Name of Paper : **Mathematical Physics-I**
Name of the Course : **B.Sc. (Hons.) Physics**
Semester : **I**
Duration : **3 Hours**
Maximum Marks : **75**

Nov. / Dec. 2019

(*a*) *Question No. 1 is compulsory.*
(*b*) *Attempt four more questions out of the rest.*
(*c*) *Non-programmable calculators are allowed.*

Q. 1. Do any five of the following : **(5 × 3 = 15)**

(*a*) Determine the linear independence/linear dependence of e^x, xe^x, x^2e^x.

(*b*) Determine the order, degree and linearity of the following differential equation.

$$\frac{d^3y}{dx^3} + x^2\left(\frac{d^2y}{dx^2}\right)^2 = 0$$

(*c*) Find the area of the triangle having vertices at P (1, 3, 2) Q (2, – 1, 1) and (–1, 2, 3).

(*d*) Let $\vec{A}$ be constant vector. Prove that $\vec{\nabla}\left(\vec{r}\,.\,\vec{A}\right) = \vec{A}$.

(*e*) Find the acute angle between the surfaces $xy^2z - 3x - z^2 = 0$ and $3x^2 - y^2 + 2z = 1$ at the point (1, – 2, 1).

(*f*) A random variable X has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-(x-5)^{3/2}} \quad -\infty < x < \infty$$

Find the mean.

Ans. (*a*) e^x, xe^x, x^2e^x

Since none of above are linear combination to each other is the only way to make their linear combination zero is take all the scalars zero that is

$$a_1e^x + a_2xe^x + a_3x^2e^x = 0$$
$$0\,.\,e^x + 0\,.\,xe^x + 0\,.\,x^2e^x = 0$$
$$\Rightarrow \qquad a_1 = a_2 = a_3 = 0$$

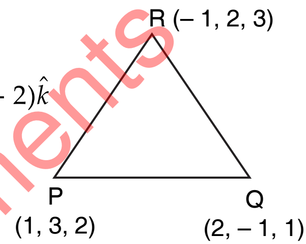
Hence these are linear independent.

(b) $\dfrac{d^3y}{dx^3} + x^2\left(\dfrac{dy}{dx^2}\right) = 0$

Its an ordinary differential equation in y with highest derivative is **order 3** and since degree is power of highest derivative term $= \left(\dfrac{d^3y}{dx^3}\right)^{1}$. Therefore, **degree is 1**

But since derivative of y has power 2 its **not linear.**

(c) Area of $\Delta = \dfrac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$



$$\overrightarrow{PQ} = (2-1)\hat{i} + (-1-3)\hat{j} + (1-2)\hat{k}$$

$$\overrightarrow{PQ} = \hat{i} - 4\hat{j} - \hat{k}$$

and

$$\overrightarrow{PR} = (-1-1)\hat{i} + (2-3)\hat{j} + (3-2)\hat{k}$$

$$\overrightarrow{PR} = -2\hat{i} - \hat{j} + \hat{k}$$

Now,

$$\overrightarrow{PQ} \times \overrightarrow{PR} \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -4 & -1 \\ -2 & -1 & 1 \end{vmatrix}$$

$\Rightarrow \quad \hat{i}\,(-4-1) - \hat{j}\,(1-2) + \hat{k}\,(-1-8)$

$\Rightarrow \quad -5\hat{i} + \hat{j} - 9\hat{k}$

$$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \sqrt{(-5)^2 + (1)^2 + (-9)^2}$$

$$= \sqrt{25 + 1 + 81}$$

$$= \sqrt{107}$$

So, Area of $\Delta = \dfrac{1}{2} \times \sqrt{107}$

(d) $\overrightarrow{A}$ is constant vector

$$\nabla(\vec{A} \cdot \vec{r}) = \vec{A} \times (\vec{\nabla} \times \vec{r}) + \vec{r} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla)\vec{r} + (\vec{r} \cdot \nabla)\vec{A}$$

$$= 0 + 0 + \vec{A} + 0 = \vec{A}$$

OR

$$\nabla(\vec{r} \cdot \vec{A}) = \left(\frac{\hat{i}\partial}{\partial x} + \frac{\hat{j}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right)(A_1x + A_2y + A_3z)$$

$$= A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

as $\quad (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) = (A_1x + A_2y + A_3z)$.

