Name of Paper	:	Mechanics
Name of the Course	:	B.Sc. (Prog.) Physics
Semester	:	I
Duration	:	3 Hours
Maximum Marks	:	75

Nov. / Dec. 2019

Attempt five questions in all.

Questions No. 1 is compulsory

Q. 1. For the vectors $A = 3i - 2j + k$ and $B = 2i - k$, determine A . B and $(A \times B) - B$. (2½ + 2½)

(*b*) Prove $(A \times B) \times C = (A \cdot C)\, B - (B \cdot C) \cdot A$. 5

(*c*) Solve the differential equation :

$$(x^2 - y^2)dy - 2xy\,dx = 0.$$

Ans. (*a*)

$$A.B = (3\hat{i} - 2\hat{j} + \hat{k}) \cdot (2\hat{i} - \hat{k})$$

$$= 6 - 1 = 5$$

$$(A \times B) \cdot A = ?$$

$$A \times B = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ 2 & 0 & -1 \end{vmatrix} = 2\hat{i} + 5\hat{j} + 4\hat{k}$$

$$(\vec{A} \times \vec{B}) \cdot \vec{A} = 0.$$

(*b*) Let three vectors, such as $A = A_1 i + A_2 j + A_3 k$, similarly other. Solve them for right and left hand side and prove them to be equal.

(*c*)
$$(x^2 - y^2)dy - 2xydx = 0$$

$$M\,dx + N\,dy = 0$$

$$M = -2xy$$

$$N = x^2 - y^2$$

To find integrating factor (IF) = $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$$= \frac{1}{-2xy}\left(2x - (-2x)\right) = \frac{-2}{y}$$

$$\text{I.F.} = e^{\int f(x) dy} = \frac{1}{y^2}$$

On multiplying If, the equation becomes –

$$\left(\frac{x^2}{y^2} - 1\right) dy - \frac{2x}{y} dx = 0$$

Now, $$M' = -\frac{2x}{y} \qquad N' = \frac{x^2}{y^2} - 1$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$ [Now this equation is exact]

$$\Psi = \int M' dx = \frac{-x^2}{y} + f(y)$$

$$\frac{\partial \Psi}{\partial y} = \frac{x^2}{y^2} + f'(y)$$

$$f'(y) = -1$$

$$f(y) = -y + C$$

Then, $$\Psi(x, y) = \frac{-x^2}{y} - y + C = k$$

$$x^2 + y^2 = k'y$$

Q. 2. (*a*) State Kepler's laws of planetary motion. (1 + 1 + 1)

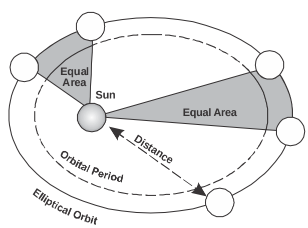
(*b*) What is a central force ? Give examples of central forces. Prove that under the influence of a central force, the motion of a particle is always confined to a plane. (2 + 1 + 4)

(*c*) A satellite revolves around a planet of mean density 104 kg/m3. If the radius of its orbit is only slightly greater than the radius of the planet, find the time of revolution of the satellite. [$G = 6.67 \times 10^{-11}$ S.I. units] (3 + 2)

Ans. (*a*) (*i*) The orbit of a planet is an ellipse with the Sun at one of two foci.

(*ii*) A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.

(*iii*) The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.



(*b*) Central force is the force that is radially pointing and the magnitude is dependent on the distance from the source. The examples of central forces are gravitational force, electrostatic forces and spring force.

Gravitational force is an example of central forces. Mathematically if we consider central point as origin

$$\vec{f}(r) = f(r)\,(\pm\hat{r}) \qquad \ldots (1)$$

where + stands for repulsion force and – for attractive force, $f(r)$ is the magnitude of the central forces and $\hat{r}$ is unit vector in the direction of central forces.

