11 Electromagnetics

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Displacement current.
- Equation of continuity.
- Maxwells equations (integral and differential forms).
- Poynting vector and Poynting theorem.
- Electromagnetic wave equation and its propagation characteristics in free space.
- Non-conducting and in-conducting media.
- Skin depth.

11.1 Introduction

You have already studied static (i.e. time independent) electric and magnetic fields in electrostatics and magnetostatics, respectively in previous classes. These fields are produced by the charges at rest and steady currents respectively, and can be analyzed independently. But if these fields vary with time, one cannot analyze them independently. Now the question arises: Why? The answer is: Faraday's law of electromagnetic induction shows that a time-varying magnetic field produces an electric field while Ampere's law shows that a time-varying electric field produces a magnetic field. Thus, changing of electric and magnetic field with time, a field of other kind is induced in the adjacent space which produces electromagnetic waves consisting electric and magnetic fields.

11.1.1 Laws of Electromagnetics Before Maxwell

There are four basic laws of electricity and magnetism before Maxwell which are as follows:

1. Gauss' law of electrostatics

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \text{or} \qquad \oint \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

Here q is charge and ρ is the volume charge density.

2. Gauss' law of magnetostatics

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \text{or} \qquad \oint \vec{B} \cdot d\vec{S} = 0$$

3. Faraday's law of induction

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{or} \qquad \oint \vec{E} \cdot d\vec{l} = -\frac{\partial \phi_B}{\partial t}$$

where ϕ_B is the magnetic flux.

4. Ampere's law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \qquad \text{or} \qquad \oint \vec{B} \cdot d\vec{l} = \mu_0 I = \mu_0 \int_S \vec{J} \cdot d\vec{S}$$

Here I is current and J is current density.

These equations are the relation between the field and their source and are used to solve the problems of electromagnetic theory since long, even before the Maxwell started his work. Among the above four equations, the Ampere's law in the present form is true only for steady case. Maxwell noticed this inconsistency in equation during his study while applying Ampere's law to a capacitor. Thus, Maxwell formulated the concept of displacement current to remove this inconsistency and modified the Ampere's law which will be discussed in following sections.

11.2 Displacement Current

The concept of displacement current was first conceived by Maxwell to explain the production of magnetic field in empty space. According to him, it is not only the current in a conductor that produces a magnetic field, but a changing electric field in a vacuum or in a dielectric also produces a magnetic field. This means that a changing electric field is equivalent to a current and gives same effect to magnetic field as the conduction current. This equivalent current is known as *displacement current* which exists in the space as long as the electric field is changing and is expressed as

$$\varepsilon_o \frac{d\phi_E}{dt}$$

In order to explain the displacement current mathematically, we consider the case of parallel-plate capacitor. Let at any particular instant, q be the charge on capacitor plate. According to the definition, conduction current at any instant is

$$i_c = \frac{dq}{dt} \tag{11.1}$$

We have already discussed about electrical displacement ($D = \varepsilon_o E$) in dielectrics (Chapter 9). Therefore,

$$D = \sigma = \frac{q}{A} \tag{11.2}$$

where σ is the surface charge density and A is the area of the parallel-plate capacitor. From Eq. (11.2) we have

$$q = DA \tag{11.3}$$

Now substituting the value of q from Eq. (11.3) in Eq. (11.1), we get

$$i_c = \frac{d}{dt}(DA) = A\frac{dD}{dt} \tag{11.4}$$

Maxwell suggested that the term $i_d = \varepsilon_o \dfrac{d\phi_E}{dt}$ should be considered as the current inside the dielectric. This current is called as displacement current and is denoted by i_d. Hence,

$$i_d = \varepsilon_o \frac{d\phi_E}{dt} \quad \Rightarrow \quad i_d = A\varepsilon_o \frac{dE}{dt} \quad \Rightarrow \quad i_d = A\frac{dD}{dt} \quad \Rightarrow \quad i_d = EA \tag{11.5}$$

We know that $J_d = i_d / A$ is current density. Therefore Eq. (11.5) may be written in terms of current density J_d as

$$\vec{J}_d = \frac{d\vec{D}}{dt} \tag{11.6}$$

or

$$\vec{J}_d = \varepsilon_0 \frac{dE}{dt} \qquad [\because \vec{D} = \varepsilon_0 E] \tag{11.7}$$

Thus, the current arising due to time-varying electric field between the plates of a capacitor is called the displacement current.

11.2.1 Characteristics of Displacement Current

1. Displacement current is a current only in the sense that it produces a magnetic field. It has none of the other properties of current because it is not related to the motion of charges.
2. Inside the dielectric there will be a displacement current which is equal to conduction current.
3. Displacement current is only an apparent current representing the rate at which flow of charge takes place from one plate to another plate.
4. Displacement current in good conductors is almost nil as compared to conduction current below the frequency 10^{15} Hz.

11.3 Equation of Continuity

Continuity equation is the consequence of conservation of charge. Law of conservation of charges states that electric charges can neither be created nor destroyed. Therefore, the total current flowing out of the system of some volume must be equal to the rate of decrease of charge within the volume. Therefore, when the current flows at any region of volume V, bounded by a closed surface S then

$$i = -\frac{dq}{dt} = \oint_S \vec{J} \cdot d\vec{S} \tag{11.8}$$

But we know that total charge is enclosed by the close surface in terms of volume charge density ρ with in volume V, that is,

$$q = \int_V \rho \, dV \tag{11.9}$$

Therefore

$$i = -\frac{dq}{dt} = \oint_S \vec{J} \cdot d\vec{S} = -\int_V \frac{\partial \rho}{\partial t} dV$$

or

$$\oint_S \vec{J} \cdot d\vec{S} + \int_V \frac{\partial \rho}{\partial t} dV = 0 \tag{11.10}$$

From the fundamental theorem of divergence, which is a relation between surface integral to volume integral, we have

$$\oint_S \vec{J} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{J} \, dV$$

$$\Rightarrow \int_V \left(\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right) dV = 0$$

which is true for any arbitrary volume, therefore,

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{11.11}$$

This equation is the continuity equation and is based on the conservation of charge. When we use time derivative term

$$\frac{\partial \rho}{\partial t} = 0$$

the above continuity equation is reduced to

$$\vec{\nabla} \cdot \vec{J} = 0 \tag{11.12}$$

That is, the net flux of current through any closed surface is zero which is the case of steady state.

11.4 Modification of Ampere's Law

1. **Integral form of Ampere's law:** Maxwell modified the Ampere's law by introducing the term of displacement current from the study of charging and discharging of a capacitor. If we look at the simple circuit with a capacitor C in Fig. 1, the current flows in the circuit after proper connection, the charges start accumulating on the capacitor plates and the magnetic field between the plates as well as outside plate (around wire) is observed. As there is no actual flow of charges between plates, there is no conduction current as well, but the electric field in space due to charges on plates continuously changes with the time as long as the charges on plates change. This changing electric field cause the generation of magnetic field between the plates.

   Now in Fig. 1, we consider a small loop around the wire just to analyze the magnetic field due to conduction current i in wire, then according to present Ampere's law "The line integral of magnetic induction B around a closed path is equal to μ_0 times the current enclosed by the path." Mathematically

$$\oint \vec{B} \cdot d\vec{l} = \mu_o i \tag{11.13}$$

   If the loop encloses a surface area S_1 then according to Stokes' theorem

$$\oint \vec{B} \cdot d\vec{l} = \int_{S_1} (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \mu_o i \tag{11.14}$$

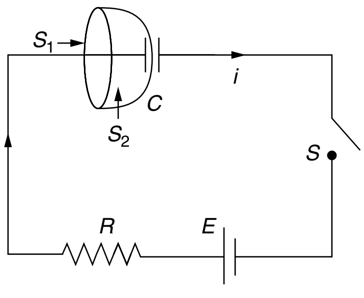

Figure 1 Modification in Ampere's law.

