Name of Paper	**:**	**Mathematical Physics-II**
Name of the Course	**:**	**B.Sc. (Hons.) Physics**
Semester	**:**	**III**
Duration	**:**	**3 Hours**
Maximum Marks	**:**	**75**

Dec. 2019

Attempt any five questions in all.

SECTION – A

Q. 1. (*a*) **Write a general expression for the Fourier series of a function $f(x)$, such that $f(x) = f(x + 2L)$, $-L < x < L$. Which terms will be missing if $f(x)$ is an even function ? Justify mathematically.** **(6)**

Ans. The solution to this question is in your textbook.

OR

Evaluate : $\int_{-L}^{L} \cos \frac{p\pi x}{L} \cos \frac{q\pi x}{L} \, dx$ **for :** **(6)**

(*i*) $p = q \neq 0$

(*ii*) $p \neq q$.

Ans. $\int_{-L}^{L} \cos \frac{p\pi x}{L} \cos \frac{q\pi x}{L} dx$

Using Identify

$$\cos A \cos B = \frac{1}{2}\left[\cos(A + B) + \cos(A - B)\right]$$

$$\Rightarrow \quad \frac{1}{2}\int_{-L}^{L}\left[\cos\left(\frac{p\pi x}{L} + \frac{2\pi x}{L}\right) + \cos\left(\frac{p\pi x}{L} - \frac{2\pi x}{L}\right)\right]dx$$

$$\Rightarrow \quad \frac{1}{2}\int_{-L}^{L}\left[\cos\frac{\pi}{L}(p + q)x + \cos\frac{\pi}{L}(p - q)x\right]dx$$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{\sin\frac{\pi}{L}(p + q)x}{\frac{\pi}{L}(p + q)} + \frac{\sin\frac{\pi}{L}(p - q)x}{\frac{\pi}{L}(p - q)}\right]_{-L}^{L}$$

$$\Rightarrow \quad \frac{L}{2\pi}\left[\left(\frac{\sin\frac{\pi}{L}(p + q)L}{p + q} + \frac{\sin\frac{\pi}{L}(p - q)L}{(p - q)}\right)\right.$$

$$-\left(\frac{\sin\frac{\pi}{L}(p+q)(-L)}{p+q}+\frac{\sin\frac{\pi}{L}(p-q)(-L)}{(p-q)}\right)\Bigg]$$

$$\Rightarrow \frac{L}{2\pi}\left[\left(\frac{\sin(p+q)\pi}{p+q}+\frac{\sin(p-q)\pi}{p-q}\right)-\left(\frac{\sin(p+q)(-\pi)}{p+q}+\frac{\sin(p-q)(-\pi)}{p-q}\right)\right]$$

Using $\sin(-\theta) = -\sin\theta$

$$\Rightarrow \frac{L}{2\pi}\left[\left(\frac{\sin(p+q)\pi}{p+q}+\frac{\sin(p-q)\pi}{p-q}\right)+\left(\frac{\sin(p+q)\pi}{p+q}+\frac{\sin(p-q)\pi}{p-q}\right)\right]$$

$$\Rightarrow \frac{L}{2\pi}\times 2\left[\frac{\sin(p+q)\pi}{p+q}+\frac{\sin(p-q)\pi}{p-q}\right]$$

$$\Rightarrow \frac{L}{\pi}\left[\frac{\sin(p+q)\pi}{p+q}+\frac{\sin(p-q)\pi}{p-q}\right]$$

Case (*i*) $p = q \neq 0$

So, $p - q = 0 \Rightarrow$ 2nd term vanisher

$$\text{Integeral} = \frac{L}{\pi}\frac{\sin 2p\pi}{2p} \qquad [\sin 2n\pi = 0]$$

$$= 0$$

Case (*ii*)

$$p = q$$

$$\text{Integeral} = \frac{L}{\pi}\frac{\sin(p+q)\pi}{p+q}+\frac{\sin(p-q)\pi}{p-q}$$

(*b*) Plot the periodic function defined by : **(2, 6, 4)**

$$f(x) = -\pi, \qquad -\pi < x < 0$$
$$f(x) = x, \qquad 0 < x < \pi$$

Find the Fourier series of this function and hence prove that :

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots..$$

Ans. The solution to this question is in your textbook.

(*c*) What is the period of sin *nx* and that of tan *x*. **(2)**

Ans. The solution to this question is in your textbook.

