Name of Paper : Mechanics

Name of the Course : B.Sc. (Hons.) Physics

Semester : I

Duration : 3 Hours

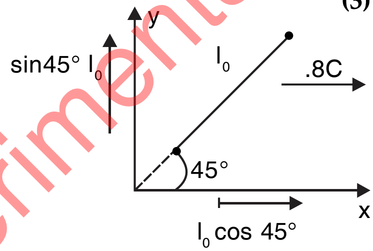
Maximum Marks : 75

Nov. / Dec. 2019

Questions No. 1 is compulsory and carriers 19 marks.
Answer any four of the remaining six, each carrying 14 marks,
attempting any two parts out of three from each question.

Q. 1. Attempt all parts of this questions :

(*i*) Calculate the percentage contraction of a rod moving with a velocity 0.8c in a direction inclined at 45° to its own length. (3)



Ans.

$$l_x = l_0 \cos 45 = \frac{l_0}{\sqrt{2}}$$

$$l_y = \frac{l_0}{\sqrt{2}}$$

$$l'_y = l_y \qquad l'x = \frac{l_x}{y} = \frac{l_0}{\sqrt{2}}\sqrt{1 - \frac{v^2}{c^2}}$$

$$l'_y = \frac{l_0}{\sqrt{2}}\sqrt{1 - \frac{.64c^2}{c^2}} = \frac{l_0}{\sqrt{2}}\sqrt{(.36)}$$

$$= \frac{l_0}{\sqrt{2}}(.6)$$

$$l' = \sqrt{(l'_y)^2 + (l'_x)^2} = \sqrt{\frac{l_0^2}{2} + \frac{l_0^2}{2}(.36)}$$

$$= \frac{l_0}{\sqrt{2}}\sqrt{1 + .36} \quad \Rightarrow \quad \frac{l_0}{\sqrt{2}}(1.166)$$

$\Rightarrow$ $= .82 l_0$

$$\% \text{ Change} = \frac{\Delta l}{l} \times 100 = \left(\frac{l_0 - .82 l_0}{l_0}\right) \times 100$$

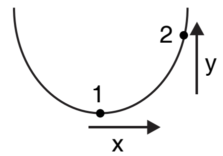
$$= 18\%.$$

(ii) A particle slides back and forth on a frictionless track whose height as a function of horizontal position x is given by $y = ax^2$, where $a = 0.92\text{m}^{-1}$. If the particle's maximum speed is 8.5 m/s, find the turning points of its motion. (3)

Ans. $y = ax^2$; $a = .92\text{m}^{-1}$

max speed = 8.5m/s^2

Applying conservation of energy at position 1 and 2.



$$\frac{m}{2}v_1^2 + mg.(0) = \frac{m_2}{2}v_2^2 + mg(y)$$

at position 2, particle stops and reverse trace its path $\therefore v_2 = 0$

Max speed is at equilibrium *i.e.*, 1

$$\frac{1}{2}m(8.5)^2 + 0 = 0 + mgax^2$$

$$\left(\frac{8.5}{2}\right)^2 = 9.8 \times .92 \times x^2$$

$$36.12 = 9.016 .\ x^2$$

$$4 = x^2$$

$$2m = x.$$

(iii) A space traveller weight 80 kg on earth. Find the weight of the traveller on another planet whose radius is twice that of the earth and whose mass is 3 times that of the earth. (3)

Ans. Weight on earth = 80 kg = mg

Another planet $R_p = 2R_e$ and $M_p = 3M_e$

Weight on another planet $\Rightarrow mg' = \dfrac{GM_e m}{R_p^2}$

Putting values :

$$mg' = \frac{G3M_e m}{(2R_e)} = \frac{3}{4}$$

$$= \frac{3}{4}mg \Rightarrow \frac{3}{4} \times 80 = 60 \text{ kg}$$

(iv) A rigid body is rotating about its axis of symmetry, its moment of inertia about the axis of rotation being 1 kg m^2 and its rate of rotation 2 rev/s. What is its angular momentum about the given axis ? what additional work will have to be done to double its rate of rotation ?