But if the loop encloses a surface area S_2 (according to fundamental theorem of curl, i.e. Stokes' theorem, no matter what surface you consider, if it is bounded with the same loop), no conduction current passes through this surface. Then,

$$\oint \vec{B} \cdot d\vec{l} = \int_{S_2} (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = 0 \tag{11.15}$$

The above two equations for the same loop with different surfaces are not same and the right-side values of equations contradict, therefore, both cannot be true. Hence, the present form of Ampere's law is inconsistent or not true for all cases.

Now from the definition of displacement current which is

$$i_d = \varepsilon_o \frac{d\phi_E}{dt}$$

which is developed in the space between capacitor plate at surface S_2 and equal to the conduction current in magnitude. Hence, either the conduction or the displacement current is present at any surface under consideration, therefore both currents are to be considered in the Ampere's law and equation is modified in following form:

$$\oint \vec{B} \cdot d\vec{l} = \mu_o \left(i + \varepsilon_o \frac{d\phi_E}{dt} \right) = i + i_d \tag{11.16}$$

Now after modification of equation as in above case, when S_1 surface is considered, i_d is absent and if S_2 surface is considered, the only i_d is present and anomaly or inconsistency in equation is removed.

2. To look at the differential form of Ampere's law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I = \mu_0 \int_S \vec{J} \cdot d\vec{S} \tag{11.17}$$

where $\vec{J}$ is the current density in the conductor having cross-sectional area $\vec{S}$. Using Stokes' law which is a relation between line integral and surface integral, we have

$$\oint \vec{B} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \mu_0 \int_S \vec{J} \cdot d\vec{S} \tag{11.18}$$

Since surface is arbitrary, so we have

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \qquad \text{or} \qquad \vec{\nabla} \times \vec{H} = \vec{J} \tag{11.19}$$

Taking divergence on both sides of Eq. (11.19), we have

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J}$$

Since $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = 0$, then also

$$\vec{\nabla} \cdot \vec{J} = 0 \tag{11.20}$$

Equation (11.20) is valid only for steady current. For other non-steady cases $\vec{\nabla} \cdot \vec{J} \neq 0$. In other words, J is not always a solenoidal vector, hence Eq. (11.19) is inconsistent.

Also from the equation of continuity

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0 \qquad \text{or} \qquad \frac{\partial\rho}{\partial t}=0$$

Here ρ is constant that shows charge density is not changing with the time. As a result, Ampere's law should be modified for time-varying field using a quantity $\vec{J}_D$ which is to be added to the right-hand side of Eq. (11.19), so that J together with $\vec{J}_D$ becomes the solenoidal vector whose divergence is always zero. Therefore, the following equation after introducing $\vec{J}_D$ is true for all cases.

$$\vec{\nabla}\times\vec{H}=\vec{J}+\vec{J}_D \tag{11.21}$$

It can be explained in the following way: The equation of continuity

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$$

from differential form of Gauss law (first Maxwell equation)

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0} \qquad \text{or} \qquad \rho=\vec{\nabla}\cdot\vec{D}$$

Then

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\vec{\rho}}{\partial\vec{t}}=\vec{\nabla}\cdot\vec{J}+\vec{\nabla}\cdot\frac{\partial\vec{D}}{\partial\vec{t}}=\vec{\nabla}\cdot\left(\vec{J}+\frac{\partial\vec{D}}{\partial\vec{t}}\right)=0 \tag{11.22}$$

Here $\vec{J}+(\partial\vec{D}/\partial\vec{t})$ is the solenoidal vector whose divergence is always zero. To remove the inconsistency in Ampere's law, Maxwell suggested that the current density $\vec{J}$ should be replaced by $\vec{J}+(\partial\vec{D}/\partial\vec{t})$ in Eq. (11.19). Hence, by introducing the term $\vec{J}+(\partial\vec{D}/\partial\vec{t})$ in Eq. (11.19), the following is the correct modified differential form of Ampere's law which is true for time varying as well as for steady currents.

$$\vec{\nabla}\times\vec{H}=\vec{J}+\left(\frac{\partial\vec{D}}{\partial t}\right)$$

or

$$\vec{\nabla}\times\vec{H}=\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t} \tag{11.23}$$

11.5 Maxwell's Equations

Maxwell, in 1864, theoretically derived the connection between the charges at rest (electrostatics), charges in motion (current electricity), electric and magnetic field (electromagnetic) and summarized in terms of four equations: Gauss' law in electrostatic, Gauss' law in magnetostatics, Ampere's law and Faraday's laws. These equations are called Maxwell's equations. Table 1 gives the four Maxwell's equations in differential and integral forms.

Table 1 Maxwell's equations in differential and integral form

S. No.	*Differential Form*			*Integral Form*
1.	$\text{div}\,\vec{E} = \dfrac{\rho}{\varepsilon_0}$	or	$\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$	$\oint \vec{E}\cdot d\vec{S} = \dfrac{q}{\varepsilon_0}$ or $\oint \vec{E}\cdot d\vec{S} = \dfrac{1}{\varepsilon_0}\int \rho\, dV$
2.	$\text{div}\,\vec{B} = 0$	or	$\vec{\nabla}\cdot\vec{B} = 0$	$\oint \vec{B}\cdot d\vec{S} = 0$
3.	$\text{curl}\,\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$	or	$\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$	$\oint \vec{E}\cdot d\vec{l} = -\dfrac{\partial\phi_B}{\partial t}$
4.	$\text{curl}\,\vec{H} = \vec{J} + \left(\dfrac{\partial\vec{D}}{\partial t}.\right)$	or	$\vec{\nabla}\times\vec{H} = \vec{J} + \left(\dfrac{\partial\vec{D}}{\partial t}.\right)$	$\oint \vec{H}\cdot d\vec{l} = \int_S \left(\vec{J} + \dfrac{\partial\vec{D}}{\partial t}\right)\cdot d\vec{S}$
	where ρ is the charge density. $\vec{D} = \varepsilon_0\vec{E}$, electric displacement vector, ε_0 is the permittivity of the free space and $\vec{E}$ is the electric field strength. $\vec{B} = \mu_0\vec{H}$, where μ_0 is the magnetic permeability of free space and $\vec{H}$ is the magnetic field intensity.			

11.5.1 Derivation of Maxwell's First Equation

According to Gauss' law in electrostatics 'The net flux passing through a closed surface is equal to $1/\varepsilon_0$ times the total charge q contained in the volume enclosed by surface.' Mathematically,

$$\phi_E = \oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0} \tag{11.24}$$

where $\oint \vec{E}\cdot d\vec{S}$ represents the total flux passing through closed surface S. But we know that total charge enclosed in the surface in terms of volume charge density ρ with in volume V is

$$q = \int_V \rho\, dV \tag{11.25}$$

From Eqs. (11.24) and (11.25), we get that

$$\oint \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\int \rho\, dV$$

or

$$\int_S \vec{D}\cdot d\vec{S} = \int_V \rho\, dV \qquad (\because \vec{D} = \varepsilon_0\vec{E})$$

By Gauss' divergence theorem

$$\int_S \vec{D}\cdot d\vec{S} = \int_V (\text{div}\vec{D})\cdot d\vec{V} = \int_V \rho\, dV$$

$$\Rightarrow \int_V (\text{div}\,\vec{D} - \rho)\, dV = 0$$

Since volume is arbitrary, hence

$$\text{div}\,\vec{D} - \rho = 0 \qquad \text{or} \qquad \text{div}\,\vec{D} = \rho \tag{11.26}$$

In free space, volume charge density ρ is zero. Therefore, Maxwell's first equation in free space is

$$\text{div}\,\vec{D} = \vec{\nabla}\cdot\vec{D} = 0 \tag{11.27}$$

11.5.2 Maxwell's Second Equation

We know that magnetic monopole does not exist in the nature. Since magnetic lines of force entering or leaving a closed surface are equal, therefore, the net magnetic flux passing through the area $d\vec{S}$ of a closed surface S is zero:

$$\int_S \vec{B}\cdot d\vec{S} = 0 \tag{11.28}$$

Using Gauss' divergence theorem which is a relation between surface integral to volume integral as given below

$$\oint_S \vec{B}\cdot d\vec{S} = \int_V (\vec{\nabla}\cdot\vec{B})\, dV$$

$$\int_V (\vec{\nabla}\cdot\vec{B})\, dV = 0$$

Since the volume is arbitrary

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{11.29}$$

This is the requirement of Maxwell's second equation and it is true for free as well as material medium.