OR

If $f(t+T) = f(t)$, then show that :

$$\int_a^b f(t)\,dt = \int_{a+T}^{b+T} f(t)\,dt$$

Ans. $\int_a^b f(-t)\,dt$ ... (i)

$f(t+T) = f(t)$ [given] ... (ii)

So, Now put $t \to t + T$ in (i)

Then lower limit changes from

$$a \to a + T$$

and upper limit changes from

$$b \to b + T$$

$$\Rightarrow \int_{a+T}^{b+T} f(t+T)\,d(t+T)$$

$$\Rightarrow \int_{a+T}^{b+T} f(t+T)\,dt \qquad [dT = 0]$$

Now using (ii)

So, $$\int_a^b f(t)\,dt = \int_{a+T}^{b+T} f(t)\,dt$$

SECTION – B

Q. 2. (*a*) Classify the point $x = 0$ as a regular or irregular singular point for the differential equation : **(3)**

$$x^2 \frac{d^2y}{d^2x} + \sin x \frac{dy}{dx} + e^{-x} y = 0.$$

Ans. The solution to this question is in your textbook.

(*b*) Solve the following differential equation about $x = 0$, using Frobenius method : **(12)**

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + xy = 0.$$

Ans. The solution to this question is in your textbook.

OR

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left[x^2 - \frac{1}{4}\right]y = 0$$

Ans. The solution to this question is in your textbook.

Q. 3. Attempt any two parts : **(2 × 7.5 = 15)**

(*a*) Prove that :

$$\mathbf{J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x \sin\theta)d\theta, \; n = 0, 1, 2 .....}$$

Ans.

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x \sin\theta)d\theta$$

We know that :

$$\cos(x \sin\theta) = J_0 + 2J_2 \cos 2\theta + 2J_4 (\cos\theta) + ... \quad ....(i)$$

$$\sin(x \sin\theta) = 2J_1 \sin\theta + 2J_3 \sin 3\theta + 2J_5 (\sin 5\theta) + ... \quad ....(ii)$$

Multiply (*i*) by cos $n\theta$ and integrate between the limits 0 and π, we get

$$\int_0^{\pi} \cos(x \sin\theta) \cos n\theta \, d\theta = \int_0^{\pi} [J_0 \cos n\theta + 2J_2 \cos 2\theta \cos n\theta + 2J_4 \cos 4\theta \cos n\theta + ...] \, d\theta$$

= 0, if n is odd.

= πJ_n, if n is even.

Now, multiplying (*ii*) by sin $n\theta$ and integrate between limits 0 and π, we get

$$\int_0^{\pi} \sin(x \sin\theta) \sin n\theta \, d\theta = \int_0^{\pi} (2J_1 \sin\theta \sin n\theta + 2J_3 \sin 3\theta \sin n\theta + ...) d\theta$$

$$= 2J_1 \int_0^{\pi} \sin\theta \sin n\theta \, d\theta + 2J_3 \int_0^{\pi} \sin 3\theta \sin n\theta \, d\theta$$

= 0 if n is even ... (*iv*)

= πS_n if n is odd.

Adding (*iii*) and (*iv*)

$$\int_0^{\pi} \left[\cos(x \sin\theta) \cos n\theta + \sin(x \sin\theta) \sin\theta\right] d\theta = T_1 J_n$$

and

$$\int_0^{\pi} \cos(n\theta - x \sin\theta) d\theta = \pi J_n$$

$$J_n = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x \sin\theta) d\theta$$

(*b*) Expand $f(x) = x^2 - 3x + 2$ in a series of the from $\sum_{k=0}^{\infty} A_k P_k(x)$, using

$$\mathbf{P_0(x) = 1, \; P_1(x) = x, \; P_2(x) = \frac{3x^2 - 1}{2}.}$$

Ans.

$$f(x) = x^2 - 3x + 2$$

$$P_0(x) = 1,\ P_1(x) = x \text{ and}$$

$$P_2(x) = \frac{3x^2 - 1}{2} \qquad \text{[given]}$$

it $$x^2 - 3x + 2 = aP_2(x) + b\,P_1(x) + c\,P_0(x)$$

$$= a\left[\frac{3x^2 - 1}{2}\right] + bx + c$$

$$= \frac{3a}{2}x^2 - bx + \left(c - \frac{a}{2}\right)$$

Equating cofficients of like power x :

$$\frac{3a}{2} = 1 \Rightarrow a = \frac{2}{3}$$

$$-b = -3$$

$$c - \frac{a}{2} = 2$$

$$c - \frac{2}{3} \times \frac{1}{2} = 2$$

$$c = 2 + \frac{1}{3} \Rightarrow c = \frac{7}{3}$$

So, $$f(x) = x^2 - 3x + 2 = \frac{2}{3}P_2(x) + 3P_1(x) + 3P_1(x) + \frac{7}{3}P_0(x)$$

(*c*) Using the generating function for Bessel's Polynomials or otherwise, prove that :

$$\mathbf{xJ'_n(x) = -nJ_n(x) + xJ_{n-1}(x)}$$

Ans. The solution to this question is in your textbook

(*d*) Obtain an expression for $P_4(x)$ using appropriate formula.