Ans. I, moment of inertia = kgm^2

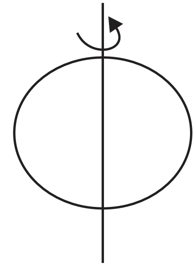
f, rotation per sec. = 2rev/s.

$$w = 2\pi f = 2\pi \times 2 = 4\pi \text{ rad/s.}$$

$$\alpha = \mathrm{I}w \;;\; w = 2\pi f$$

$$= 1 \times 4\pi$$

$$= 4\pi \text{ kgm}^2\text{/sec.}$$



Now the work done is to change its kinetic energy.

$$\Delta E = \mathrm{I}w_f - \mathrm{I}w_i$$

$$= \mathrm{I}\,(2w_i - w_i) \qquad (\because w \to 2w)$$

$$= \mathrm{I}w_i$$

$$= 1\,.\,2\pi$$

$$= 2\pi$$

(*v*) A particle, moving in a straight line with S.H.M. of period $2\pi/\omega$ about a fixed point O, has a velocity $\sqrt{3}b\omega$ when at a distance b from O. Calculate its amplitude and the time it takes to cover the rest of its distance. (3)

Ans. $$T = \frac{2\pi}{\omega},\; V = b\omega\sqrt{3} \text{ at } x = b$$

Substituting in $V = \omega\sqrt{A^2 - x^2}$

$$b\omega\sqrt{3} = \omega\sqrt{A^2 - b^2}$$

$$b^2\omega^2 3 = \omega^2(A^2 - b^2)$$

$$3b^2 + b^2 = A^2$$

$$A = \pm 2b$$

Now the time taken to travel $x = b$ can be found $x = A \sin \omega t$ $\therefore$ motion is considered as it started from mean position $x = b$, $A = 2b$

$$\therefore \qquad b = 2b \sin \omega t$$

$$\frac{1}{2} = \sin \omega t$$

$$\therefore \qquad \omega t = \frac{\pi}{6}$$

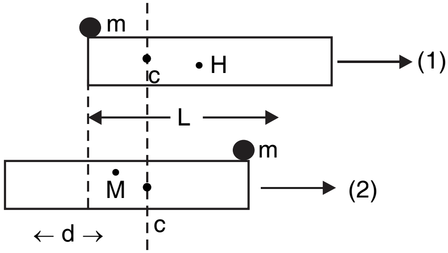
$$t = \frac{\pi}{6\omega}$$

Now time taken for extreme position $= \dfrac{T}{4} - \dfrac{\pi}{6\omega}$

$$= \frac{2\pi}{4\omega} - \frac{\pi}{6\omega} = \frac{\pi}{3\omega}$$

(*vi*) A 4800 kg elephant is standing at one end of a 15000 kg rail car, which is at rest all by itself, on a frictionless horizontal track. The elephant walks 19m towards the other end of the car. How far does the car move ?

Ans. Condring length of Rail car = 19m



$m =$ mass of elephant

M = Mass of rail car

C = Centre of mass

L = 19 m

Let rail car moves d distance after moving of elephant and without any frictional force centre of mass remains at rest.

$\therefore \qquad x_{cm} = \text{Constant}$

For condition (1)

$$x_{cm} = \frac{m.0 + M\frac{L}{2}}{m + M} = \frac{M\frac{L}{2}}{m + m}$$

For condition (2), (wrt initial positions)

$$x_{cm} = \frac{m(L - d) + M\left(\frac{L}{2} - d\right)}{m + M}$$

$$\therefore \qquad \frac{M\frac{L}{2}}{m + m} = \frac{m(L - d) + M\left(\frac{L}{2} - d\right)}{m + M}$$

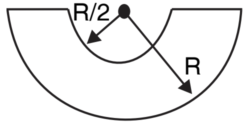
$$\Rightarrow \qquad M\cancel{\frac{L}{2}} = mL - md + M\cancel{\frac{L}{2}} - Md$$

putting values, $d = 4.60$m, in opp. direction to motion of elephant.

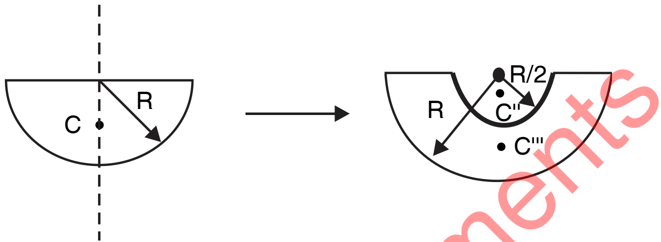
Q. 2. (*i*) (*a*) Find the location of the centre of mass of a solid hemisphere of uniform density and radius R.

Ans. Find the answer to this question in the text portion of this book and solved papers.