11.5.3 Maxwell's Third Equation

According to Faraday's law of electromagnetic induction the induced electromagnetic force around a closed circuit is equal to the negative time rate of charge of magnetic flux linked with the circuit. Thus,

$$e = -\frac{d\phi}{dt} \tag{11.30}$$

But we know that

$$e = \oint_c \vec{E}\cdot d\vec{l} = -\frac{d\phi}{dt}$$

$$\Rightarrow \oint_c \vec{E}\cdot d\vec{l} = -\frac{d\int_S \vec{B}\cdot d\vec{S}}{dt} = -\int \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} \tag{11.31}$$

From Stokes' fundamental theorem

$$\oint_c \vec{E}\cdot \vec{dl} = \int (\vec{\nabla}\times\vec{E})\cdot d\vec{S}$$

$$\Rightarrow \int (\vec{\nabla}\times\vec{E})\cdot d\vec{S} = -\int \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}$$

Since surface S is arbitrary, hence

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{11.32}$$

This is Maxwell's third equation for free as well as for material medium.

11.5.4 Maxwell's Fourth Equation

The integral form of Maxwell fourth equation is

$$\oint \vec{B} \cdot d\vec{l} = \mu_o \left(i + \varepsilon_o \frac{d\phi}{dt} \right) \tag{11.33}$$

The current i in term of J and electric flux ϕ in terms of E for any surface can be expressed as

$$i = \int \vec{J} \cdot d\vec{S}$$

$$\phi = \int \vec{E} \cdot d\vec{S}$$

The right side of equation can be expressed as

$$i + \varepsilon_o \frac{d\phi}{dt} = \int \vec{J} \cdot d\vec{S} + \varepsilon_o \int \frac{d\vec{E}}{dt} \cdot d\vec{S} \tag{11.34}$$

From Stokes' theorem

$$\oint \vec{B} \cdot d\vec{l} = \int \left(\vec{\nabla} \times \vec{B} \right) \cdot d\vec{S} \tag{11.35}$$

Therefore

$$\int (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \mu_0 \left[\int \vec{J} \cdot d\vec{S} + \varepsilon_o \int \frac{d\vec{E}}{dt} \cdot d\vec{S} \right]$$

$$= \mu_0 \left[\int \left(\vec{J} + \varepsilon_o \frac{d\vec{E}}{dt} \right) \cdot d\vec{S} \right] \tag{11.36}$$

Since surface S is arbitrary, hence

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_o \frac{d\vec{E}}{dt} \right)$$

$$\vec{\nabla} \times \vec{H} = \left(\vec{J} + \frac{d\vec{D}}{dt} \right) \qquad [\because (\vec{D} = \varepsilon_o \vec{E})] \tag{11.37}$$

This is Maxwell's fourth equation in differential form.

11.6 Maxwell's Equation in Integral Form

1. Maxwell's first equation in differential form is

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{11.38}$$

Integrating it with respect to volume V, we get

$$\int_V (\vec{\nabla} \cdot \vec{D})\, dV = \int_V \rho \, dV$$

The volume integral can be changed into surface integral with the help of Gauss divergence theorem as

$$\int_V \left(\vec{\nabla} \cdot \vec{D} \right) dV = \oint \vec{D} \cdot d\vec{S} \tag{11.39}$$

$$\oint \vec{D} \cdot d\vec{S} = \int_V \rho \, dV$$

or $$\oint \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0} \tag{11.40}$$

This is the integral form of Maxwell's first equation.

2. Maxwell's second equation in differential form is

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{11.41}$$

Integrating second equation with respect to the volume V, we get

$$\int_V (\vec{\nabla} \cdot \vec{B}) \, dV = 0$$

By Gauss' divergence theorem

$$\int_V (\vec{\nabla} \cdot \vec{B}) \, dV = \oint \vec{B} \cdot d\vec{S}$$

$$\oint \vec{B} \cdot d\vec{S} = 0 \tag{11.42}$$

where S is the surface enclosing volume V. This is the integral form of Gauss' divergence theorem in magnetostatics.

3. Maxwell's third equation in differential form is

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{11.43}$$

Integrating the above equation over an open surface S, we get

$$\int_S \left(\vec{\nabla} \times \vec{E}\right) \cdot d\vec{S} = -\int_S \left(\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{S}$$

The surface integral can be converted into line integral through Stokes' theorem as

$$\int_S \left(\vec{\nabla} \times \vec{E}\right) \cdot d\vec{S} = \oint \vec{E} \cdot d\vec{l} = -\int_S \left(\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{S}$$

Therefore

$$\oint \vec{E} \cdot d\vec{l} = -\int_S \left(\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{S}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial \phi_B}{\partial t} \qquad \left(\because \phi_B = \int_S \vec{B} \cdot d\vec{S}\right) \tag{11.44}$$

This is the integral form of Faraday's law of electromagnetic induction. Equation (11.44) is the integral form of Maxwell's third equation.

4. Maxwell's fourth equation in differential form is

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{11.45}$$

Integrating above equation with respect to S, we get

$$\int_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{S} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S}$$

From Stokes' theorem,

$$\int_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{S} = \oint \vec{H} \cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S}$$

$$\oint \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{S} + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$$

$$\oint \vec{H} \cdot \vec{dl} = \left(i + \varepsilon_o \frac{d\phi_E}{dt} \right) \quad \left(\because \phi_E = \int_S \vec{E} \cdot d\vec{S} \right) \tag{11.46}$$

Equation (11.46) is the integral form of Maxwell's fourth equation.

11.7 Physical Significance of Maxwell's Equations

11.7.1 Maxwell's First Electromagnetic Equation

Because of time independence, *Maxwell's first electromagnetic equation* is a steady-state equation. It represents the Gauss' law in electrostatics which states that the electric flux through any closed hypothetical surface is equal to $1/\varepsilon_0$ times the total charge enclosed by the surface.

11.7.2 Maxwell's Second Electromagnetic Equation

Maxwell's second electromagnetic equation represents Gauss' law in magnetostatics. It states that the net magnetic flux through any closed surface is zero (i.e., the number of magnetic lines of flux entering any region is equal to the lines of flux leaving it). It also explains that no isolated magnetic pole exists.

11.7.3 Maxwell's Third Electromagnetic Equation

Maxwell's third electromagnetic equation represents Faraday's law in electromagnetic induction. It states that an electric field is induced in the form of close lines when magnetic flux (or lines of magnetic force) changes through an open surface. The line integral of induced electric field around a close path is equal to the negative rate of change of magnetic flux.