(*e*) let
$$\phi_1 = xy^2z - 3x - z^2$$
$$\phi_2 = 3x^2 - y^2 + 2z$$

$$[\cos\theta] = \left.\frac{(\nabla\phi_1)(\nabla\phi_2)}{|\nabla\phi_1|\,|\nabla\phi_2|}\right|_{(1,\,-2,\,1)}$$

$$\cos\theta = \frac{[(y^2z - 3)\hat{i} + (2xyz)\hat{j} + (xy^2 - 2z)\hat{k}]\,[(6x)\hat{i} - (2y)\hat{j} + 2\hat{k}]}{\sqrt{(y^2z-3)^2 + (2xyz)^2 + (xy^2 - 2z)^2}\,\sqrt{(6x)^2 + (2y)^2 + (2)^2}}$$

Putting co-ordinates given in question

$$\theta = \cos^{-1}\left(\frac{24 - 18 - 16 + 8 - 4}{\sqrt{21}\quad\sqrt{56}}\right)$$

$$= \cos^{-1}\left(\frac{-14}{\sqrt{21}\,\sqrt{56}}\right)$$

(*f*)
$$f(x) = \frac{1}{2\pi}e^{-(x-5)^{2/3}}$$

$$\langle x\rangle = \frac{\dfrac{1}{2\pi}\displaystyle\int_{\infty}^{\infty} xe^{-(x-5)^{2/3}}\,dx}{\dfrac{1}{2\pi}\displaystyle\int_{\infty}^{\infty} e^{-(x-5)^{2/3}}\,dx}$$

The soution of this term will provide you the mean of x.

Q. 2. (*a*) Solve the simultaneous differential equations given below :

$$\frac{dy}{dt} = y \qquad\qquad \frac{dx}{dt} = 2y + x$$

(*b*) Two independent random variables X and Y have probability density functions $f(x) = e^{-x}$ and $g(y) = 2e^{-2y}$ respectively. What is the probability that X and Y lie in the intervals $1 < x \leq 2$ and $0 < x \leq 1$.

The time rate of change of the temperature of a body at an instant t is proportional to the temperature difference between the body and its surrounding medium at that instant.

(*c*) Box A contains 8 times out of which 3 are defective. Box B contains 5 items out of which 2 are defective. An item is drawn randomly from each box.

(5 + 5 + 5)

(*i*) What is the probability that both the items are non-defective ?

(*ii*) What is the probability that only one item is defective ?

(*iii*) What is the probability that the defective item came from box A ?

Ans. (*a*) Find the similar question in the text book.

(*b*) Find the probability of each function by integration and then multiply both probabilities.

(*c*) (*i*) 5 items of 8 items in a Box A are non defective $= \frac{5}{8}$

3 items out of 5 in box B are non defective $= \frac{3}{5}$

Probabilility that both the items are not defective $= \frac{5}{8} \times \frac{3}{5} = \frac{3}{8}$.

(*c*) (*ii*) Probability of defected form A

Join it with last the and non defected from B $= \frac{3}{8} \cdot \frac{3}{5} = \frac{9}{40}$

Probability of non-defected from A

Join it with last the and defected from B $= \frac{5}{8} \cdot \frac{2}{5} = \frac{10}{40}$

Probability of only one defected $= \frac{9}{10} + \frac{10}{40} = \frac{19}{40}$.

(*c*) (*iii*) We want to calculate the probility such that the items has been found defective and then we want the chances that if came from bag A. [P(defect|A)]

According to Baye's theorem

$$P(A|defect) = \frac{P(A)\,P(defect\,|\,A)}{P(A)\cdot P(defect\,|\,A) + P(B)\cdot P(defect\,|\,B)}$$

$$= \frac{\frac{1}{2}\times\frac{3}{8}}{\left(\frac{1}{2}\times\frac{3}{8}\right)+\left(\frac{1}{2}\times\frac{2}{5}\right)} = \frac{\frac{3}{16}}{\frac{3}{16}+\frac{1}{5}}$$

$$= \frac{\frac{3}{16}}{\frac{15+16}{80}} = \frac{3}{16}\times\frac{80}{31} = \frac{15}{31}$$

Q. 3. Solve the following differential equations :

(*a*) $y'' + y = \sec x$ (8)

(*b*) $(z + ye^{xy})dx + (xe^{xy} - 2y)dy = 0$ (7)

Ans. (*a*) This is linear differential equation. Can be solved based on direct formula.

