in a substance.

Que5: What is a polarimeter?

Ans: It is a device which is used to measure the optical rotation produced by an optically active substance. By measuring the angle e the specific rotation of an optically active substance can be determined.

Two types of polarimeters are generally used in the laboratory now a days:

(a) Laurent's Half Shade Polarimeter

(b) Biquartz Polarimeter

Que6: Explain working of Laurent's Half Shade Polarimeter in detail.

Ans: Construction: It consists of a monochromatic source S which is placed at focal point of a convex lens L. Just after the convex lens there is a Nicol Prism P which acts as a polariser. H is a half shade device which divides the field of polarised light emerging out of the Nicol P into two halves generally of unequal brightness. T is a glass tube in which optically active solution is filled. The light after passing through T is allowed to fall on the analyzing Nicol A which can be rotated about the axis of the tube. The rotation of analyser can be measured with the help of a scale C. Laurent's half shade polarimeter is shown in Figure 6.22.

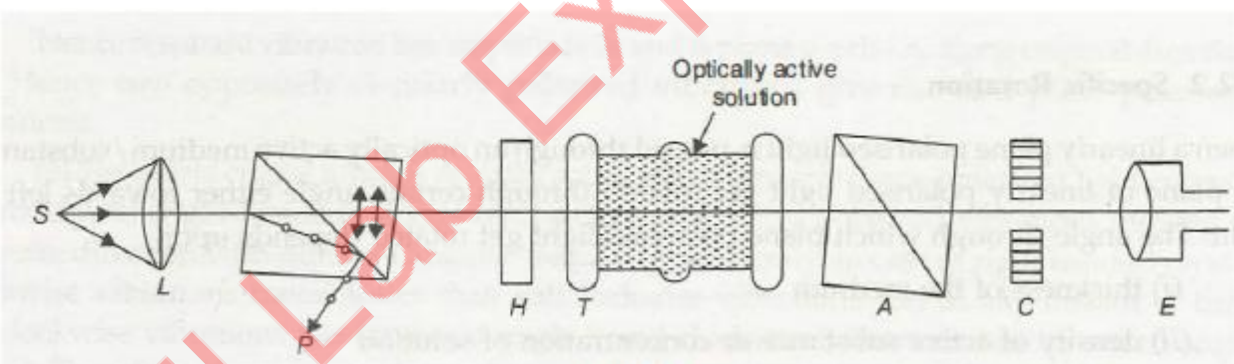


FIGURE 6.22

Working: In order to understand the need of a half shade device, let us suppose that half shade device is not present. The position of the analyzer is so adjusted that the field of view is dark when tube is empty. The position of the analyzer is noted on circular scale. Now the tube is filled with optically active solution and it is set in its proper position. The optically active solution rotates the plane of polarization of the light emerging out of the polariser P by some angle. So the light is transmitted by analyzer A and the field of view of telescope becomes bright. Now the analyzer is rotated by a finite angle so that the field of view of telescope again become dark. This will happen only when the analyzer is rotated by the same angle by which plane of polarization of light is rotated by optically active solution.

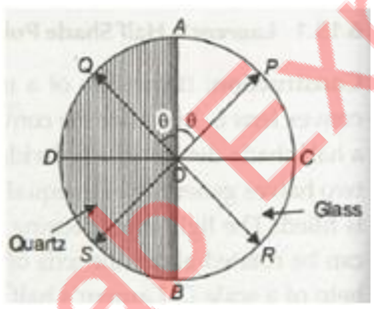
The position of analyzer is again noted. The difference of the two readings will give you angle of rotation of plane of polarization.

A difficulty is faced in the above procedure that when analyzer is rotated for the total darkness, then it is attained gradually and hence it is difficult to find the exact position correctly for which complete darkness is obtained.

To overcome above difficulty half shade device is introduced between polariser P and glass tube T.

Half Shade Device

It consists of two semicircular plates ACB and ADC. One half ACB is made of glass while other half is made of quartz. Both the halves are cemented together. The quartz is cut parallel to the optic axis. Thickness of the quartz is selected in such a way that it introduces a path difference of ‘A/2 between ordinary and extraordinary ray. The thickness of the glass is selected in such a way that it absorbs the same amount of light as is absorbed by quartz half.



Let us consider that the vibration of polarisation is along OP. On passing through the glass half the vibrations remain along OP. But on passing through quartz half these vibrations will split into 0- and £-components. The £-components are parallel to the optic axis while O- component is perpendicular to optic axis. The O-component travels faster in quartz and hence an emergence 0-component will be along OD instead of along OC. Thus components OA and OO will combine to form a resultant vibration along OQ which makes same angle with optic axis as OP. Now if the Principal plane of the analyzing Nicol is parallel to OP then the light will pass through glass half unobstructed. Hence glass half will be brighter than quartz half or we can say that glass half will be bright and the quartz half will be dark. Similarly if principal plane of analyzing Nicol is parallel to OQ then quartz half will be bright and glass half will be dark.