11.7.4 Maxwell's Fourth Electromagnetic Equation

Maxwell's fourth electromagnetic equation represents the modified form of Ampere's circuital law which states that a changing electric field produces a magnetic field and an electric field can also be produced by changing magnetic field. Therefore, *Maxwell's fourth electromagnetic equation* gives the new concept of generation of magnetic field by displacement current.

11.8 Poynting Vector and Poynting Theorem

The moving oscillating coupled electric and magnetic fields behave as electromagnetic waves. These waves are transverse in nature where electric and magnetic vectors oscillate perpendicular to the direction of motion. During propagation, these waves also transport energy and momentum. The waves, when strike any surface, exert a pressure on the surface.

Poynting Vector is defined as the energy transported by wave per unit area per unit time. It is denoted by a vector P and can be expressed by the cross product of electric and magnetic field in the following way

$$\vec{P} = \vec{E} \times \vec{H} \qquad \text{or} \qquad \vec{P} = \frac{\vec{E} \times \vec{B}}{\mu_o} \tag{11.47}$$

The direction of the flow of this power through unit area is in the direction of propagation of wave. Its SI unit is Watt/m^2.

Poynting theorem is a work–energy theorem of electromagnetics and expressed as *work done on the charges by the electromagnetic forces is equal to the decrease in energy stored in the fields, and less than the energy that flows out through the surface*. To derive and explain the Poynting theorem, let us take third and fourth Maxwell equations as follows:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{11.48}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{11.49}$$

Taking the dot product of $\vec{H}$ with Eq. (11.48) and that of $\vec{E}$ with Eq. (11.49), we have

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \tag{11.50}$$

$$\vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \tag{11.51}$$

Subtracting Eq. (11.50) from Eq. (11.51), we get

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \vec{J} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = -\left[\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right] - \vec{E} \cdot \vec{J}$$

From vectors product

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot (\vec{E} \times \vec{H}) \tag{11.52}$$

Therefore,

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\left[\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right] - \vec{E} \cdot \vec{J} \tag{11.53}$$

But $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$. Therefore

$$\nabla \cdot (\vec{E} \times \vec{H}) = -\left[\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} + \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}\right] - \vec{E} \cdot \vec{J}$$

$$= -\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right)\right] - \vec{E} \cdot \vec{J} \quad \left(\because \mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) \text{and } \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right)\right)$$

$$\vec{E} \cdot \vec{J} \quad = -\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right)\right] - \nabla \cdot (\vec{E} \times \vec{H})$$

Taking the volume integral over a volume V enclosed by surface S, we get

$$\int_V (\vec{E}\cdot\vec{J})\,dV = -\int_V \left[\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right)\right] dV - \int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dV \tag{11.54}$$

Using Gauss divergence theorem

$$\int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dV = \oint(\vec{E}\times\vec{H})\cdot dS$$

Hence,

$$\int_V \left(\vec{E}\cdot\vec{J}\,\right) dV = -\int_V \left[\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right)\right] dV - \oint(\vec{E}\times\vec{H})\cdot dS \tag{11.55}$$

Equation (11.55) represents the work energy theorem of electromagnetic and is called Poynting theorem for the flow of energy in an electromagnetic field.

1. The term $\int_V (\vec{E}\cdot\vec{J}\,)\,dV$ represents the work done per unit time on the charges by electromagnetic fields.
2. The term $-\int_V \left[\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right)\right] dV$ represents the rate of decrease of stored energy in electric and magnetic fields in volume V.
3. $-\oint(\vec{E}\times\vec{H})\cdot dS$ represents the rate of flow of energy through surface area S enclosing volume V.

Here $\vec{P} = \vec{E}\times\vec{H}$ is the energy flowing through unit area and unit time and is known as the Poynting vector.

11.9 Plane Electromagnetic Waves in Free Space

We describe one of the important applications of Maxwell's equations to derive electromagnetic wave equations for field vectors $\vec{E}$ and $\vec{B}$. In free space, where there is no charge or current (i.e. $\rho = 0$, $\vec{J} = 0, \varepsilon = \varepsilon_0$, $\mu = \mu_0, B = \mu_0 H$ and $D = \varepsilon_0 E$), Maxwell's equations are as follows:

$$\vec{\nabla}\cdot\vec{E} = 0 \tag{11.56}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{11.57}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{11.58}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \tag{11.59}$$

Taking the curl on both sides of Eq. (11.58) we get

$$\vec{\nabla}\times(\vec{\nabla}\times E) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2 E = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

Using Eq. $\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ and $\vec{\nabla}\cdot\vec{E} = 0$

$$\nabla^2 E = \mu_0\varepsilon_0 \frac{\partial^2 E}{\partial t^2} \tag{11.60}$$

Similarly taking the curl of fourth equation (11.59) we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times\left(\mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2 B = -\mu_0\varepsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

Using $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and $\vec{\nabla}\cdot\vec{B} = 0$ we have

$$\nabla^2 B = \mu_0\varepsilon_0 \frac{\partial^2 B}{\partial t^2} \tag{11.61}$$

In vector form

$$\nabla^2 \vec{E} = \mu_0\varepsilon_0\frac{\partial^2 \vec{E}}{\partial t^2}, \qquad \nabla^2 \vec{B} = \mu_0\varepsilon_0\frac{\partial^2 \vec{B}}{\partial t^2} \tag{11.62}$$

The general wave equation for any function like u moving with speed v is

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{11.63}$$

Therefore, from the above equation, Eq. (11.62) represents wave equations for ***E*** and ***B*** in free space. Each Cartesian component of ***E*** and ***B*** satisfies the three-dimensional wave equation.

So Maxwell's equations imply that empty space supports the propagation of electromagnetic waves, travelling at a speed

$$\mu_0\varepsilon_0 = \frac{1}{v^2} \Rightarrow v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{(4\pi\times10^{-7}\ \text{Weber/A-m})(8.85\times10^{-12}\ \text{C}^2\text{N-m}^2)}} = 2.99\times10^8\ \text{m/s}$$

Hence, *electromagnetic waves propagate in free space with the speed of light*:

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \tag{11.64}$$

In some other medium, velocity is given as

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{(\mu_0\mu_r)(\varepsilon_0\varepsilon_r)}} \tag{11.65}$$

where μ_r and ε_r are relative permeability and relative permittivity, respectively. Using Eq. (11.64), Eq. (11.65) can be written as

$$v = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

or $$v = \frac{c}{\sqrt{\varepsilon_r}} \qquad \text{[For non-magnetic material } \mu_r = 1\text{]}$$

As we know that the refractive index n of the medium is

$$n = \frac{c}{v}$$

Therefore,

$$n = \sqrt{\varepsilon_r} \tag{11.66}$$

The speed of light in a material is always less than in vacuum because ε_r has a value greater than one.