(*b*) This is a kind of non-exact differential equation and can be solved by making it exact and then solve using standard formula. Find similar questions in the textbook.

Q. 4. (*a*) Solve the initial value problem. (8)

(*i*) $y'' + 4y' + 8y = \sin x$

(*ii*) $y(0) = 1,\ y'(0) = 0$

(*b*) A metal bar at a temperature 100°F is placed in a room at a constant temperature of 0°F. After 20 minutes the temperature of the bar is 50°F. Find : (7)

(*i*) The time it will take the bar to reach a temperature of 25°F.

(*ii*) Temperature of the bar after 10 minutes.

Ans. (*a*) Find complimentary function and particular integral for this equation and so0lve this.

(*b*)

$$T(t) = T_{sur} + (T_0 - T_{sur})e^{-kt}$$

K : Constant

T_0 : Initial temperature = 100°F

T_{sur} : Surrounding temperature = 0°F

$t = 20$ min

$$T(20) = 0 + (100 - 0)e^{-k\,.\,20}$$

$$50 = 100\ e^{-k20}$$

$$\frac{1}{2} = e^{-k.20}$$

$$2 = e^{k20}$$

$$\ln 2 = 20 \, . \, k$$

$$\frac{.693}{20} = k \qquad \Rightarrow \qquad k = .03465.$$

(*i*) $$T(t) = 0 + (100 - 0)e^{-k.t}$$

$$25 = 100 \; e^{-kt}$$

$$\frac{1}{4} = e^{-kt}$$

$$4 = e^{kt}$$

Taking log

$$\log 4 = k \, . \, t$$

$$0.6020 = .03465 \, . \, t$$

$$2\,.06 = t$$

(*ii*) $$T\,(10) = 0 + (100 - 0)$$

$$= 100e^{-k\,.\,10}$$

$$= 100 \, . \, e^{-(0.03465\,.\,10)}$$

$$= 271.48\text{m}$$

Q. 5. (*a*) If v denotes the region inside that semicircular cylinder

$$0 \le x \le \sqrt{a^2 - y^2} \quad 0 \le z \le 2a$$

Evaluate $\iiint_v x dv$ **(7)**

Ans. $$x = \sqrt{a^2 - y^2}$$

Squaring

$$x^2 + y^2 = a^2 \qquad\qquad 0 \le z \le 2a$$

$$\iiint x dv = \iiint x dx\, dy\, dz$$

$$= \int_0^x x\,dx \int_0^{\sqrt{a^2 - x^2}} dy \, . \int_0^{2a} dz$$

as $$x^2 + y^2 = a^2 \qquad \Rightarrow \qquad y^2 = a^2 - x^2$$

$$y = \sqrt{a^2 - x^2}$$

$$2a\int_0^x \sqrt{a^2 - x^2}\ .\ xdx$$

put $\quad a^2 - x^2 = t\,;$

derivating, $-2xdx = dt$

$$-a.\int \sqrt{t}\ dt$$

$$-a.\,t^{3/2}.\frac{2}{3} = \frac{-2a}{3}(a^2 - x^2)^{3/2}$$

Q. 6. (*a*) Find the directional derivative of φ = $4xz^3 - 3x^2y^2z$ at (2, –1,2) in the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$ **(5)**

(*b*) Find the value of $\nabla^2(\ln r)$

(*c*) Prove that :

$$\iiint \frac{dv}{r^2} = \oiint_s \frac{\vec{r}\,.\,\bar{n}}{r^2}\,ds$$

Where *v* is the volume of region enclosed by surface.

Ans. (*a*) Find the gradient of this function phi and then dot product it with $2i - 3j + 6k$ vector.