When the principal plane of analyzer is along AOB then both halves will be equally bright. On the other hand if the principal plane of analyzer is along DOC. then both the halves will be equally dark.

Thus it is clear that if the analyzing Nicol is slightly disturbed from DOC then one half becomes brighter than the other. Hence by using half shade device, one can measure angle of rotation more accurately.

Determination of Specific Rotation

In order to determine specific rotation of an optically active substance (say sugar) the polarimeter tube T is first filled with pure water and analyzer is adjusted for equal darkness (Both the halves should be equally dark) point. The position of the analyzer is noted with the help of scale. Now the polarimeter tube is filled with sugar solution of known concentration and again the analyser is adjusted in such a way that again equally dark point is achieved. The position of the analyzer is again noted. The difference of the two readings will give you angle of rotation θ. Hence specific rotation S is determined by using the relation.

$$[S]t\ \lambda = \theta / LC$$

The above procedure may be repeated for different concentration.

Chapter 6

Wave Guides: Planar optical wave guides. Planar dielectric wave guide. Condition of continuity at interface. Phase shift on total reflection. Eigenvalue equations. Phase and group velocity of guided waves. Field energy and Power transmission. **(8 Lectures)**

Que1: What is an optical waveguide.

Ans: An optical waveguide is a physical structure that guides electromagnetic waves in the optical spectrum. They are used as components in integrated optical circuits, as the transmission medium in long distances for light wave communications, or for biomedical imaging.

Fig. 1 shows the configuration of a typical planar dielectric waveguide. A slab of dielectric material, called film or core, surrounded by media of lower refractive indexes, called cover and subtract as the upper and lower, respectively

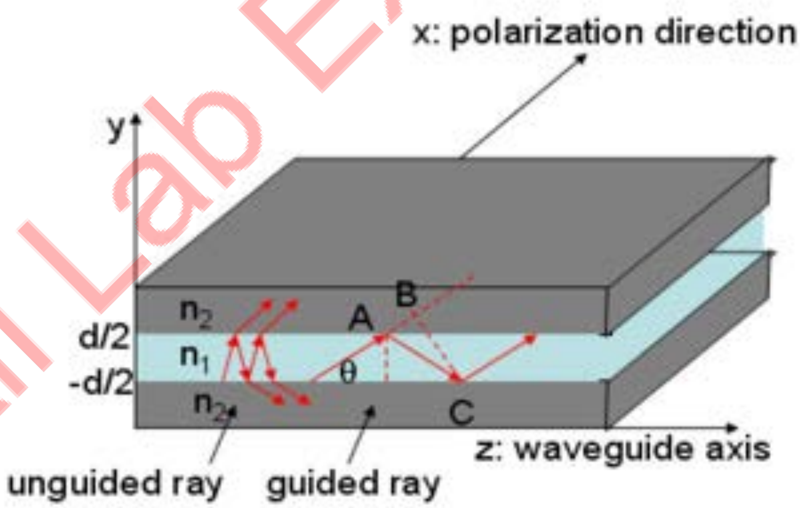


Fig.1 (Planar dielectric waveguide configuration. The width of the slab is d and refraction index is n_1, and the cover and subtract have same refraction index n_2.)

A light ray can be guided inside the slab by total internal reflection in the zigzag fashion. Only certain reflection angle θ will constructively interfere in the waveguide and hence only certain waves can exist in the waveguide (this will be discussed more in section 2 waveguide modes).

Case 1: θ smaller than complement of the critical angle

$$\theta < \bar{\theta}_c = \pi/2 - \sin^{-1}(n_2/n_1) = \cos^{-1}(n_2/n_1) \qquad \text{case 1}$$

Total internal reflection will happen at the boundaries. Then the rays can travel in z direction by bouncing between the slabs surfaces without loss of energy (figure showed in the right of Fig.1). And we also assume that all the materials are lossless.

Case 2: θ larger than complement of the critical angle

$$\theta > \bar{\theta}_c = \pi/2 - \sin^{-1}(n_2/n_1) = \cos^{-1}(n_2/n_1) \qquad \text{case 2}$$

Total internal reflection can not happen at the boundaries. Then rays will lose a portion of their power at each reflection, and eventually they will vanish.

Que2: Derive the wave equation for a slab waveguide.