(*b*) (*i*) Mass in the shape of a hemisphere of radius R/2 is removed from the hemisphere in part (*a*), as shown in the figure. Where is the centre of mass of the remaining mass ? **(4 + 3)**



Ans. C is centre of mass of full hemisphere $x_c = \dfrac{3R}{8}$



C'' and C''' are centre of mass of removed and remaing hemisphere resp.

$$\therefore \qquad x_{c''} = \frac{3}{8}\left(\frac{R}{2}\right) = \frac{3R}{16}$$

Now, since the case is symmetric

$$\frac{m_{\text{removed}}}{m_{\text{remaining}}} \Rightarrow \frac{m}{M} = \frac{x_{c'''}}{x_{c''}}$$

$$x_{c'''} = \frac{m}{M}\,\frac{3R}{16} = \frac{\frac{2}{3}\pi\left(\frac{R}{2}\right)^3 . \rho}{\frac{2}{3}\pi R^3 . \rho} \times \frac{3R}{16}$$

Since density is same = ρ for both and $\rho = \dfrac{\text{mass}}{\text{vol.}}$

$$x_{c'''} = \frac{R^3}{8}\,\frac{1}{R^3} \times \frac{3R}{16} = \frac{3R}{128}$$

(*ii*) Two particles having masses m_1 and m_2 move so that their relative velocity is v and the velocity of their centre of mass is v_{cm}. Prove that the total kinetic energy of system is $(Mv^2_{cm} + \mu v^2)2$, where M is the total mass and μ is the reduced mass of the system. **(7)**

Ans. Find the similar question in the next portion of this book.

(*iii*) An empty freight car of mass 500 kg starts from rest under an applied force of 100N. At the same time sand beings to run into the car at a steady rate of 20 kg/s from a hopper at rest on the track. Find the speed of the car when 100 kg of sand has been transferred. **(7)**

Ans. $M = 500$ kg (mass of freight car)

$U = 0$m/s

$F = 100$ N

$b = 20$ kg/s

$m = 100$

$v = ?$

at $t = 0$, car is empty and at rest

x component of momentum $= P_x(0) = 0$

at $t = t_f$, $m = b\,t_f$ of sand put into car

at speed v_f, thus, $P_x(t_f) = (M + m)v_f$

$$P_x(t_f) = (M + b\,t_f)v$$

Now $$\int_0^{t_f} F_x\,dt = \Delta P_x$$ (*i.e.*, change in momentum)

$$= P_x(t_f) - P_x(0)$$

$$F(t_f) = (M + bt_f)v_f$$

$$\therefore \quad v_f = \frac{F\,t_f}{M + b\,t_f}$$

$bt_f = 100$ and $b = 20$, $M = 500$, $F = 100$

$$\therefore \quad t_f = \frac{100}{20} = 5 \qquad \left(\because 100 = \frac{dm}{dt}.t\right)$$

Putting values

$$v_f = \frac{100 \times 5}{(500 + 100)} = .833 \text{ m/s.}$$

$$\frac{1}{\mu}\left[\ln(m_0 + \mu t)\right]_0^t = \frac{-1}{\mu}\left[\ln(F - \mu v)\right]_0^0$$

Solving further

$$v = \frac{Ft}{m_0 + \mu t} = \frac{100 \times 5}{500 + 100} = \frac{500}{600} = \frac{5}{6}\text{m/s}$$

Calculation of t :

According to Questions :

$$100 \text{ kg of sand} = \frac{dm}{dt} \cdot t$$

$$100 = 20 \,.\, t$$

$$t = \frac{100}{20} = 5 \text{ sec.}$$

Q. 3. (*i*) Obtain an expression for the moment of inertia of a solid cylinder about an axis through its centre and perpendicular to its axis of cylindrical symmetry. (7)

Ans. Find its solution in this text book portion.