Multiplying equation (1) by $\vec{r}$ on both the sides we get

$$\vec{r} \times \vec{f}(r) = f(r)\,\{\vec{r} \times \hat{r}\} \qquad \ldots (2)$$

This gives

$$f(r) = 0 \qquad \ldots (3)$$

Since $\vec{r} \times \hat{r} = 0$

but we know that

$$\vec{f}(r) = m\frac{d^2\vec{r}}{dt^2} \qquad \ldots (4)$$

therefore,

$$\vec{r} \times m\frac{d^2\vec{r}}{dt^2} = 0 \qquad \ldots (5)$$

or,

$$\vec{r} \times \frac{d^2\vec{r}}{dt^2} = 0 \qquad \ldots (6)$$

now,

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \vec{r} \times \frac{d^2\vec{r}}{dt^2} + 0 \qquad \ldots (7)$$

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = 0 \qquad \ldots (8)$$

$$\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \text{cons tan}\,t\ (\vec{h}) \qquad \ldots (9)$$

where $\vec{h}$ is a vector which does not depend on time and is perpandicular to the plane formed by the position vector $\vec{r}$ and velocity $\frac{d\vec{r}}{dt}$. Thus the plane formed by $\vec{r}$ and velocity $\frac{d\vec{r}}{dt}$ will also remains constant. Therefore, the particle will always move in the same plane.

(*c*)
$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

and
$$\frac{GM}{r} = r^2\omega^2 \quad \text{as} \quad v = r\omega$$

$$G\delta\left(\frac{4}{3}\pi R^3\right) = r^3\omega^2$$

when R ~ *r* then

$$\frac{4}{3}\pi G\delta = \left(\frac{2\pi}{T}\right)^3$$

$$T = \sqrt{\frac{3\pi}{\delta G}}$$

$$= \sqrt{\frac{3 \times 3.14}{10^4 \times 6.67 \times 10^{-11}}} = 3758 \text{ sec.}$$

Q. 3. (*a*) What do you understand by the centre of mass of a system of particles? Show that in the absence of external forces the velocity of the centre of mass remains constant. (2 + 3)

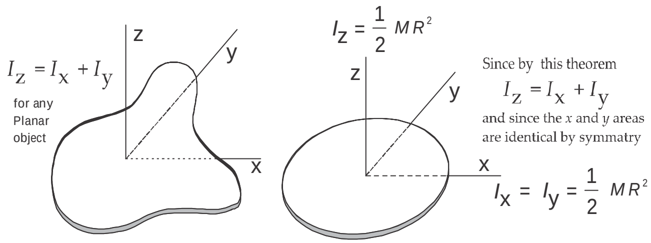
(*b*) What is moment of inertia? State parallel and perpendicular axis theorems. (2 + 1½ + 1½)

(*c*) The angular momentum of a rotating body is conserved, while its moment of inertia is decreased. Show that its rotational kinetic energy increases. (2 + 3)

Ans. (*a*) Please refer your text book for its solution.

(*b*) Moment of inertia is the name given to rotational inertia, the rotational analog of mass for linear motion. It appears in the relationships for the dynamics of rotational motion. For a point mass, the moment of inertia is just the mass times the square of perpendicular distance to the rotation axis, I = mr^2.

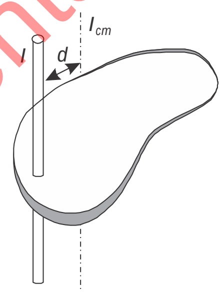
Perpendicular Axis Theorem : For a planar object, the moment of inertia about an axis perpendicular to the plane is the sum of the moments of inertia of two perpendicular axes through the same point in the plane of the object.



Parallel Axis Theorem : The moment of inertia of my object about an axis through its centre of mass is the minimum moment of inertia for an axis in the direction in space. The moment of inertia about any axis parallel to that axis through the center of mass is given by

$$I_{\text{parallel axis}} = I_{cm} + Md^2$$

The expression added to the center of mass moment of inertia will be recognized as the moment of inertia of a point mass the moment of inertia about a parallel axis is the center of mass moment plus the moment of inertia of the entire treated as a point mass at the center of mass.



(*c*)

$$I_1 w_1 = I_2 w_2$$

Here as I increases w decreases and K.E. = $\frac{1}{2} I w^2$ also decreases.

Q. 4. (*a*) State and prove work-energy theorem.

(*b*) What are conservative and non-conservative forces ? Show that work done by a conservative force along a closed path is zero. (2 + 3)

(*c*) Establish the equation of motion of a rocket and obtain the velocity of the rocket at time *t* taking into account the effect of gravity. (2 + 3)

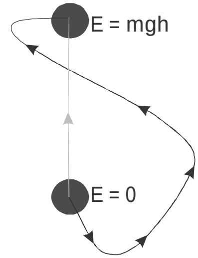
Ans. (*a*) Use your text book to solve this.

(*b*) Conservative Forces : Conservative forces are those for which work done depends only on initial and final points for example, Gravitational Force, Electrostatic Forces.

Non-Convervative Forces : Non-convervative forces are those where the work done or the kinetic energy did depend on other factors such as the velocity or the particular path taken by the object. For example, Frictional Force.