Ans:

Consider the asymmetric slab waveguide shown in figure 2.2. Maxwell's equations can be written in terms of the refractive index n_i $(i = 1, 2, 3)$ of the three layers and by assuming that the material of each layer is non-magnetic and isotropic, that is $\mu = \mu_0$ and ϵ is a scalar, we have

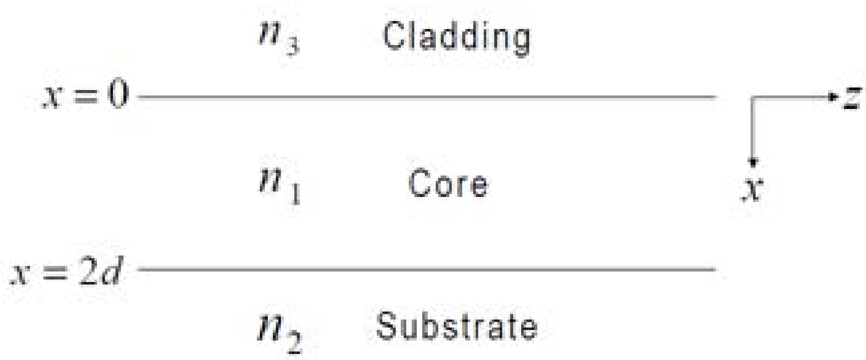


Figure 2.2: A Three-Layer Dielectric Slab Waveguide

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t} \quad (2.5)$$

$$\nabla \times \mathbf{H} = n_i^2 \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \quad (2.6)$$

$$\nabla \cdot \mathbf{E} = 0 \quad (2.7)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (2.8)$$

If we apply the curl operator to equation 2.5, we get:

$$\nabla \times \nabla \times \mathbf{E} = -\mu_o \nabla \times \frac{\partial \mathbf{H}}{\partial t} \quad (2.9)$$

$$= -\mu_o n_i^2 \epsilon_o \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (2.10)$$

where equation 2.6 has been used to eliminate **H**. To simplify further, we use the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (2.11)$$

where **A** is an arbitrary vector field. Using equations 2.7 and 2.11, equation 2.10

can be simplified to:

$$\nabla^2 \mathbf{E} = \mu_o \epsilon_o n_i^2 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2.12}$$

Writing the above equation in phasor notation (assuming a time-harmonic field of the form $e^{-j\omega t}$) we obtain

$$\nabla^2 \mathbf{E} + k_o^2 n_i^2 \mathbf{E} = 0 \tag{2.13}$$

which is the familiar three-dimensional vector wave equation for a uniform dielectric with refractive index n_i. Here k_o is the free-space wave number given by $k_o = \omega\sqrt{\mu_o \epsilon_o}$. The electric field vector **E** in equation 2.13 is a phasor quantity, which is complex and has both a magnitude and a phase. In addition, **E** is in general a function of space co-ordinates x, y, z and angular frequency ω. **E** is independent of time since the time dependence has been removed by the phasor transformation. We may simplify equation 2.13 by assuming that the structure is uniform in the y-direction (see figure 2.1) and extends to infinity in the y-direction. This allows us to assume that the field **E** is also uniform in this direction. Thus $\frac{\partial}{\partial y}$ is replaced by zero. If we further assume a z-dependence of the form $e^{j\beta z}$, with β as the longitudinal propagation constant, equation 2.13 is simplified and takes the form:

$$\frac{d^2 \mathbf{E}}{dx^2} + (k_o^2 n_i^2 - \beta^2)\mathbf{E} = 0 \tag{2.14}$$

The above equation is known as Helmholtz equation. In this case $\mathbf{E}$ is a function of x only and the equation is a second order ordinary differential equation. The propagation constant β can be expressed as $\beta = k_o n_{eff}$, where n_{eff} is called the effective index. The field of a slab waveguide is in general a superposition of Transverse Electric (TE) polarized field and Transverse Magnetic (TM) polarized field.

Que3: Derive Transverse Electric (TE) and Transverse Magnetic modes of waveguides.

Ans:

Transverse Electric (TE) Guided Modes

By using equation 2.14, the TE scalar wave equation for the three waveguide regions takes the following form:

$$\frac{d^2E_y}{dx^2} - r^2E_y = 0 \quad , \quad x \leq 0 \tag{2.15}$$

$$\frac{d^2E_y}{dx^2} + q^2E_y = 0 \quad , \quad 0 \leq x \leq 2d \tag{2.16}$$

$$\frac{d^2E_y}{dx^2} - p^2E_y = 0 \quad , \quad x \geq 2d \tag{2.17}$$

where $r^2 = \beta^2 - k_o^2 n_3^2$, $q^2 = k_o^2 n_1^2 - \beta^2$ and $p^2 = \beta^2 - k_o^2 n_2^2$. For guided modes, we require that the power to be confined largely to the central region of the guide and no power escapes from the structure. The form of equations 2.15, 2.16 and 2.17 then implies that this requirement will be satisfied for an oscillatory solution in the core region ($q^2 \geq 0$) with evanescent tails in the cladding and substrate regions ($r^2, p^2 \geq 0$) (see figure 2.4). Assuming $n_1 > n_2 \geq n_3$, it is straightforward to show that for guided modes, the possible range of β is given by $k_o n_1 \geq \beta \geq k_o n_2 \geq k_o n_3$.