(*ii*) A ring of mass 0.3 kg and radius 0.1 m and a solid cylinder of mass 0.4 kg and of the same radius are given the same kinetic energy and released simultaneously on a flat horizontal surface such that they begin-to roll as soon as released towards a well which is at the same distance from the ring and the cylinder. Assuming that the rolling friction in both cases is negligible, find out which object reaches the wall first ? (7)

Ans. $[M_r]$ Mass of ring = 0.3 kg

$[R_r]$ Radius of ring = 0.1 m

$[M_c]$ Mass of cylinder = 0.4 kg

$[R_c]$ Radius of cylinder = 0.1 m

K.E of ring = K.E of cylinder [given in question]

As both ring and cylinder is in pure rolling so K.E. of ring/cylinder is simply sum of K.E. of translation and K.E of rotation.

$$\text{K.E} = \text{K.E of translation} + \text{K.E. of rotation}$$

$$= \frac{1}{2}MV_c^2 + \frac{1}{2}I_c w^2$$

As ring and cylinder is initially in pure rolling so it doesn't matter, whether friction is present or not.

$$\text{K.E. of ring} = \frac{1}{2}M_r V_r^2 + \frac{1}{2}I_{ring}\, w^2$$

$$= \frac{1}{2}M_r V_r^2 + \frac{1}{2}M_r \cancel{R_r^2}\left[\frac{V_r}{\cancel{R_r}}\right]^2$$

$$\text{K.E. of ring} = M_r V_r^2$$

$$\text{K.E. of cylinder} = \frac{1}{2}M_c V_c^2 + \frac{1}{2}I_c w^2$$

$$= \frac{1}{2}M_c V_c^2 + \frac{1}{2} \times \frac{1}{2}M_c \cancel{R_c^2}\left(\frac{V_c}{\cancel{R_c^2}}\right)^2$$

$$= \frac{1}{2}M_c V_c^2 + \frac{1}{4}M_c V_c^2$$

$$\text{K.E. of cylinder} = \frac{3}{4}M_c V_c^2$$

$$M_r V_r^2 = \frac{3}{4}M_c V_c^2$$

$$V_r = \sqrt{\frac{k}{M_r}}$$

$$V_c = \sqrt{\frac{4k}{3M_c}}$$

To calculate time, let's say the distance between initial position of ring and cylinder and wall be l , then

$$t = \frac{l}{v} \text{ [so } t_1 \text{ and } t_2 \text{ be time taken by ring \& cylinder, then]}$$

$$t_1 = \frac{l}{v_r} = \sqrt{\frac{M_r}{k}}$$

$$t_2 = \frac{l}{v_c} = \sqrt{\frac{3Mc}{4k}}$$

(*iii*) A uniform rod of mass M and length L lies on a smooth horizontal plane. A particle of mass m moving at a speed v perpendicular to the length of the rod strikes it a distance L/4 from the centre and stops after the collision . Find :

(*a*) The velocity of the centre of the rod.

(*b*) The angular velocity of the rod about its centre just after collision. (4 + 3)

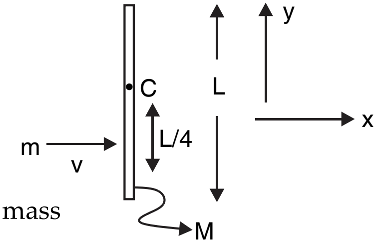
Ans. Mass of rod = M

Length of rod = L

Mass of particle = m

Initial velocity of particle = v

Final velocity of particle = 0



Let V be the translational velocity of centre of mass of rod and angular velocity w after collision, then

from conservation of linear momentum along x-axis

$$mv = MV$$

$$V = \frac{mv}{M}$$

P ring conservation of angular momentum about centre of mass of rod (which is at C)

$$mv \cdot \frac{\cancel{L}}{\cancel{4}} = \frac{ML^{\cancel{2}}}{\cancel{12}\,_3} w$$

Q. 4. (*i*) Derive the expression for the gravitational potential due to a spherical shell of radius R and mass M at a point outside the shell and also at a point inside the shell. Give its graphical representation. (7)

Ans. Find its solution in the text portion.

(*ii*) A bead of mass *m* slides without friction on a smooth rod along the *x*-axis. The rod is equidistant between two spheres of mass M. the spheres are located at $x = 0$, $y = \pm a$ and attract the attract the bead gravitationally.

(*a*) Find the potential energy of the bead.

(*b*) The bead is released at $x = 3a$ with velocity v_0 towards the origin. Find the speed as it passes the origin.

(*c*) Find the frequency of small oscillations of the bead about the origin.