Ans. The solution to this question is in your textbook

SECTION – C

Q. 4. Attempt any one part : **(1 × 5 = 5)**

(*a*) Evaluate : $\displaystyle\int_0^1 \frac{dx}{\sqrt{-\ln x}}$

Ans. $\displaystyle\int_0^1 \frac{dx}{\sqrt{-\ln x}}$

$\Rightarrow$ $$\int_0^1 (-\ln x)^{-1/2}\, dx$$

Put, $-\ln x = y \quad \Rightarrow \quad x = e^{-y}$

$$dx = \theta^{-y}\, dy$$

So, Integral becomes :

$\Rightarrow$ $$\int_0^\infty y^{1/2}\, e^{-y}\, dy$$

This is a gamma function with

$$n = \frac{1}{2}$$

$\Rightarrow$ $$\sqrt{\frac{1}{2}} = \sqrt{\pi}$$

(*b*) Evaluate : $\int_0^a y^4 (a^2 - y^2)^{1/2}\, dy$

Ans. The solution to this question is in your textbook

(*c*) Prove that : $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\, dx = \frac{1}{a^n b^m} \beta(m, n)$

Ans. The solution to this question is in your textbook

SECTION – D

Q. 5. (*a*) The solutions to 2-D wave equation are obtained as trigonometric functions as well as in terms of Bessel functions. Explain how trigonometric cosine function is different from the Bessel Function of Order Zero. Compare them in terms of :

(*i*) Periodicity

(*ii*) Amplitude

(*iii*) Zeros.

Indicate difference using a plot. **(5)**

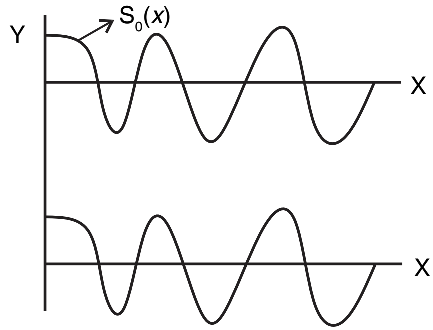
Ans. Cosine series or function

$$= \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots.$$

Bessel function of zero

$$\text{order} = S_0(x) = \frac{1 - x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2}$$

So, this is clear from the both series that both have even power of x. But there are certain difference also.



***(i)* Periodicity :** Both functions have varying period depending on the value of x. They can have period 1, 2 or infinite.

***(ii)* Amplitude :** Bessel function hasdecreasing amplitude and of large value of x, it is apporximately zero. But cosine function has fix amplitude.

***(iii)* Zeros :** Both cosine and Bessel function have equal number of zeros. The number of zeros can be calcualted from the points on x-axis where the value of y is zero.

OR

Using the method of separation of variable, solve : (5)

$$\frac{\partial u}{\partial y} = 2\frac{\partial^2 u}{\partial x^2};\ 0 < x < 3,\quad y > 0$$

Given $u(0, y) = u(3, y) = 0$, and $u(x, 0) = 5 \sin 4\pi x - 3 \sin 8\pi x$.

Ans. The solution to this question is in your textbook

***(b)* Find the steady state temperature, $u(x, y)$ of a rectangular plate ($0 < x < 1$; $0 < y < 2$) subject to the boundary conditions : $u(x, 0) = 0$, $u(0, y) = 0$, $u(1, y) = 0$, and $u(x, 2) = x$. (10)**

Ans. For finding steady state temperature we will use laploce equation in two dimension.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...}(i)$$