A direct consequence of the closed path test is that the work done by a conservative force on a particle moving between any two points does not depend on the path taken by the particle.

This is illustrated in the figure to the right. The work done by the gravitational force on an object depends only on its change in height because the gravitational force is conservative. The work done by a conservative force is equal to the negative of change in potential energy during that process. For a proof, imagine two paths 1 and 2, both going from point A to point B. The variation of energy for the particle, taking path 1 from A to B and then path 2 backwards from B to A, is 0; thus, the work is the same path 1 and 2 *i.e.*, the work is independent of the path followed, as long as it goes from A to B.



(*c*) Rocket motion is based on Newton's third law, which states that "for every action there is an equal and opposite reaction". Hot gases are exhausted through a nozzel of the rocket and produce the action force. The reaction force acting in the opposite direction is called the thrust force. The thrust force just causes the rocket acceleration.

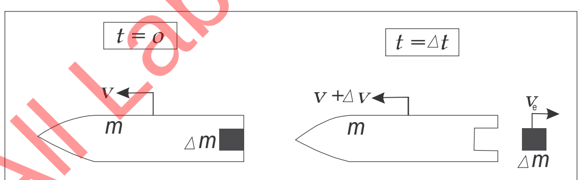
Let the initial mass of the rocket be m and its initial velocity be v. In certaintime dt, the mass of the rocket decreases by dm as a result of the fuel combustion. This leads the rocket velocity to be increased by dv. We apply the law of conservation of momentum to the system of the rocket and gas flow. At the initial moment the momentum of the system is equal to mv. In a small time dt the momentum of the rocket becomes

$$p_1 = (m - dm)(v + dv)$$

and the momentum of the exhaust gases in the Earth's coordinate system is

$$p_2 = dm\,(v - u)$$

where u is the exhaust gas velocity with respect to the rocket. Here we took into account that the exhaust velocity is in the opposite direction to the rocket movement. Therefore, we have put the minus sign in front of u.



By the law of conservation of the total momentum of the system, we can write:

$$p = p_1 + p_2, \quad \text{Þ} \quad mv = (m - dm)(v + dv) + dm\,(v - u).$$

By transforming the given equation, we obtain :

$$\cancel{mv} = \cancel{mv} - \cancel{vdm} + mdv - dmdv + \cancel{vdm} - udm.$$

We can neglect the term $dmdv$ in the last expression considering small increments of these values. As a result, the equation is written as

$$mdv = udm$$

We divide both sides by dt to convert the equation into the form of Newton's second law :

$$m\frac{dv}{dt} = u\frac{dm}{dt}$$

The given equation is called the differential equation of rocket motion. The right side of the equation represents the thrust force T :

$$T = u\frac{dm}{dt}$$

As it can be seen from the last formula, the thrust force is proportional to the exhaust velocity and the fuel burn rate.

Of course, the differential equation we derived describes an ideals case. It does not take into account the gravitational force or aerodynamic force. Their inclusion leads to significant complication of the differential equation.

If we integrate the differential equation, we can get the dependence of the rocket velocity on the burned fuel mass. The resulting formula is called the ideal equation or Tsiolkovsky rocket equation who derived it in 1897.

To get the this formula it's convenient to use the differential equation in the form :

$$mdv = udm$$

Separating the variables and integrating gives :

$$dv = u\frac{dm}{m}, \Rightarrow \int_{v_0}^{v_1} dv = \int_{m_0}^{m_1} u\frac{dm}{m}.$$

Take into account that *dm* denotes mass decrease. Therefore we take the increment *dm* with the negative sign. As a result, the equation is written as follows :

$$v\Big|_{v_0}^{v_1} = -u(\ln m)\Big|_{m_0}^{m_1}, \Rightarrow \quad v_1 - v_0 = u \ln \frac{m_0}{m_1}.$$

where v_0 and v_1 are the initial and final velocities of the rocket, m_0 and m_1 are the initial and final masses of the rocket, respectively.

By setting $v_0 = 0$, we obtain the formula derived by Tsiolkovsky :

$$v = u \ln \frac{m_0}{m}.$$

This formula determines the rocket velocity depending on its mass change while the fuel is burning. It allow rough estimation of the fuel capacity necessary to accelerate the rocket to a given velocity.