11.10 Transverse Nature of Electromagnetic Waves

The electromagnetic waves are transverse in nature where E and B vector oscillate perpendicular to the propagation direction. To explain the transverse nature, let us have the solution of wave equations which are mathematically second order differential equations. The equations are

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \quad \text{and} \quad \nabla^2 B - \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = 0 \tag{11.67}$$

The general solution of these equation are respectively

$$\vec{E}(r,t) = \vec{E}_o e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

and $$\vec{B}(r,t) = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \tag{11.68}$$

where $\vec{E}_o$ and $\vec{B}_0$ are the complex amplitudes for electric and magnetic fields, respectively, whose real part represent the physical value. $\vec{k}$ is the wave vector and $\vec{r}$ is position vector which are expressed as

$$\vec{k} = \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi}{c/\upsilon}\hat{n} = \frac{2\pi\upsilon}{c}\hat{n} = \frac{\omega}{c}\hat{n}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$\hat{n}$ is unit vector represents the wave propagation direction. Then

$$\vec{k}\cdot\vec{r} = (k_x\hat{i} + k_y\hat{j} + k_z\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k}) = (k_x x + k_y y + k_z z)$$

Now considering solution $\vec{E}$, we find the divergence of Eq. (11.56), that is, $\vec{\nabla}\cdot\vec{E}=0$.

$$\vec{\nabla}\cdot\vec{E}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}E_{0x}+\hat{j}E_{0y}+\hat{k}E_{0z}\right)e^{i(k_x x+k_y y+k_z z-\omega t)}$$
$$=i(E_{0x}k_x+E_{0y}k_y+E_{0z}k_z)e^{i(k_x x+k_y y+k_z z-\omega t)}$$
$$=i(\hat{i}k_x+\hat{j}k_y+\hat{h}k_z)\cdot(\hat{i}E_{0x}+\hat{j}E_{0y}+\hat{h}E_{0z})e^{i(k_x x+k_y y+k_z z-\omega t)}$$
$$=i[\vec{k}\cdot E_0 e^{i(k_x x+k_y y+k_z z-\omega t)}]$$
$$=i\vec{k}\cdot\vec{E}$$

Since $\vec{\nabla}\cdot\vec{E}=0$. So

$$\vec{k}\cdot\vec{E}=0 \tag{11.69}$$

or $\vec{E}$ is perpendicular to $\vec{k}$. Now $\vec{k}$ has direction of wave propagation, so $\vec{E}$ is perpendicular to the direction of propagation. Similarly, consider second equation (11.57), $\vec{\nabla}\cdot\vec{B}=0$. We get

$$\vec{\nabla}\cdot\vec{B}=i(\vec{k}\cdot\vec{B})\Rightarrow\vec{k}\cdot\vec{B}=0. \tag{11.70}$$

So $\vec{B}$ is perpendicular to the direction of wave propagation. Therefore EM wave is transverse in nature.

11.11 Characteristic Impedance

Consider Maxwell's third equation

$$\vec{\nabla}\times\vec{E}=-\frac{\partial B}{\partial t}$$

We solve it by considering $\vec{E}$ and $\vec{B}$ as given by Eq. (11.68):

$$\vec{\nabla}\times\vec{E}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ E_{0x}e^{i(\vec{k}\cdot\vec{r}-\omega t)} & E_{0y}e^{i(\vec{k}\cdot\vec{r}-\omega t)} & E_{0z}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}\left(E_{0z}e^{i(k_x x+k_y y+k_z z-\omega t)}\right)-\frac{\partial}{\partial z}\left(E_{oy}e^{i(k_x x+k_y y+k_z z-\omega t)}\right)\right]$$
$$+\hat{j}\left[\frac{\partial}{\partial z}\left(E_{0x}e^{i(k_x x+k_y y+k_z z-\omega t)}\right)-\frac{\partial}{\partial x}\left(E_{oz}e^{i(k_x x+k_y y+k_z z-\omega t)}\right)\right]$$
$$+\hat{k}\left[\frac{\partial}{\partial x}\left(E_{0y}e^{i(k_x x+k_y y+k_z z-\omega t)}\right)-\frac{\partial}{\partial y}\left(E_{ox}e^{i(k_x x+k_y y+k_z z-\omega t)}\right)\right]$$
$$=i\left[\hat{i}(k_y E_{0z}-k_z E_{0y})+\hat{j}(k_z E_{0x}-k_x E_{0z})+\hat{k}(k_x E_{0y}-k_y E_{0x})\right]e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= i[\vec{k} \times \vec{E}_0]e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i[\vec{k} \times \vec{E}]$$

and $$\frac{\partial B}{\partial t} = \frac{\partial}{\partial t}(B_0 e^{i(k_x x + k_y y + k_z z - \omega t)} = -i\omega B_0 e^{i(k_x x + k_y y + k_z z - \omega t)} = -i\omega \vec{B}$$

Hence from $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$, we have

$$i(\vec{k} \times \vec{E}) = i\omega\vec{B} \quad \Rightarrow \quad \vec{k} \times \vec{E} = \omega\vec{B} \tag{11.71}$$

Similarly from Maxwell's fourth equation $\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$, we have

$$\vec{k} \times \vec{B} = -\omega\mu_0\varepsilon_0\vec{E} \tag{11.72}$$

From Eqs. (11.71) and (11.72) it can be concluded that electric and magnetic vectors $\vec{E}$ and $\vec{B}$ are mutually perpendicular to each other and perpendicular to the direction of propagation vector $\vec{k}$ (see Fig. 2). Further from Eq. (11.71) we have

$$\vec{k} \times \vec{E} = \omega\vec{B}$$

$$k(\hat{n} \times \vec{E}) = \omega\vec{B}$$

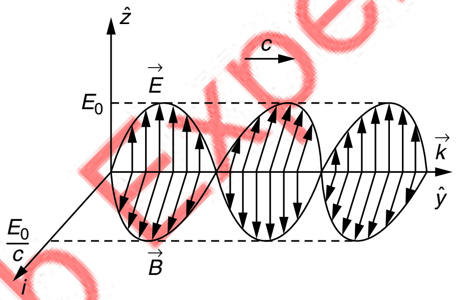

Figure 2

The mod of the above equation is

$$k\left|\hat{n} \times \vec{E}\right| = \omega\left|\vec{B}\right|$$

$$\frac{\vec{E}}{\vec{B}} = \frac{\omega}{k} = c \qquad \left(k = \frac{\omega}{z}\right)$$

or $$\frac{\vec{E}}{\vec{H}} = \mu_0 c = \sqrt{\frac{\mu_0}{\varepsilon_0}} \qquad [\vec{B} = \mu_0 H]$$

E/H is the characteristic impedance or intrinsic impedance of free space denoted by Z_0 and has the unit electrical resistance. Its value is

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{45 \times 10^{-7}}{8.86 \times 10^{-12}}} = 376.7\,\Omega \tag{11.73}$$

This implies that electric vector $\vec{E}$ and magnetic field vector $\vec{B}$ are in the same phase.

11.12 Electromagnetic Waves in Dielectric Medium

Since we are familiar that there is no free charge in dielectric medium therefore, $\rho = 0$, $\sigma = 0$, and hence $\vec{J} = \sigma E = 0$. However, μ and ε have finite values. So Maxwell's equations are as follows:

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{11.74}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{11.75}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{11.76}$$

$$\vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t} \tag{11.77}$$

Taking curl on both sides of Eq. (11.76) we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right)$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 E = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

Using Eq. (11.77) we have

$$\nabla^2 E = \mu\varepsilon \frac{\partial^2 E}{\partial t^2} \qquad [\because \vec{\nabla} \cdot \vec{E} = 0]$$

Similarly taking curl of Eq. (11.77) we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times \left(\mu\varepsilon \frac{\partial \vec{E}}{\partial t} \right)$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 B = -\mu\varepsilon \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$$

Using Eq. (11.76) we have

$$\nabla^2 B = \mu\varepsilon \frac{\partial^2 B}{\partial t^2} \qquad [\because \vec{\nabla} \cdot \vec{B} = 0]$$

In vector form

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}, \qquad \nabla^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} \tag{11.78}$$

Equation (11.78) represents wave equations for ***E*** and ***B*** in dielectric medium.