From equation 2.5, the other field components of the TE modes are obtained in terms of E_y as follows:

$$H_x = -\frac{\beta}{\omega\mu_o} E_y \tag{2.18}$$

$$H_z = -\frac{j}{\omega\mu_o} \frac{\partial E_y}{\partial x} \tag{2.19}$$

Thus, for guided modes the solution of E_y in the three regions is

$$E_y = \begin{cases} Ae^{rx} & , x \leq 0 \\ A\cos(qx) + B\sin(qx) & , 0 \leq x \leq 2d \\ \left(A\cos(2dq) + B\sin(2dq)\right) e^{-p(x-2d)} & , x \geq 2d \end{cases} \tag{2.20}$$

where A and B are constants. By examining equation 2.20, the boundary condition on E_y is satisfied by its continuity at both $x = 0$ and $x = 2d$. The other tangential field component to the waveguide interfaces, namely H_z, must also be continuous at these interfaces. From equations 2.19 and 2.20, we have:

$$H_z = \frac{-j}{\omega\mu_o}\begin{cases} rAe^{rx} & , x \leq 0 \\ q\left(-A\sin(qx) + B\cos(qx)\right) & , 0 \leq x \leq 2d \\ -p\left(A\cos(2dq) + B\sin(2dq)\right)e^{-p(x-2d)} & , x \geq 2d \end{cases} \quad (2.21)$$

The continuity condition of H_z yields two equations. One at $x = 0$ and the second at $x = 2d$, that is:

$$rA = qB \quad (2.22)$$

and

$$q\left(-A\sin(2dq) + B\cos(2dq)\right) = -p\left(A\cos(2dq) + B\sin(2dq)\right) \quad (2.23)$$

Eliminating the ratio A/B from these equations yields [39, 40]:

$$\tan(2dq) = \frac{q(p+r)}{q^2 - pr} \quad (2.24)$$

This is the eigenvalue equation for the TE modes of the asymmetric slab waveguide. Equation 2.24 is an implicit relationship which involves the wavelength, refractive indices of the layers and core thickness as known quantities, and the propagation constant β as the only unknown quantity.

The symmetric waveguide $(n_2 = n_3)$
can only support modes with even or odd electric field patterns. In this case it can be easily shown that the eigen-value equation 2.24 reduces to $(p = r)$:

$$\tan(2dq) = \frac{2pq}{q^2 - p^2} \tag{2.25}$$

An example of the field pattern of the TE modes for a three-layer slab waveguide is given in figure 2.4.

Transverse Magnetic (TM) Guided Modes

The wave equation for this polarization is obtained in terms of the magnetic field component H_y as:

$$\frac{d^2H_y}{dx^2} - r^2 H_y = 0 \quad , \quad x \leq 0 \tag{2.26}$$

$$\frac{d^2H_y}{dx^2} + q^2 H_y = 0 \quad , \quad 0 \leq x \leq 2d \tag{2.27}$$

$$\frac{d^2H_y}{dx^2} - p^2 H_y = 0 \quad , \quad x \geq 2d \tag{2.28}$$

From equation 2.6, the other field components of the TM modes are obtained in terms of H_y as:

$$E_x = \frac{\beta}{\omega n_i^2 \epsilon_o} H_y \tag{2.29}$$

$$E_z = \frac{j}{\omega n_i^2 \epsilon_o} \frac{\partial H_y}{\partial x} \tag{2.30}$$

Thus, the solution of H_y in the three regions for the guided modes is

$$H_y = \begin{cases} Ce^{rx} & ,x \leq 0 \\ C\cos(qx) + D\sin(qx) & ,0 \leq x \leq 2d \\ (C\cos(2dq) + D\sin(2dq))\, e^{-p(x-2d)} & ,x \geq 2d \end{cases} \quad (2.31)$$

where C and D are constants. The field component E_z is obtained from equations 2.30 and 2.31 as follows:

$$E_z = \frac{j}{\omega\epsilon_o} \begin{cases} \frac{rC}{n_3^2} e^{rx} & ,x \leq 0 \\ \frac{q}{n_1^2}\left(-C\sin(qx) + D\cos(qx)\right) & ,0 \leq x \leq 2d \\ \frac{-p}{n_2^2}\left(C\cos(2dq) + D\sin(2dq)\right) e^{-p(x-2d)} & ,x \geq 2d \end{cases} \quad (2.32)$$