(*b*) $\qquad \nabla^2(\ln \bar{r}) = \nabla\,.\,\nabla(\ln \bar{r})$

$$\nabla(\ln \bar{r}) = \hat{i}\frac{\partial}{\partial x}(\ln \bar{r}) + \hat{j}\frac{\partial}{\partial y}(\ln \bar{r}) + \hat{k}\frac{\partial}{\partial z}(\ln \bar{r})$$

$$= \hat{i}\,\frac{1}{x}\,\frac{d\vec{r}}{dx} + \hat{j}\,\frac{1}{r}\,\frac{d\vec{r}}{dy} + \hat{k}\,\frac{1}{r}\,\frac{d\vec{r}}{dz}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r^2 = (x^2 + y^2 + z^2)$$

$$\nabla\frac{dr}{dx} = 2x \qquad \text{and} \qquad \frac{dr}{dx} = \frac{x}{r}$$

Similarly $\qquad \dfrac{dt}{dy} = \dfrac{y}{r} \qquad \dfrac{dr}{dx} = \dfrac{z}{r}$

$$\nabla(\ln r) = \frac{1}{r^2}(x\hat{i} + y\hat{j} + 3\hat{k})$$

$$= \frac{\vec{r}}{r^2}$$

$$\nabla \cdot \nabla(\ln r) = \nabla \cdot \left(\frac{\vec{r}}{r^2}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial k}\right)\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^2}\right)$$

$$= \frac{r^2 . 1 - 2xr\frac{dr}{dx}}{r^4} + \frac{r^2 . 1 - 2yr\frac{dr}{dx}}{r^4} + \frac{r^2 . 1 - 2zr\frac{dr}{dx}}{r^4}$$

$$= \left(\frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^2}\right) - \frac{2}{r^3}\left(x\frac{dr}{dx} + y\frac{dr}{dy} + z\frac{dr}{dz}\right)$$

$$= \frac{3}{r^2} - \frac{2}{r^3}\left(x . \frac{x}{r} + y . \frac{y}{r} + z . \frac{z}{r}\right)$$

$$= \frac{3}{r^2} - \frac{2}{r^3}\left(\frac{x^2 + y^2 + z^2}{r}\right)$$

$$= \frac{3}{r^2} - \frac{2}{r^3} . \frac{r^2}{r} \Rightarrow \frac{3}{r^2} - \frac{2}{r^2} \Rightarrow \frac{1}{r^2}$$

Q. 7. (*a*) Suppose $\vec{A} = (2y + 3)\hat{i} + xz\hat{j} + (yz - x)\hat{k}$

Evaluate $\int_e \vec{A} . d\vec{r}$ **along the following paths : (9)**

(*i*) $x = 2t^2, y = t, z = t^3$ **from** $t = 0$ **to** $t = 1$

(*ii*) The straight line from (0,0,0) to (0,0,1) then to (0,1,1) and then to (2,1,1)

(*b*) Evaluate $\iint \vec{A} . \hat{n}\, dS$

where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ **and S is the surface of the cylinder** $x^2 + y^2 = 16$ **included in the first octant between** $z = 0$ **to** $z = 5$.

Ans. (*a*) $\int_c \vec{F} \cdot \vec{dr} = \int_c (2y + 3)\, dx + (xz)\, dy + (yz - x)\, dz$

$$\left[\begin{array}{llll} \text{since } x = 2t & y = t & z = t^3 \\ \therefore \dfrac{dx}{dt} = 2 & \dfrac{dy}{dt} = 1 & \dfrac{dz}{dt} = 3t^2 \end{array}\right]$$

$$= \int_0^1 (2t + 3)\,(2\,dt) + (2t)\,(t^3)\,dt + (t^4 - 2t)\,(3t^2\,dt)$$

$$= \int_0^1 (4t + 6 + 2t^4 + 3t^6 - 6t^3)\,dt$$

$$= \left[4\frac{t^2}{2} + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{6}{4}t^4\right]_0^1$$

$$= \left[2t^2 + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{3}{2}t^4\right]_0^1$$

$$= 2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2} = 7.32857$$

Integral will remain the same only the limits will change according to the condition in the questions mentioned further.