Q. 5. (*a*) Define kinetic energy of rotation. Develop an expression of kinetic energy involving both translation and rotation. (2 + 4 + 4)

(*b*) A torque of 1 Nm is applied to a wheel of mass 10 kg and radius of gyration 50 cm. What is the resulting translational acceleration ? (2 + 3)

Ans. (*a*) Find the solution to this problem in your text book.

(*b*) $t = I a$

$I = M r^2$

$$\alpha = \frac{\tau}{Mr^2} = \frac{1Nm}{10 \times (0.5)^2}$$

$$= 0.4 \text{ rad / sec}^2$$

$$a = r\alpha$$

$$= 0.5 \times 0.4 = 0.2 \text{ m/sec}^2.$$

Q. 6. (*a*) What do you understand by simple harmonic motion ? Set up the differential equation of motion for a simple harmonic motion and obtain solution. Find the expression for time period and angular frequency. (2 + 6 + 2)

(*b*) At what displacement the kinetic and potential energies are equal ? (2 + 3)

Ans. (*a*) The solution to this question is in your text book.

(*b*) $$\text{K.E. of an oscillator} = \frac{1}{2} m\omega^2 (a^2 - x^2)$$

$$\text{P.E. of an oscillator} = \frac{1}{2} m\omega^2 x^2$$

When $$\frac{1}{2} m\omega^2 (a^2 - x^2) = \frac{1}{2} m\omega^2 x^2$$

$$(a^2 - x^2) = x^2 \quad \text{then} \quad \frac{1}{\sqrt{2}} a = x$$

Q. 7. (*a*) Differentiate between inertial and non-inertial frames. (1 + 1)

(*b*) State Einstein's postulates of special theory of relativity. Derive the Lorentz transformation equations. (2 + 6)

(*c*) A rod 1m long is moving along its length with a velocity 0.6c. Calculate its length as it appears to an observer on the earth. (2 + 3)

Ans. (*a*) Inertial References Frames : Interial reference frames in which Newton first law of motion holds *i.e.*, an object at rest and an object in motion remains motion unless acted by a net force. An intertial reference frame is either at rest or moves with a constant velocity.

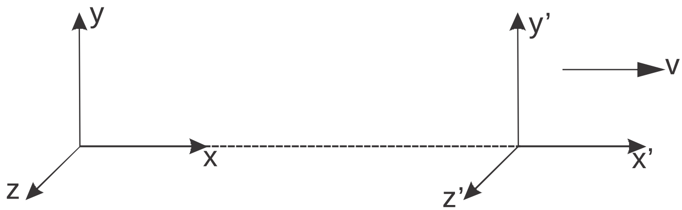
Non-intertial Reference Frames : Non-intertial refence frame is a reference frame that is accelerating either in linear fashion or rotating around some axis.

Examples : inertial reference frames – A train moving with constant velocity.

non-intertial reference frames – A turning car with constant speed.

(*b*) The first postulate of special relativity is the idea that the laws of physics are the same and can be stated in their simplest from in all inertial frames of reference. The second postulate of special relativity is the ideas that the speed of light *c* is a constant, independent of the relative motion of the source.

Consider two coordinate systems (x, y, z, t) and (x', y', z', t') that coincide at $t = t' = 0$. The unprimed system is stationary and the primed system moves to the right the x–direction with speed v.



At time $t = t' = 0$, an isotropic light pulse is generated at

$$x = x' = 0,\ y = y' = 0,\ z = z' = 0$$

Since, the speed of light is the same (= c) in both systems, the wave front will satisfy both

$$x^2 + y^2 + z^2 = (ct)^2$$
$$x'^2 + y'^2 + z'^2 = (ct')^2.$$

If we substitute Galilcan transformation $x' = x - vt;\ y' = y;\ z' = z;\ t' = t$ into eq. 2, we obtain:

$$x^2 + y^2 + z^2 \underbrace{-2xvt + v^2t^2}_{\text{not compatible with Eq. 1}} = (ct)^2.$$

The Galilean transformation does not work. Note that the unwanted terms above involve both space and time. If we stick with the reasonable assumption that $y = y'$; $z = z'$ (implying that there is no effect perpendicular to the relative motion), then the only way we can avoid an unwanted terms such as ($-2xvt + v^2t^2$) above is to assume that t' is a function of both x and t.

The questions arises then how (x, y, z, t) and (x', y', z', t') are related. If both coordinate systems are inertial (that is, no relative acceleration), then a particle moving along a staight line in the other. Otherwise we would introduce spurious forces into the system with a curved trajectory. Hence we require linear transformations.