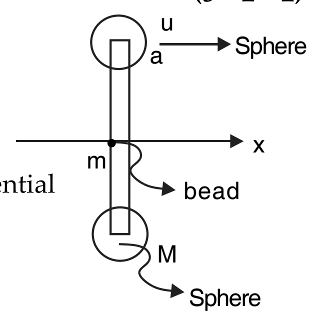
(3 + 2 + 2)

Ans. (*a*) Mass of each sphere = M

Mass of bead = m

Co-ordinates of sphere = $(0, \pm a)$

Co-ordinates of bead = (0, 0)



In this position bead would have gravitational potential energy due to two spheres

- So gravitational potential energy (U) = $\frac{-GM_1M_2}{r}$

Gravitational P.E. of bead = P.E due to sphere at $(0, +a)$

+ P.E. due to sphere at $(0, -a)$

$$= \frac{-GMm}{a} + \left(\frac{-GMm}{a}\right) = \frac{-2GMm}{a}$$

(*b*) Applying consumption of energy for bead sphere system between the two points $x = 3a$ and $x = 0$, will get our answer as

$$\text{M.E. at } x = 3a = \text{M.E. at } x = 0$$

$$\text{K.E. of bead} + \text{P.E. of bead} = \text{K.E. of bead} + \text{P.E of bead}$$

$$\frac{1}{2}\cancel{m}v_0^2 + \left[\frac{-GM\cancel{m}}{\sqrt{10a^2}}\right] = \frac{1}{2}\cancel{m}v^2 + \left[\frac{-2GM\cancel{m}}{a}\right]$$

$$\frac{V_0^2}{2} - \frac{GM}{\sqrt{10}a} = \frac{V^2}{2} - \frac{2GM}{a}$$

$$\frac{V_0^2}{2} - \frac{GM}{\sqrt{10}\,a} + \frac{2GM}{a} = \frac{V^2}{2}$$

$$V_2 = V_0^2 - \frac{2GM}{a}\left[\frac{1}{\sqrt{10}} - 2\right]$$

$$V = \sqrt{V_0^2 - \frac{2GM}{a}\left[\frac{1}{\sqrt{10}} - 2\right]}$$

(*c*) • Please refer diagram for better understanding of solution.

Let bead of mass is at x_m from the origin on x – axis and $x << a$, two spheres of mass M each attracts the bead gravitational towards themselves.

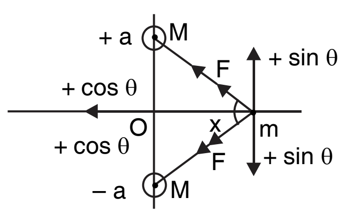
• As masses and distances for the two sphere is same so the force experienced by bead of mass m due to each sphere is also same and let it be F.

• Resolving F into two components one along horizontal towards origin and other perpendicular to x-axis.

• The y-component perpendicular to x-axis of F will cancel out due to force of two spheres on bead

• The x-component will adds up as providing net force along origin (let is be I_n net)

$$\vec{F}_{x\,net} = -2F\cos q$$



where $$F = \frac{GMm}{\left(\sqrt{x^2 + a^2}\right)^2}$$

$$\vec{F}_{x\,\text{net}} = -2\frac{GMm}{(x^2+a^2)} \cdot \frac{x}{\sqrt{x^2+a^2}} \qquad \left[\cos\theta = \frac{x}{\sqrt{x^2+a^2}}\right]$$

$$\vec{F}_{x\,\text{net}} = -2\frac{GMm\,x}{(x^2+a^2)^{\frac{3}{2}}} \qquad \ldots (1)$$

According to Newton's 2nd law :

$$\vec{F}_{\text{net}} = m\,\vec{a}$$

Using equation (1) we get

$$\cancel{m}\,a = -2\frac{GM\cancel{m}\,x}{(x^2+a^2)^{\frac{3}{2}}}$$

considering $x << a$ the $(x^2+a^2)^{\frac{3}{2}} \approx a^3$, so

$$a = -2\frac{GMm}{a^3}x \qquad \left[\text{where } a = \frac{d^2x}{dt^2}\right]$$

$$\frac{d^2x}{dt^2} + \frac{2GMm}{a^3}x = x$$ Which is differential equation of SHM

So, $$w^2 = \frac{2GMm}{a^3}$$ and

$$w = \sqrt{\frac{2GMm}{a^3}} \quad \text{or} \quad f = \frac{1}{2\pi}\sqrt{\frac{2GMm}{a^3}}$$

(*iii*) A particle of mass *m* moves in the central force field with the force function *f*(*r*) = – K*r*, with K > 0. Find the effective potential energy and hence show that all the orbits are bounded. Find the radius and period of circular orbits, if any. (7)

Ans. Find its solution in this text book portion.