Continuity of E_z at $x = 0$ and $x = 2d$ leads to:

$$\frac{rC}{n_3^2} = \frac{qD}{n_1^2} \quad (2.33)$$

and

$$\frac{q}{n_1^2}\left(-C\sin(2dq) + D\cos(2dq)\right) = \frac{-p}{n_2^2}\left(C\cos(2dq) + D\sin(2dq)\right) \quad (2.34)$$

Eliminating the ratio C/D from these two equations results in

$$\tan(2dq) = \frac{qn_1^2\left(n_3^2 p + n_2^2 r\right)}{n_2^2 n_3^2 q^2 - n_1^4 pr}$$

which is the eigenvalue equation for TM modes of an asymmetric slab waveguide. An example of the TM mode patterns for a symmetric slab waveguide is given in figure 2.5. As evident from the figure, H_y is continuous across a layer interface but its derivative is discontinuous there, causing a sudden change in the slope of H_y there.

Que4: Derive the mode number and cut-off condition in optical waveguide.

Ans:

The notation TE_N (and similarly TM_N) is used to refer to a mode possessing N nodes in the distribution of E_y for TE modes and H_y for TM modes. The value of N can be obtained by taking the argument of the tangent in the eigenvalue equations 2.24 and 2.35 to be $(2dq - N\pi)$. Since $n_1 > n_2 > n_3$, the cut-off condition is given by

$$\beta = k_o n_2 \tag{2.36}$$

This corresponds to loss of optical confinement due to loss of exponential decay away from the waveguide in the substrate. The resultant effect is a field-spreading throughout the substrate region.

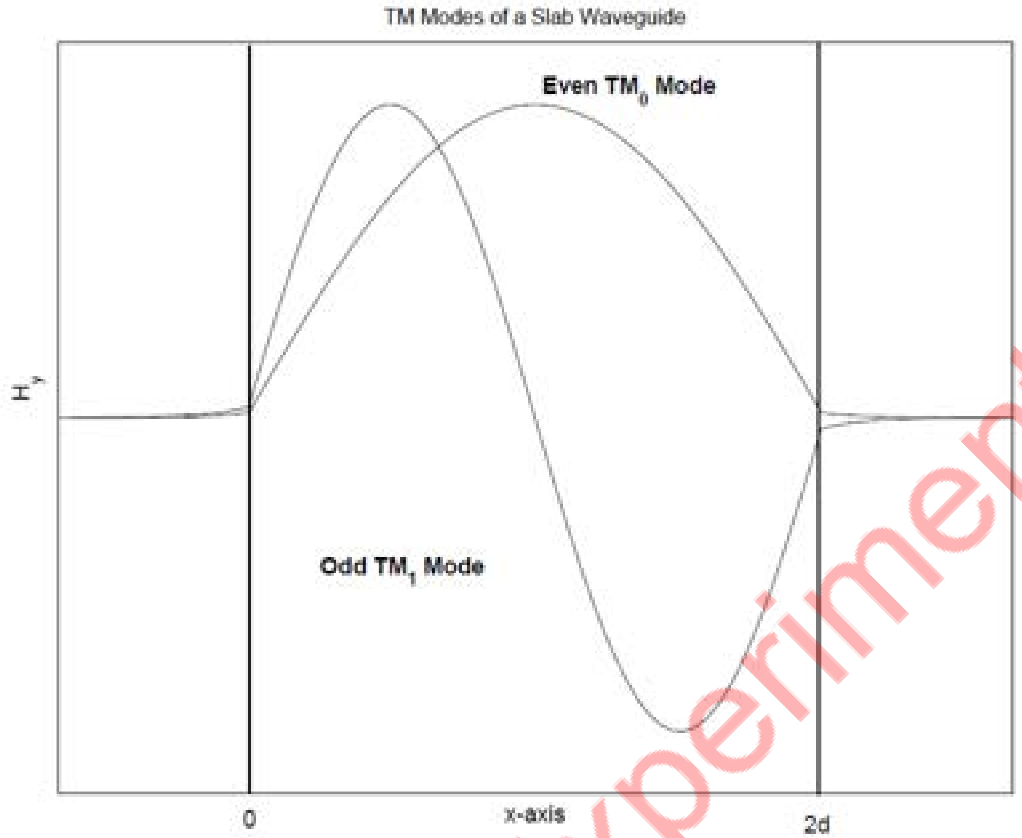


Figure 2.5: TM Mode Patterns of a Slab Waveguide

The cut-off conditions for TE_N and TM_N modes can be found by using the above definitions for the mode numbers and cut-offs. Substituting equation 2.36 into equation 2.24 along with the appropriate expressions for p, q, r at cut-off, the cut-off condition for the TE modes is stated as [39]:

$$\tan(2dk_c(n_1^2 - n_2^2)^{1/2} - N\pi) = \left(\frac{n_2^2 - n_3^2}{n_1^2 - n_2^2}\right)^{1/2} \tag{2.37}$$

where k_c corresponds to the cut-off wave number for TE_N. In terms of the normalized frequency (v), given by:

$$v = k_o d(n_1^2 - n_2^2)^{1/2} \tag{2.38}$$

the cut-off value v_c for the TE_N mode is [39]:

$$v_c = \frac{1}{2}\tan^{-1}\left[\left(\frac{n_2^2 - n_3^2}{n_1^2 - n_2^2}\right)^{1/2}\right] + \frac{N\pi}{2} \tag{2.39}$$

where $\tan^{-1}$ is restricted to the range $0 - \pi/2$. Equation 2.39 can be used to obtain M, the number of TE guided modes and is found to be

$$M = \left\{\frac{1}{\pi}\left(2v - \tan^{-1}\left[\left(\frac{n_2^2 - n_3^2}{n_1^2 - n_2^2}\right)^{1/2}\right]\right)\right\}_{int} \tag{2.40}$$

where the subscript int indicates the next largest integer.

The corresponding cut-off condition and number of guided TM modes are given as follows

$$v_c = \frac{1}{2}\tan^{-1}\left[\left(\frac{n_1}{n_3}\right)^2\left(\frac{n_2^2 - n_3^2}{n_1^2 - n_2^2}\right)^{1/2}\right] + \frac{N\pi}{2} \tag{2.41}$$

$$M = \left\{\frac{1}{\pi}\left(2v - \tan^{-1}\left[\left(\frac{n_1}{n_3}\right)^2\left(\frac{n_2^2 - n_3^2}{n_1^2 - n_2^2}\right)^{1/2}\right]\right)\right\}_{int} \tag{2.42}$$

Chapter 7

Optical Fibres: Numerical Aperture. Step and Graded Indices (Definitions Only). Single and Multiple Mode Fibres. **(3 Lectures)**

Que 1: Explain Optical fibre using a schematic.

Ans: Optical fibers are fine transparent glass or plastic fibers which can propagate light. They work under the principle of total internal reflection from diametrically opposite walls. In this way light can be taken anywhere because fibers have enough flexibility. This property makes them suitable for data communication, design of fine endoscopes, micro sized microscopes etc. An optic fiber consists of a core that is surrounded by a cladding which are normally made of silica glass or plastic. The core transmits an optical signal while the cladding guides the light within the core. Since light is guided through the fiber it is sometimes called an optical wave guide. The basic construction of an optic fiber is shown in figure (1).

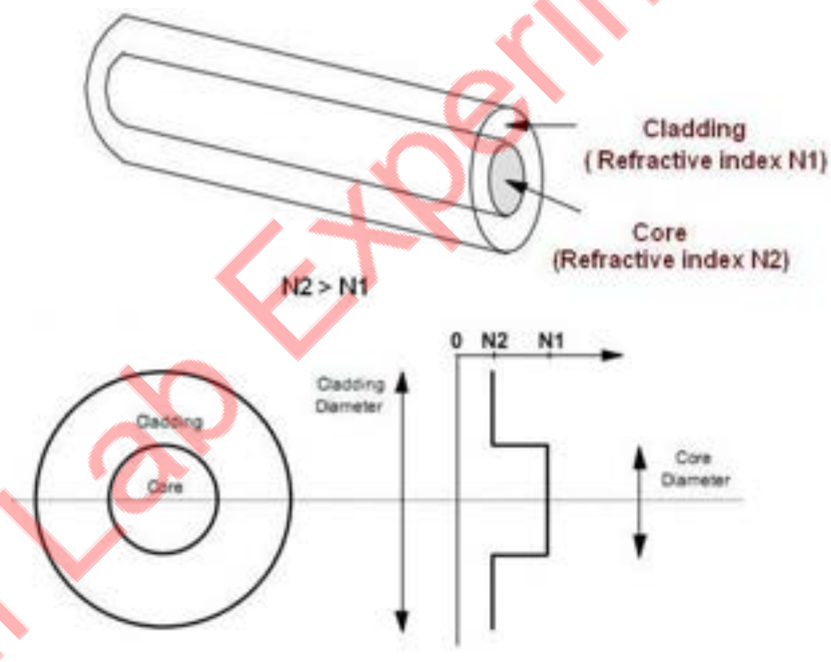


Fig (1)

Que 2: Explain and calculate the numerical aperture of an optical fibre.