Let $u = X(x)\ Y(y)$

Put this in *(i)*

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = P^2$$

$$X'' = -P^2 X \quad \text{or } X'' + P^2 X \qquad \text{...}(ii)$$

$$Y'' = P^2X \quad \text{or} \quad Y'' - P^2Y = 0 \qquad \ldots (iii)$$

Awcillory equation for (*ii*) is

$$m^2 + P^2 = 0 \quad \text{or} \quad m = \pm iP$$

$$X = C_1 \cos Px + C_2 \sin Px$$

A.E. for (*iii*) is

$$m^2 - P^2 = 0 \quad \text{or} \quad m = \pm P$$

$$V = C_3 e^{py} + C_u e^{-py}$$

So, $$u = (C_1 \cos Px + C_2 \sin Px)\,(C_3 P^{Py} + C_4 P^{-Py})$$

$$x = 0, \quad u = 0$$

So, $$C_1 = 0$$

Now, $$u = C_2 \sin Px\,(C_3 e^{Py} + C_4 P^{-Py})$$

$$x = 1, \quad u = 0$$

$$C_2 = 0, \text{ So, } \sin P = 0 = \sin n\pi$$

$$P = n\pi$$

Now, $$u = C_2 \sin n\pi \times (C_3 e^{nn\pi} + C4 e^{-nn\pi})$$

Now, $$u = 0 \text{ and } y = 0$$

$$0 = C_2 \sin n\pi x\,(C_3 + C_4)$$

$$C_3 + C_4 = 0 \quad \text{or} \quad C_3 = -C_4$$

Now, $$u = C_2 C_3 \sin n\pi x\,(e^{n\pi y} - e^{-n\pi y})$$

Now, $$y = 2, \quad u = x$$

$$x = C_2\,C_3 \sin n\pi x\,(e^{2n\pi} - e^{-2n\pi})$$

$$C_2 C_3 = \frac{x}{\sin n\pi x\,(e^{2n\pi} - e^{-2n\pi})}$$

Put this in u

$$u = \frac{x\,(e^{n\pi y} - e^{-n\pi y})}{(e^{2n\pi} - e^{-2n\pi})}$$

$$u = x\,\frac{\sin n\pi y}{\sin 2n\pi}$$

OR

Using the method of separation of variable, solve 1-D wave equation :

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Subject to conditions $y(0, t) = 0$, $y(L, t) = 0$ and

$$y(x, 0) = \begin{cases} x, & 0 < x < \frac{L}{2} \\ L - x, & \frac{L}{2} \le x \le L \end{cases}, \; y_t\,(x, 0) = 0$$

where $y_t = \dfrac{\partial y}{\partial t}$.

Ans. The solution to this question is in your textbook

(*c*) Show that $u(x, t) = e^{-8t} \sin 2x$ is a solution to $\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}$ with the conditions $u(0, t) = u(\pi, t) = 0$, $u(x, 0) = \sin 2x$.

Ans. The solution to this question is in your textbook

OR

Using the method of separation of variable, prove that the general solution of $\dfrac{\partial f}{\partial t} = 4\dfrac{\partial f}{\partial x}$ is given by :

$$f(x, t) = Ae^{k\left[\left(\frac{x}{4}\right) + t\right]}$$

where A and *k* are some constants. **(5)**

Ans. $$\frac{\partial f}{\partial t} = \frac{4\partial f}{\partial x} \qquad \dots (i)$$

Let $$f = X(x)\ T(t)$$

where X is a function of x only and T is a function of t only.

Put this in (*i*) we get

$$\frac{\partial (X\,.\,T)}{\partial t} = 4\,\frac{\partial\,(X\,.\,T)}{\partial\, x}$$

$$X\,\frac{\partial\, T}{\partial\, t} = 4T\,\frac{\partial X}{\partial x}$$

$$4T\,.\,X^1 = XT^1$$

$$\frac{X^1}{X} = \frac{T^1}{4T} = C$$

$$\frac{X^1}{X} = C$$

$$\frac{1}{X}\frac{dX}{dx} = C \quad \text{or} \quad \frac{dX}{X} = C\,dx$$

$$\log X = Cx + \log a$$

$$\log\left(\frac{X}{a}\right) = Cx$$

$$X = a\,e^{cx}$$

$$\frac{T^1}{4T} = C$$

$$\frac{1}{T}\frac{dT}{dt} = 4C$$

$$\frac{dT}{T} = 4\,C\,dt$$

$$\log T = 4\,Ct + \log b$$

$$T = b\,e^{4tC}$$

$$f(x, t) = XT$$

$$= ab\,e^{[Cx + 4\,tC]}$$

$$ab = A$$

$$f(x, t) = A\,e^{4C\left[\frac{x}{4} + t\right]}$$

$$k = 4C$$

So, $$f(x, t) = A\,e^{k\left[\frac{x}{4} + t\right]}$$