Q. 5. (*i*) What do you understand by 'logarithmic decrement', 'relaxation time' and 'quality factor' of a weakly damped harmonic oscillator? Show that the average energy of a weakly damped harmonic oscillator decays exponentially with time. (3 + 4)

Ans. If T is period of as C illation, we can write equation of damped oscillator as $a_1 = a_0 e^{-bT}$

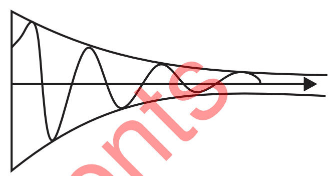
So that $\frac{a_0}{a_1} = e^{bT}$

Since $a_0 > a$, thus the ratio called decrement and its denoted by 'd'

For $t = T$ to $t = 2T$

equation of weakly damped oscillator can be written as

$$\frac{a_1}{a_2} = \frac{a_0 e^{(-bT)}}{a_0 e^{(-2bT)}} = e^{bT}$$



$\therefore$ in general $\frac{a_{n-1}}{a_n} = e^{bT} = d$

The logarithmic of ratio of successive amplitudes of oscillation separated by one period is called logarithmic decrement and its denoted by λ

$$\lambda = l_n \left(\frac{a_{n-1}}{a_n} \right) = \lambda \frac{T}{2m}$$

Relaxation Time : Its a way of expressing the damping effects by means of time taken by amplitude to decay to $e{-1} = 0.368$ of its original value again equation damped oscillation $a(t)\ a_0 e^{-bT}$

and $a(t_1 + t) = a_0 e^{[-b(t1 + t)]}$

Now $\frac{a(t_1 + t)}{a(t)} = e^{-bt_1}$

Quality factor : It another way of expressing damping effects by means of decay of energy for such oscillator, $\langle E \rangle = E_0 e^{-2bt}$

and oscillator decays to $E_0 e^{-1}$ in time $t = \frac{1}{2b} = \frac{m}{y}$ sec.

If w_0 is angular frequency, then in this time oscillation will vibrate through w_0m/y rad.

The no. of radians through which a weakly damped system oscillator as its average energy decays to Eoe^{-1} is measure of quality factor Q

$$Q = \frac{w_0 m}{y} = \frac{w_0}{2b} = \frac{w_0 t}{2}$$

undammped oscillation ($y = 0$) has infinite for weakly. Damped oscillation.

$$w_0 \approx w - \sqrt{\frac{k}{m}} \qquad \therefore \qquad Q = \sqrt{\frac{km}{y^2}}$$

Average Energy of W.D Oscillation :

Total Energy $E = KE(t) + U(t)$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} kx^2$$

instantaneous disp of weakly damp oscillation is given

$$x(t) = a_0 e^{(-bt)} \quad \cos \ (w_0 t + \phi)$$

$$x(t) = V = -a_0 \, e^{-bt} \, [b \cos (w_0 t + \phi) + w_0 \sin(w_0 t + \phi)]$$

$$E = \frac{1}{2} ma^2 \, . \, e^{(-2bt)} \, [b^2 \cos^2 (w_0 t + \phi) + w_0^2 \sin^2 (w_0 t + \phi)$$

$$+ bw_0 \sin 2 \, (w_0 t + \phi)]$$

$$KE = \frac{1}{2} ma^2 e^{(-2bt)} \, [b \cos (w_0 t + \phi) + w_0 \sin (w_0 t + \phi)]^2$$

$$v = \frac{1}{2} kx^2 = \frac{1}{2} nw^2 \, . \, x^2$$

$\therefore$ $K = mw^2$

Substituting x

$$v = \frac{1}{2} ma^2 w^2 \, . \exp(-2bt) \, [(b^2 + w^2) \cos^2 (w_0 t + \phi)$$

$$E(t) = \frac{1}{2} ma^2 \, e^{(-2bt)} \, [(b^2 + w^2) \cos^2 (w_0 t + \phi)$$

$$w_0^2 \sin^2 (w_0 t + \phi) + bw_0 \sin 2(w_0 t + \phi)]$$

if damping is small, amplitude decrease very less $\therefore$ e^{-2bt} is nearly constant

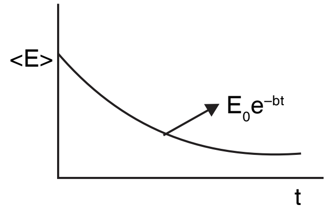
and $\langle \sin^2(w_0 t + \phi) \rangle = \langle \cos^2 (w_0 t + \phi) \rangle = 1/2$