11.13 Electromagnetic Waves in Conducting Medium

In conducting medium, the charge given to material is always lie at the surface and no charge stay inside the conducting material, hence charge density $\rho = 0$. So, for a conducting medium Maxwell's equations are as follows:

$$\vec{\nabla} \cdot \vec{D} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

In conducting medium $\vec{D} = \varepsilon \vec{E}$, $\vec{B} = \mu \vec{H}$, $\vec{J} = \sigma \vec{E}$, where σ represents the conductivity of the isotropic and homogeneous medium. Thus, Maxwell's equations reduced to

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{11.79}$$

$$\vec{\nabla} \cdot B = 0 \tag{11.80}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{11.81}$$

$$\vec{\nabla} \times \vec{B} = \mu\left(\sigma\ \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right) \tag{11.82}$$

To derive wave equation in conducting medium take the curl on both sides of Eq. (11.81), we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 E = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

Substituting the value of $\vec{\nabla} \times \vec{B}$ from Eq. (11.82) in the above equation, we get

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 E = -\mu \frac{\partial}{\partial t}\left(\sigma E + \varepsilon \frac{\partial E}{\partial t}\right)$$

Since $\vec{\nabla} \cdot \vec{B} = 0$ is from Maxwell's first equation, we have

$$-\nabla^2 E = -\mu\sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2}$$

or

$$\nabla^2 \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{11.83}$$

Similarly, we can obtain wave equation for ***B*** by taking curl of Eq. (11.82) and using Eq. (11.80) as

$$\nabla^2 B - \mu\sigma \frac{\partial B}{\partial t} - \mu\varepsilon \frac{\partial^2 B}{\partial t^2} = 0 \tag{11.84}$$

The above equations are wave equations in conducting medium. If we take $\sigma = 0$ and permeability and permittivity for free space, the above equations will be for free space. In conducting medium, the wave vector k is a complex and the real part of it determines the physical values of wave such as wavelength and speed of wave. The imaginary part of wave vector results in an attenuation of wave (decreasing amplitude of E and B with depth of penetration in medium). Here unlike in free space, the electric and magnetic field vectors are no longer in phase, rather magnetic field lags behind the electric field (Fig. 3).

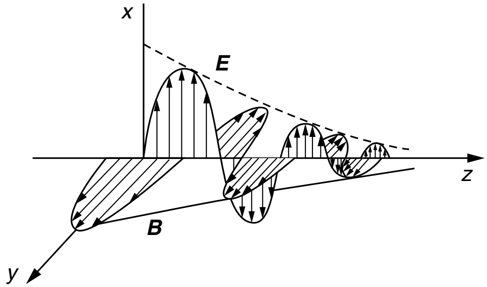

Figure 3 Phase-diagram of electric and magnetic field vectors.

11.14 Skin Depth

Skin depth is an essential parameter for the wave when the electromagnetic waves penetrate in conducting medium. *It is the depth in conducting medium in which the strength of electric field is reduced* $1/e$ *times of its original values*.

The skin depth is frequency dependent for good conductor and frequency independent for poor conductor. Consider the solution of wave equation (11.83) as

$$E(r,t) = E_o e^{j(k.r-\omega t)}$$

where k is complex and can be expressed with real and imaginary term α and β respectively as $k = \alpha + j\beta$. Now if wave is moving along z direction with E vector parallel to x, then E will be

$$E_x(z,t) = E_{ox} e^{j((\alpha + j\beta)z-\omega t)}$$

or

$$E_x(z,t) = E_{ox} e^{-\beta z} e^{j(\alpha z-\omega t)}$$

the attenuation factor is $e^{-\beta z}$. E_x should be $(1/e)$ times its original value if $\beta z = 1$. In this case z, the depth in the medium becomes skin depth and is denoted by δ as shown in Fig. 4. Hence,

$$z = \delta = \frac{1}{\beta} \tag{11.85}$$

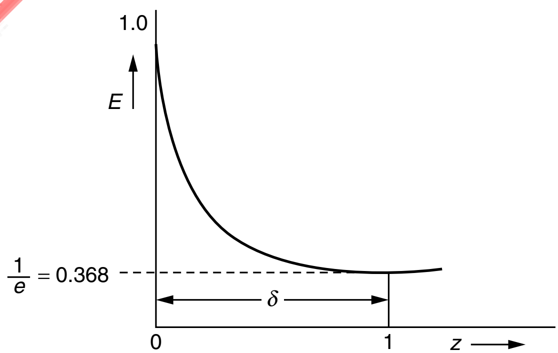

Figure 4 Skin depth.

The value of α and β can be obtained with wave equation

$$\nabla^2 \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

and its solution

$$E_x(z,t) = E_{ox} e^{j[(\alpha + j\beta)z - \omega t]}$$

which will be

$$\alpha = \omega \sqrt{\sqrt{\frac{\mu\varepsilon}{2}} \left[\left(1 + \frac{\sigma^2}{\omega^2 \varepsilon^2} \right)^{1/2} + 1 \right]}; \quad \beta = \omega \sqrt{\sqrt{\frac{\mu\varepsilon}{2}} \left[\left(1 + \frac{\sigma^2}{\omega^2 \varepsilon^2} \right)^{1/2} - 1 \right]} \tag{11.86}$$

For good conductor $(\sigma \gg \varepsilon\omega)$, Hence

$$\beta = \omega \sqrt{\frac{\mu\varepsilon}{2} \left(\frac{\sigma}{\omega\varepsilon} \right)} \quad \text{or} \quad \beta = \sqrt{\frac{\mu\sigma\omega}{2}}$$

Thus, skin depth

$$\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\mu\sigma\omega}} \tag{11.87}$$

In terms of frequency (f) skin depth will be

$$\delta = \sqrt{\frac{2}{2\pi f \mu\sigma}} = \sqrt{\frac{1}{\pi f \mu\sigma}} \tag{11.88}$$

From Eq. (11.88), we can conclude that *skin depth or penetration depth is inversely proportional to the root of frequency of wave.*

For poor conductor $(\sigma \ll \varepsilon\omega)$

$$\beta = \omega \sqrt{\sqrt{\frac{\mu\varepsilon}{2}} \left[\left(1 + \frac{1}{2} \frac{\sigma^2}{\omega^2 \varepsilon^2} \right) - 1 \right]}$$

or

$$\beta = \omega \sqrt{\frac{\mu\varepsilon}{2} \times \frac{1}{2} \frac{\sigma^2}{\omega^2 \varepsilon^2}}$$

or

$$\beta = \frac{\sigma}{2} \sqrt{\frac{\mu}{\varepsilon}}$$

Thus, skin depth

$$\delta = \frac{1}{\beta} = \frac{2}{\sigma} \sqrt{\frac{\varepsilon}{\mu}} \tag{11.89}$$

From the above equation, we can conclude that *skin depth or penetration depth is independent of frequency of wave.*

Solved Examples

Example 1

Prove that electromagnetic waves propagate with speed of light.

Solution: The wave equations for $\vec{E}$ and $\vec{B}$ in free space are as follows:

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}, \qquad \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

In vacuum, then, each Cartesian component of $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfies the three dimensional wave equation. Hence

$$\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}$$

So Maxwell's equations imply that empty space supports the propagation of electromagnetic waves, travelling at a speed

$$\mu_0 \varepsilon_0 = \frac{1}{v^2}$$

$$\Rightarrow v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{(4\pi \times 10^{-7} \text{ Weber/A-m})(8.85 \times 10^{-12} \text{ C}^2\text{N-m}^2)}} = 2.99 \times 10^8 \text{ m/s}$$

Hence, *electromagnetic waves propagate with the speed of light*:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

From this result we can conclude that light is an electromagnetic wave.

Example 2

Prove that the speed of light in a material is always less than that in vacuum.

Solution: We know that in vacuum material travels with velocity of light. In some other medium, velocity is given as

$$v = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{1}{\sqrt{(\mu_0 \mu_r)(\varepsilon_0 \varepsilon_r)}}$$

where μ_r and ε_r are relative permeability and relative permittivity, respectively. Since

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

therefore

$$v = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

or

$$v = \frac{c}{\sqrt{\varepsilon_r}} \qquad \text{[For non-magnetic material } \mu_r = 1]$$

The speed of light in a material is always less than that in vacuum because ε_r has a value greater than one.