Ans: In order to understand the propagation of light through an optical fibre, consider the figure (2). Consider a light ray (i) entering the core at a point A , travelling through the core until it reaches the core cladding boundary at point B. As long as the light ray intersects the core-cladding boundary at a small angles, the ray will be reflected back in to the core to travel on to point C where the process of reflection is repeated .ie., total internal reflection takes place. Total internal reflection occurs only when the angle of incidence is greater than the critical angle. If a ray enters an optic fiber at a steep angle(ii), when this ray intersects the core-cladding boundary, the angle of intersection is too large. So, reflection back in to the core does not take place and the light ray is lost in the cladding. This means that to be guided through an optic fibre, a light ray must enter the core with an angle less than a particular angle called the acceptance angle of the fibre. A ray which enters the fiber with an angle greater than the acceptance angle will be lost in the cladding.

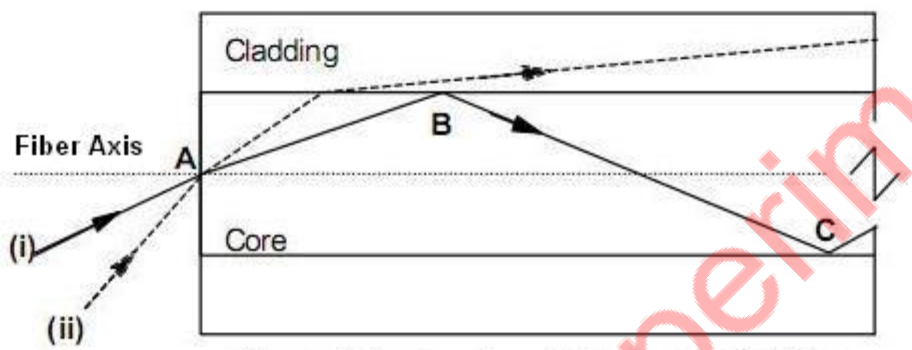


Figure 2 Propagation of light in an optical fibre

Consider an optical fibre having a core of refractive index n_1 and cladding of refractive index n_2. let the incident light makes an angle i with the core axis as shown in figure (3). Then the light gets refracted at an angle ϑ and fall on the core-cladding interface at an angle where,

$$\theta' = (90 - \theta) \quad \text{-----------------------} \ (1)$$

By Snell's law at the point of entrance of light in to the optical fiber we get,

$$n_0 \sin i = n_1 \sin\theta \quad \text{--------------------} \ (2)$$

Where n_0 is refractive index of medium outside the fiber. For air n_0 =1.

When light travels from core to cladding it moves from denser to rarer medium and so it may be totally reflected back to the core medium if θ' exceeds the critical angle θ'c. The critical angle is that angle of incidence in denser medium (n1) for which angle of refraction become 90°. Using Snell's laws at core cladding interface,

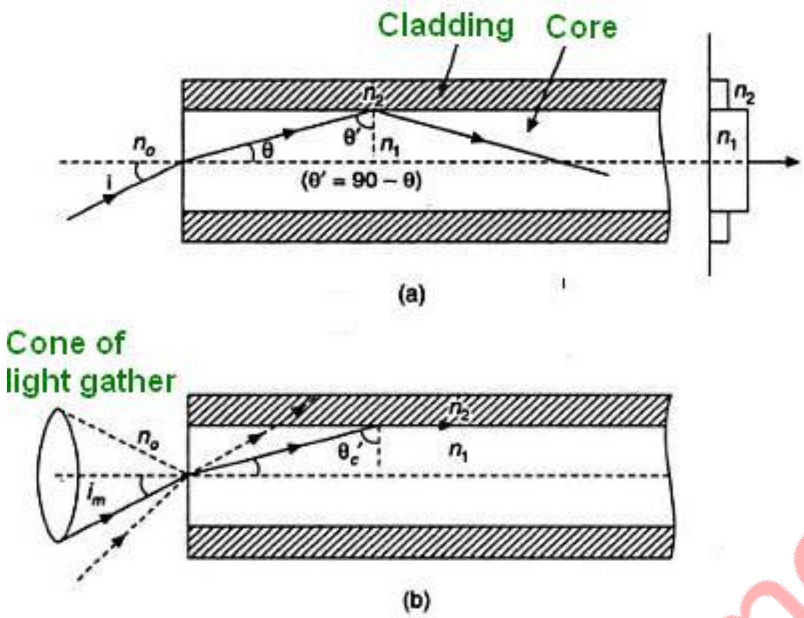


Figure 3.

$$n_1 \sin \theta'_c = n_2 \sin 90$$

or

$$Sin\, \theta'_c = \frac{n_2}{n_1} \quad \text{------------------------ (3)}$$

Therefore, for light to be propagated within the core of optical fiber as guided wave, the angle of incidence at core-cladding interface should be greater than ϑ'_c. As i increases, ϑ increases and so ϑ' decreases. Therefore, there is maximum value of angle of incidence beyond which, it does not propagate rather it is refracted in to cladding medium (fig: 3(b)). This maximum value of i say i_m is called maximum angle of acceptance and $n_0 \sin i_m$ is termed as the numerical aperture (NA).