$\langle \sin(w_0 t + \phi) \rangle = 0$

$$\therefore \qquad \langle E \rangle = \frac{1}{2} ma^2 \, e^{-2bt} \langle [b^2 + w^2) \cos^2 (w_0 t + \phi) +$$

$$w_0^2 \sin^2 (w_0 t + \phi) + b w_0 2(w_0 t + \phi)] \rangle$$

$$= \frac{1}{2}ma^2\ e^{-2bt}\left[\frac{b^2+w^2}{2}+\frac{w_0^2}{2}\right]$$

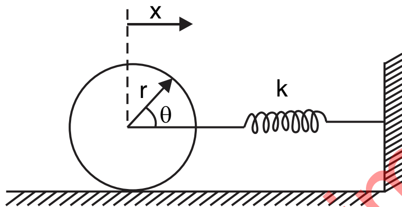


$$= \frac{1}{2}ma^2\,w^2\,e^{-2bt} \quad ; \quad E_0 = \frac{1}{2}ma^2w^2$$

$$= E_0 e^{-2bt}$$

Hence showed.

(*ii*) A circular solid cylinder of radius *r* and mass *m* is connected to a spring of spring constant *k* as shown in the figure below.



Determine the frequency of horizontal oscillations of the system if the cylinder :

(*a*) Slips on the surface without rolling.

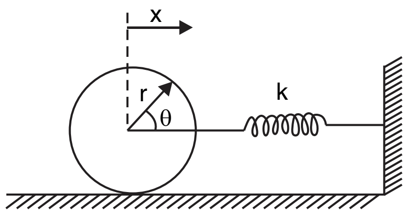
(*b*) Rolls on the surface without slipping.

(*c*) Neglect friction.

Ans. Let Mass of cylinder = M

Radius of cylinder = *r*

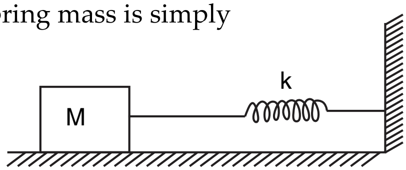
Spring constant = *k*



(*a*) As mentioned in the question, if there is no fiction then their will be no torque on cylinder, so if we displace cylinder a displacement *x* from initial position of equilibrium, then the cylinder will perform only translation motion and for simplicity we can replace cylinder with a block of mass M and the entire situation will look like this system will perform S.H.M.

And simply time period of oscillation of spring mass is simply

$$T = 2\pi\sqrt{\frac{m}{k}}$$



Or we can calculate using energy as :

At any position *x*, the total energy associated with mass M performing SHM is

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E(\text{total}) \qquad \text{... (1)}$$

For SHM total energy is constant :

Differentiating equation (1) w.r.t time

$$\frac{1}{2} m \times 2v \frac{dv}{dt} + \frac{1}{2} k \, 2x \, . \frac{dx}{dt} = 0$$

$$m \cancel{v} \, a + kx \, \cancel{v} = 0$$

$$ma = -kx$$

$$a = -\frac{k}{m} x$$

$$\frac{d^2 x}{dt^2} + \frac{k}{m} x = 0$$

$$w^2 = \frac{k}{m} \qquad \Rightarrow \qquad w = \sqrt{\frac{k}{m}}$$

or

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

(b) Here it is mentioned that cylinder a performing pure rolling or rolling without slipping initially so it doesn't matter whether friction is present or not so to calculate frequency of oscillation. Let's displace cylinder of mass M by x away from horizontal and writing the total energy of oscillation of cylinder.

$$\frac{1}{2} \text{K.E.}_{\text{translated}} + \text{K.E.}_{\text{Rotation}} + \text{P.E.}_{\text{spring}} = \text{E(total)} \qquad \text{... (1)}$$

$$\frac{1}{2} \text{MV}^2 + \frac{1}{2} \times \frac{1}{2} \text{M} r^2 w^2 + \frac{1}{2} kx^2 = \text{E} \qquad \text{... (2)}$$

$$w = \frac{\text{V}}{r} \quad r_0$$

$$\frac{1}{2} \text{MV}^2 + \frac{1}{4} \text{MV}^2 + \frac{1}{2} kx^2 = \text{E} \qquad \text{... (3)}$$