Example 3

Determine refractive index and velocity of light if the relative permittivity of distilled water is 64.

Solution: The velocity of distilled water is given by

$$v = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

where $\mu_r = 1$, $c = 3 \times 10^8$ m/sec and $\varepsilon_r = 64$. Therefore

$$v = \frac{3\times10^8}{\sqrt{64}} = 3.75\times10^7 \text{ m/s}$$

As we know, the refractive index n of the medium is $n = c/v$. Therefore,

$$n = \sqrt{\varepsilon_r} = \sqrt{64} = 8$$

Example 4

A uniform plane wave having electric field intensity in air as 7×10^3 V/m in the y-direction is propagating in the x-direction at a frequency of 2×10^8 rad/sec. Determine the frequency, wavelength, time-period and amplitude of $\boldsymbol{H}$.

Solution: We have

$$E_y = 7 \times 10^3 \cos(2 \times 10^8 t - px)$$

Here $\omega = 2 \times 10^8$rad/sec, $\mu_0 = 4\pi \times 10^{-7}$ Weber/A-m, $\varepsilon_0 = 8.85\times10^{-12}$ C^{2}N-m^2. Now frequency is given by

$$\upsilon = \frac{w}{2\pi} = \frac{2\times10^8}{2\times3.14} = 318.5 \times 10^5 \text{ Hz} = 3.18\times10^7 \text{ Hz}$$

Wavelength is given by

$$\lambda = \frac{v}{\upsilon} = \frac{3\times10^8}{3.18\times10^7} = 9.43 \text{ m}$$

Time period is given by

$$\lambda = \frac{1}{\upsilon} = \frac{1}{3.18\times10^{7}} = 3.14\times10^{-8}\ \text{sec}$$

Amplitude of ***H*** is

$$\frac{E}{H} = \frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.77 \approx 377\ \Omega$$

$$\Rightarrow H = \frac{7\times10^{3}}{377} = 18.56\ \text{A/m}$$

Therefore

$$H_z = 18.56 \cos(2\times10^{8}\, t - px)$$

Example 5

If the magnitude of ***E*** in a plane wave is 377 V/m, determine the magnitude of ***H*** for a plane wave in free space.

Solution: We have

$$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ \Omega \ \Rightarrow H = \frac{377}{377} = 1\ \text{A/m}$$

Example 6

A parallel-plate capacitor with circular plates of radius $a = 0.055$ m is being charged at a uniform rate so that the electric field between the plates changes at a constant rate

$$\frac{\partial\vec{E}}{\partial t} = 1.5\times10^{13}\ \text{V/m/s}$$

Determine the displacement current for the capacitor.

Solution: The displacement current density between the plates of the capacitor is

$$\vec{J}_D = \frac{\partial\vec{D}}{\partial t} = \varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Displacement current

$$\vec{I}_D = (\pi a^2)\,\vec{J}_D = \pi a^2\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Here

$$\frac{\partial\vec{E}}{\partial t} = 1.5\times10^{13}\ \text{V/m/s},\ a = 0.055\ \text{m and}\ \varepsilon_0 = 8.85\times10^{-12}\ \text{C}^2\text{N-m}^2$$

Displacement current

$$\vec{I}_D = \pi a^2\varepsilon_0\frac{\partial\vec{E}}{\partial t} = 1.3\ \text{A}$$

Example 7

A lamp radiates 500 W power uniformly in all directions. Calculate the electric and magnetic field intensities at 1 m distance from the lamp.

Solution: As we know Poynting vector $\vec{P} = \vec{E} \times \vec{H}$ is the energy flowing through unit area and unit time. Now

$$\text{Area} = 4\pi r^2 = 4\pi(1)^2 = 4\pi \text{ m}^2$$

Now

$$\vec{P} = \frac{500}{4\pi} \text{ Joule/m}^2\text{/sec}$$

or

$$EH = \frac{500}{4\pi}$$

But we know that

$$\frac{E}{H} = \frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.77 \approx 377\ \Omega$$

So $E = 377\ H$. Multiplying both sides by H and using the value of EH we get

$$377 \times H^2 = \frac{500}{4\pi} \Rightarrow H^2 = \frac{500}{4\pi \times 377}$$

$$\Rightarrow H^2 = 0.105$$

$$\Rightarrow H = 0.33 \text{ A-turn/m}$$

Now

$$EH = \frac{500}{4\pi} \Rightarrow E = \frac{500}{4\pi \times 0.33} = 120.63 \text{ V/m}$$

Example 8

Earth receives 2 calories of solar energy per minute per cm^2 as an average over a year for whole surface. What are the amplitudes of average electric and magnetic field radiation?

Solution: The energy received by an electromagnetic power flow is given by

$$\vec{P} = \vec{E} \times \vec{H}$$

$$\Rightarrow \vec{P} = \frac{2 \times 4.2 \times 10^4}{60} = 1400 \text{ Joule/m}^2\text{/s}$$

Now $P = EH$. So $EH = 1400$. But we know that

$$\frac{E}{H} = \frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.77 \approx 377\ \Omega$$

So

$$E \times \frac{E}{377} = 1400 \Rightarrow E^2 = 527240 \Rightarrow E_{avg} = 726.1 \text{ A-turn/m}$$

Similarly H_{avg} can be calculated as

$$H_{avg} = \frac{1400}{E}$$

$$\Rightarrow H_{avg} = 1.928 \text{ A-turn/m}$$

The amplitudes are calculated using the following expression:

$$E_0 = E_{avg}\sqrt{2} = 1.414 \times 726.1 = 1026.7 \text{ A-turn/m}$$

Similarly $H_0 = 2.726$ A-turn/m.

Example 9

Calculate the skin depth for a frequency of 10^{20} Hz for silver if $\mu_0 = 4\pi \times 10^{-7}$ Weber/A-m, $\sigma = 3 \times 10^7$ S/m.

Solution: We know that

$$\delta = \frac{1}{k} = \sqrt{\frac{2}{\mu\sigma\omega}}$$

Given that $\mu_0 = 4\pi \times 10^{-7}$ Weber/A-m, $\sigma = 3 \times 10^7$ S/m and $\omega = 2\pi f = 10^{20}$. So

$$\delta = \frac{1}{k} = \sqrt{\frac{2}{4\pi \times 10^{-7} \times 3 \times 10^7 \times 2\pi \times 10^{20}}} = 0.091 \times 10^{-10} \text{ m}$$

Short Answers of Some Important Questions

1. How was the idea of electromagnetic waves conceived?

Answer: Faraday's law suggests that a time-varying magnetic field produces an electric field while Ampere's law shows that a time-varying electric field produces a magnetic field. Using this fact, Maxwell showed that if either of the electric or magnetic field changes with time, a field of another kind is induced in the adjacent space and produces waves which are called electromagnetic waves.

2. What do you understand by electromagnetic waves?

Answer: Electromagnetic waves consist of changing electric and magnetic fields. The electric and magnetic components of plane electromagnetic wave are perpendicular to each other and also perpendicular to the direction of the propagation. These waves propagate in space from one position to another even in absence of material medium.

3. Write down some properties of electromagnetic waves.

Answer: The properties of electromagnetic waves travelling through free space are as follows:

1. Electromagnetic waves travel with the speed of light.
2. Electromagnetic waves are transverse waves.
3. The ratio of electric to magnetic field in an electromagnetic wave equals the speed of light.
4. Electromagnetic waves carry both energy and momentum.