From equation(2),

$$NA = n_0 \sin i_m = n_1 \sin \theta$$

$$= n_1 \sin(90 - \theta_c)$$

$$Or\ NA = n_1\, Cos\, \theta'_c$$

$$= n_1 \sqrt{1 - Sin^2\theta'_c}$$

$$\sin \theta'_c = \frac{n_2}{n_1}$$

From equation (2)

Therefore,

$$NA = n1\sqrt{1 - \frac{n_2^{\ 2}}{n_1^2}}$$

$$NA = \sqrt{n_1^{\ 2} - n_2^{\ 2}}$$

The significance of NA is that light entering in the cone of semi vertical angle i_m only propagate through the fibre. The higher the value of $i_m$or NA more is the light collected for propagation in the fibre. Numerical aperture is thus considered as a light gathering capacity of an optical fibre. Numerical Aperture is defined as the Sine of half of the angle of fibre's light acceptance cone. i.e. NA= Sin θ_a where θ_a, is called acceptance cone angle.

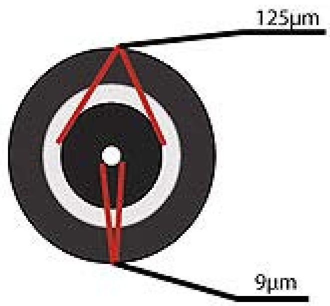
Let the spot size of the beam at a distance d (distance between the fiber end and detector) as the radius of the spot(r). Then,

$$\sin\theta = \frac{r}{\sqrt{r^2 + d^2}} \qquad \text{------------------------- (4)}$$

Que3: Differentiate between single and multiple mode optical mode fibres.
Ans:

Single Mode Fiber Optic Cable

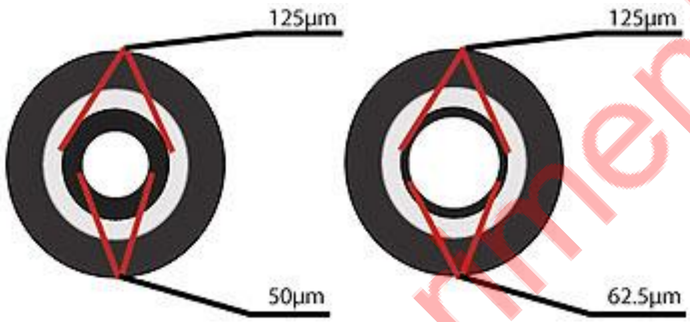
Single Mode fiber optic cable has a small diametral core that allows only one mode of light to propagate. Because of this, the number of light reflections created as the light passes through the core decreases, lowering attenuation and creating the ability for the signal to travel further. This application is typically used in long distance, higher bandwidth runs by Telcos, CATV companies, and Colleges and Universities.
Left: Single Mode fiber is usually 9/125 in construction. This means that the core to cladding diameter ratio is 9 microns to 125 microns.



Multimode Fiber Optic Cable

Multimode fiber optic cable has a large diametral core that allows multiple modes of light to propagate. Because of this, the number of light reflections created as the light passes through the core increases, creating the ability for more data to pass through at a given time. Because of the high dispersion and attenuation rate with this type of fiber, the quality of the signal is reduced over long distances. This application is typically used for short distance, data and audio/video applications in LANs. RF broadband signals, such as what cable companies commonly use, cannot be transmitted over multimode fiber.

Above: Multimode fiber is usually 50/125 and 62.5/125 in construction. This means that the core to cladding diameter ratio is 50 microns to 125 microns and 62.5 microns to 125 microns.



Que 4: Give a qualitative difference between step- index and graded-index multimode fibre.

Ans:

Step-Index Multimode Fiber

Due to its large core, some of the light rays that make up the digital pulse may travel a direct route, whereas others zigzag as they bounce off the cladding. These alternate paths cause the different groups of light rays, referred to as modes, to arrive separately at the receiving point. The pulse, an aggregate of different modes, begins to spread out, losing its well-defined shape. The need to leave spacing between pulses to prevent overlapping limits the amount of information that can be sent. This type of fiber is best suited for transmission over short distances.

Graded-Index Multimode Fiber

Contains a core in which the refractive index diminishes gradually from the center axis out toward the cladding. The higher refractive index at the center makes the light rays moving down the axis advance more slowly than those near the cladding. Due to the graded index, light in the core curves helically rather than zigzag off the cladding, reducing its travel distance. The shortened path and the higher speed allow light at the periphery to arrive at a receiver at about the same time as the slow but straight rays in the core axis. The result: digital pulse suffers less dispersion. This type of fiber is best suited for local-area networks.