(*b*) Equation of surface S

$$x^2 + y^2 - 16 = 0$$

$$\nabla S = 2x\hat{i} + 2y\hat{j}$$

$\hat{n}$ = Unit vector normal to surface S at any point $(x,y,z) = \dfrac{\nabla S}{|\nabla S|}$

$$\hat{n} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}} \Rightarrow \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4(x^2 + y^2)}}$$

$$\hat{n} = \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{x^2 + y^2}}$$

Here, $x^2 + y^2 = 16$ [Given]

$$\hat{n} = \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{16}} \Rightarrow \frac{x\hat{i} + y\hat{j}}{4}$$

$$\hat{n} = \frac{1}{4}x\hat{i} + \frac{1}{4}y\hat{j}$$

Also, $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$

$$\vec{A} \cdot \hat{n} = (z\hat{i} + x\hat{j} - 3y^2z\hat{k}) \cdot \left(\frac{1}{4}x\hat{i} + \frac{1}{4}y\hat{j}\right)$$

$$\vec{A} \cdot \hat{n} = \frac{1}{4}xz + \frac{1}{4}xy$$

Take projection on xz-plane $\rightarrow$

$$dS = \frac{dx\ dz}{|\hat{n} \cdot \hat{j}|}$$

Let R be region of projection of S on xz-plane, R is bounded by $x = 0$ to $x = 4$ and $z = 0$ to $z = 5$

Here, $\iint_S \vec{A} \cdot \hat{n}\ dS = \iint_R \vec{A} \cdot \hat{n} \frac{dx\ dz}{|\hat{x} \cdot \hat{j}|}$

$$\Rightarrow \iint_R \vec{A} \cdot \hat{n} \frac{dx\ dz}{|\hat{x} \cdot \hat{j}|}$$

$$\Rightarrow \iint_R \left(\frac{xz}{4} + \frac{xy}{4}\right) \frac{dxdz}{\frac{1}{4}y}$$

$$\left[\begin{array}{l} \hat{x} = \frac{1}{4}x\hat{i} + \frac{1}{4}y\hat{j} \\ \hat{x} \cdot \hat{j} = \frac{1}{4}y \\ |\hat{x} \cdot \hat{i}| = \frac{1}{4}y \end{array}\right]$$

$$\Rightarrow \iint_R \left(\frac{xz}{y} + \frac{xy}{y}\right) dxdz$$

$$\Rightarrow \iint_R \left(\frac{xz}{y} + x\right) dxdz$$

$$\Rightarrow \iint_R \left(\frac{xz}{y} + x\right) dxdz$$

$$\left[\begin{array}{l} \text{Here, } x^2 + y^2 = 16 \\ y = \sqrt{16 - x^2} \end{array}\right]$$

$$\Rightarrow \iint_R \left(\frac{xz}{\sqrt{16 - x^2}} + x\right) dxdz$$

$$\Rightarrow \int_{z=0}^{5} \int_{x=0}^{4} \left(\frac{xz}{\sqrt{16 - x^2}} + x\right) dxdz$$

$$\Rightarrow \int_{z=0}^{5}\left[\int_{x=0}^{4}\left(z\frac{x}{\sqrt{16-x^2}}+x\right)dx\right]dz$$

$$\Rightarrow \int_{z=0}^{5}\left[\int_{x=0}^{4}\left(z\frac{-2x\times-\frac{1}{2}}{\sqrt{16-x^2}}+x\right)dx\right]dz$$

$$\Rightarrow \int_{z=0}^{5}\int_{x=0}^{4}\left(-\frac{z}{2}\frac{-2x}{\sqrt{16-x^2}}+x\right)dx\,dz$$

$$\Rightarrow \int_{0}^{5}\left(-\frac{z}{2}\frac{\sqrt{16-x^2}}{\frac{1}{2}}+\frac{x^2}{2}\right)_{0}^{4}dz$$

$$\Rightarrow \int_{0}^{5}(8+4z)dz$$

$$\Rightarrow \int_{0}^{5}(4z+8)dz \quad\Rightarrow\quad \left.4\frac{z^2}{2}+8z\right]_{0}^{5}$$

$$\Rightarrow \left(4\times\frac{25}{2}+8\times 5\right)-(0+0)$$

$$\Rightarrow \quad 50+40-0$$

$$\Rightarrow \quad 90.$$