The most general, linear transformation between (x, t) and (x', t') can be written as :

$$x' = a_1 x + a_2 t$$
$$t' = b_1 x + b_2 t,$$

where a_1, a_2, b_1, b_2 are constants that can only depend on v, the velocity between the coordinate systems, and on c.

Before substituting Equation 3 and 4 into Equation 2 we note that the origin of the primed frame ($x' = 0$) is a point that moves with speed v as seen in the unprimed frame. its location in the unprimed frame is given by $x = vt$.

So equation 3 must satisfy:

$$0 = a_1 x + a_2 t, \qquad x = -\frac{a_2}{a_1} t = vt \qquad \text{or} \qquad \frac{a_2}{a_1} = -v.$$

Let's now rewrite Equation 3.

$$x' = a_1 \left(x + \frac{a_2}{a_1} t \right) = a_1 (x - vt)$$

Then we have eliminated a_2. Let's now substitute equation 3 and 4 into equation 2, then we obtain

$$[a_1 (x - vt)]^2 + y^2 + z^2 = c^2 [b_1 x + b_2 t]^2.$$

Expanding and rearranging gives :

$$\left(a_1^2 - c^2 b_1^2\right) x^2 + y^2 + z^2 = \left(c^2 b_2^2 - a_1^2 v^2\right) t^2 + \left(2 b_1 b_2 c^2 + 2 a_1^2 v\right) xt$$

The only way this is consistent with $x^2 + y^2 + z^2 = c^2 t^2$ (Eq. 1), is if :

$$a_1^2 - c^2 b_1^2 = 1, \qquad \text{... (5)}$$

$$c^2 b_2^2 - a_1^2 v^2 = c^2, \qquad \text{... (6)}$$

$$2 b_1 b_2 c^2 + 2 a_1^2 v = 0, \qquad \text{... (7)}$$

We can solve these three equations for the three unknowns a_1, b_1, b_2. A little bit of algebra gives :

From equation 5 and 6, we get

$$b_1^2 = \frac{a_1^2 - 1}{c^2}$$

$$b_2^2 = 1 + \frac{v^2}{c^2} a_1^2,$$

Squaring Equation 7 and substituting eqs. 5 and 6 gives :

$$b_1^2 b_2^2 c^4 = a_1^4 v^2 \rightarrow \left(\frac{a_1^2 - 1}{c^2} \right) \left(1 + \frac{v^2}{c^2} a_1^2 \right) c^4 = a_1^4 v^2$$

or

$$\left(a_1^2-1\right)\left(c^2+v^2a_1^2\right)=a_1^4v^2 \rightarrow a_1^2c^2-c^2-v^2a_1^2=0$$

and

$$a_1=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\equiv\gamma$$

we chose $+\sqrt{\ }$ since as $v\rightarrow 0$ $x'=x$, and $x'=-x$.

Substituting $a_1=\gamma$ into equation 5 and 6, gives.

$$\frac{1}{1-\frac{v^2}{c^2}}-c^2b_1^2=1 \qquad \text{or} \qquad b_1^2=\frac{v^2}{c^2}\left(\frac{1}{c^2-v^2}\right)=\frac{v^2}{c^4}\gamma^2$$

$$c^2b_2^2\,\frac{1}{1-\frac{v^2}{c^2}}v^2=c^2 \qquad \text{or} \qquad b_2^2=\frac{1}{1-\frac{v^2}{c^2}}=\gamma^2$$

We chose $b_2=\sqrt{\gamma^2}=+\gamma$, since $t'=t$ and not $t'=-t$ when $v\rightarrow 0$,

and $b_1=-\frac{v}{c^2}\gamma$, where the minus sign is required by equation 7 :

$$b_1=-\frac{a_1^2v}{b_2c^2}=-\frac{v}{c^2}\gamma$$

So we summarize our new (Lorentz) transformations:

$$x'=\gamma(x-vt)$$
$$y'=y$$
$$z'=z$$
$$t'=\gamma\left(t-\frac{v}{c^2}x\right)$$

Inverting prined and unprined coordinates ($v\rightarrow -v$) gives :

$$x=\gamma(x'-vt')$$
$$y=y'$$
$$z=z'$$

$$t = \gamma\left(t' - \frac{v}{c^2}x'\right)$$

(*c*)

$$L = L_0\left(1 - \frac{v^2}{c^2}\right)$$

$$L0 = 1\text{m}$$

$$L = 1\sqrt{\left(1 - \frac{(0.6c)^2}{c^2}\right)}$$

$$= \sqrt{1 - 0.36} = 0.8\text{m}$$