4. Give some examples of electromagnetic waves.

Answer: Radio waves, light, X-rays, γ-rays, etc. are the examples of electromagnetic waves.

5. What is displacement current?

Answer: The rate of change of electric displacement vector with time is known as displacement current. In other words, one can say that the displacement current is the current arising due to time-varying electric field between the plates of the capacitor.

6. What is the role of displacement current in electromagnetics?

Answer: On the basis of displacement current, the symmetry character of electric field and magnetic field is more prominent. With the introduction of current density, a changing electric field is now seen to produce magnetic field just as a changing magnetic field gives rise to electric field. Thus, higher degree of symmetry of electric and magnetic field is more satisfactory. Also on the basis of displacement current, both steady and non-steady current circuits may be analyzed as well as all the variations in AC circuits with a capacitor can be easily understood.

7. Differentiate between conduction current and displacement current.

Answer:

1. Conduction current is due to the actual flow of current in a conductor while displacement current is the result of time-varying electric field in a dielectric.
2. Conduction current density is the product of electrical conductivity and electric field; however, displacement current density is the rate of change of electric displacement vector with time.
3. Conduction current obeys Ohm's law while displacement does not obey Ohm's law.

8. Write down Maxwell's equations in dielectric media.

Answer: In dielectric medium there is no free charge. So $\sigma = 0$, $J = 0$ and $\rho = 0$. Therefore, Maxwell's equations are as follows:

$$\nabla \cdot \vec{D} = 0;\ \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t};\ \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

9. Show that $\vec{E}/\vec{B} = c$, where c is the velocity of electromagnetic wave.

Answer: We know that

$$\vec{k} \times \vec{E} = w\vec{B}$$

Since we have already discussed that electric field vector is perpendicular to the direction of propagation, so

$$\vec{k} \cdot \vec{E} = w\vec{B}$$

or $$\frac{\vec{E}}{\vec{B}} = \frac{w}{\vec{k}} = \frac{2\pi\nu}{2\pi/\lambda} = \nu\lambda$$

or $$\frac{\vec{E}}{\vec{B}} = c \ [\because c = \nu\lambda]$$

10. What is Poynting vector?

Answer: $\vec{P} = \vec{E} \times \vec{H}$ is the energy flowing through unit area and unit time and is known as the Poynting vector. It is also called the flux vector. The SI unit of Poynting vector is Wm^{-2}.

Important Points and Formulas

1. With the change in electric and magnetic field with time, a field of other kind is induced in the adjacent space which produces electromagnetic waves consisting electric and magnetic fields.
2. Maxwell formulated the concept of displacement current to remove the inconsistency in Ampere's law by adding the term $\vec{J}_D = \frac{\partial \vec{D}}{\partial t}$.
3. The current arising due to time-varying electric field between the plates of a capacitor is called the displacement current.

$$\vec{J}_d = \varepsilon_0 \frac{dE}{dt}$$

4. The equation of continuity is based on the conservation of charge.
5. Electromagnetic waves propagate with the speed of light

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

6. The speed of light in a material is always less than in vacuum because ε_r has a value greater than one.
7. According to Poynting theorem, the rate at which electromagnetic energy in a finite volume decreases with time is equal to the rate of dissipation of energy in the form of joule heat plus the rate at which energy flows out of the volume.
8. $\vec{P} = \vec{E} \times \vec{H}$ is the energy flowing through unit area and unit time and is known as the Poynting vector. It is also called the flux vector. The SI unit of Poynting vector is Wm^{-2}.
9. Skin depth is the depth in conducting medium in which the amplitude of the electromagnetic wave is reduced $(1/e)$ times of its original value.
10. The skin depth is frequency dependent for good conductor and frequency independent in poor conductor.

Multiple Choice Questions

1. Displacement current is due to
   (a) displacement of electric charges
   (b) time varying magnetic field
   (c) time varying electric field
   (d) Both (b) and (c)
2. Equation of continuity is based on
   (a) conservation of charges
   (b) conservation of momentum
   (c) conservation of angular momentum
   (d) None of these
3. Time varying electric field in the region between the plates is equivalent to
   (a) conduction current
   (b) displacement current
   (c) Both (a) and (b)
   (d) Neither (a) nor (b)
4. Who observed that a time varying magnetic field gave rise to an electric field.
   (a) Maxwell **(b)** Ampere
   (c) Oersted **(d)** Faraday
5. Poynting theorem represents
   (a) conservation of charges
   (b) conservation of momentum
   (c) conservation of energy
   (d) None of these
6. Maxwell observed and corrected a discrepancy in
   (a) Ampere's Law
   (b) Faraday's Law
   (c) Gauss Law for electrostatics
   (d) None of these
7. According to maxwell's equation in free space; $\nabla \cdot E = ?$
   (a) ρ **(b)** 0
   (c) ρ/ε_o **(d)** $\frac{\varepsilon_0}{\rho}$
8. In a conducting medium, the electromagnetic waves are.
   (a) amplified **(b)** attenuated
   (c) both (a) & (b) **(d)** None of the these

9. Energy density in electric and magnetic field is
(a) Different **(b)** 1.5
(c) L/C **(d)** Same

10. The wave velocity in non-conducting medium is
(a) $\frac{1}{\sqrt{\mu\varepsilon}}$
(b) $\sqrt{\mu/\varepsilon}$
(c) $\sqrt{\frac{\mu_0}{\varepsilon_0}}$
(d) $\frac{1}{\sqrt{\mu_0\varepsilon_0}}$

11. The characteristic impedance of free space is
(a) 0 **(b)** 1
(c) 377 **(d)** None of these

Short Answer Type Questions

1. What do you understand by electromagnetic waves?
2. What are Maxwell's equations?
3. What do you mean by displacement current?
4. Differentiate between conduction current and displacement current.
5. What is current density?
6. Write down Maxwell's equations for free space.
7. What is Poynting vector?
8. What do you understand by impedance?

Long Answer Type Questions

1. Explain the concept of Maxwell's displacement current and show how it led to the modification of Ampere's law.
2. Derive Maxwell's equations. Explain the physical significance of each equation.
3. Derive the electromagnetic wave equations in vacuum. Hence show that the waves travel at a speed of light.
4. Derive Poynting theorem. Explain each term.
5. Prove that electromagnetic waves propagate with speed of light.

Numerical Problems

1. Determine refractive index and velocity of light if the relative permittivity of distilled water is 81.
2. A uniform plane wave has electric field intensity in air as 7500 V/m in the *y*-direction. The wave is propagating in the *x*-direction at a frequency of 2×10^9 rad/s. Determine the frequency, wavelength, time-period and amplitude of ***H***.
3. If the magnitude of ***E*** in a plane wave is 455 V/m, determine the magnitude of ***H*** for a plane wave in free space.
4. A parallel-plate capacitor with circular plates of radius a = 0.55 cm is being charged at a uniform rate so that the electric field between the plates changes at a constant rate $\frac{\partial \vec{E}}{\partial t} = 1.5 \times 10^{13}$ V/m/s. Determine the displacement current for the capacitor.
5. A lamp radiates 400 W power uniformly in all directions. Calculate the electric and magnetic field intensities at 1.5 m distance from the lamp.

Answers

Multiple Choice Questions

1. (c)
2. (a)
3. (b)
4. (d)
5. (c)
6. (a)
7. (b)
8. (b)
9. (d)
10. (a)
11. (c)

Numerical Problems

1. 9, 3.33×10^{7} m/s

2. 3.18×10^{8} Hz, 0.94 m, 3.14×10^{-9} s and 19.89 A/m

3. 1.21 A/m

4. 1.3×10^{-2} A

5. 796.18 V/m and 0.04 A-turn/m