Differentiate equation (3) w.r.t. time and using the fact that, energy of SHM is constant.

$$\frac{1}{2} \text{MV}^2 + \frac{1}{4} \text{MV}^2 + \frac{1}{2} kx^2 = \text{E}$$

$$\frac{1}{\cancel{2}} \text{M2V} \frac{dv}{dt} + \frac{1}{\cancel{4}_2} \text{M2V} \frac{dv}{dt} + \frac{1}{\cancel{2}} k \times \cancel{2} x \frac{dx}{dt} = 0$$

$$M \cancel{v} a + \frac{1}{2} M \cancel{v} a + kx \cancel{v} = 0$$

$$\frac{3}{2} M a = -kx$$

$$a = \frac{-2k}{3M} x \qquad \left[\text{using } G = \frac{d^2x}{dt^2}\right]$$

$$\frac{d^2x}{dt^2} + \frac{2k}{3M} x = 0$$

$$w^2 = \frac{2k}{3M} \quad \Rightarrow \quad w = \sqrt{\frac{2k}{3M}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{2k}{3M}}$$

(*iii*) A particle of mass with velocity v_0 collides elastically with another particle of mass M at rest, and is scattered through angle θ in the centre of mass frame. Show that the final velocity of mass m in the laboratory frame is :

$$v_f = \left(\frac{v_0}{m + M}\right)(m^2 + M^2 + 2mM\cos\theta)^{1/2}$$

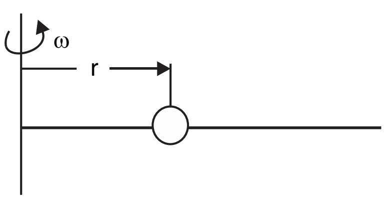
Also find the fractional loss of kinetic energy of mass m if $m = M$. (7)

~~**Ans.**~~

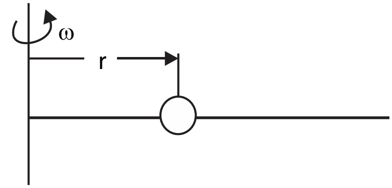
Q. 6. (*i*) How does the rotation of Earth about its axis affect the acceleration due to gravity experienced by a body at rest at a point on the surface of earth ? Support your answer with a suitable derivation and diagram. (7)

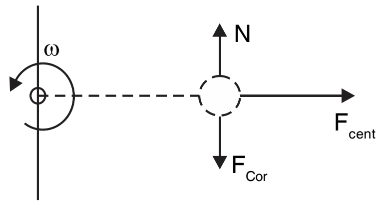
Ans. Find the solution to this question in this text book.

(*ii*) A bead of mass 'm' slides without friction on a rigid wire rotating at constant angular speed ω as shown in the figure. Find an expression for the force exerted by the wire on the bead that is initially at rest at a distance r_0 from the axis. (7)



Ans. A bead slides without friction on a rigid wire rotating at constant angular speed ω. The problem is to find the force exerted by the wire on the bead.





In a coordinate system rotating with the wire the motion is purely radial. F_{cent} is the centrifugal force and F_{cor} is the Coriolis force. Since the wire is frictionless, the contact force N is normal to the wire. In the rotating system the equations of motion are

$$F_{cost} = m\ddot{r} \text{ and } N - F_{Cor} = 0.$$

Since, $\qquad F_{cent} = m\omega^2 r, \quad m\ddot{r} - m\omega^2 r = 0,$

Hence, $\qquad r = Ae^{\omega t} + Be^{-\omega t},$

where A and B are constants depending on the initial conditions.

The other equation gives :

$$N = F_{Cor} = 2m\dot{r}\omega$$

$$= 2m\omega^2 (Ae^{\omega t} - Be^{-\omega t}).$$

To complete the problem, apply the initial conditions which specify A and B.

Consider the following initial conditions and find the final value of N.

(*i*) at $t = 0, r = 0$; (*ii*) at $t = 0, r = 0, \dot{r} = v_0$; (*iii*) at $t = 0, r = a, \dot{r} = 0$.

(*iii*) The space and time coordinates of two events as measured in frame S are :

Event 1 : $x_1 = x_0, t_1 = x_0/c, y_1 = z_1 = 0,$
Event 1 : $x_2 = 2x_0, t_2 = x_0/c, y_2 = z_2 = 0.$

Find the velocity of another frame S' in which the second event occurs by time $x_0/2c$ before the first event. **(7)**

Ans. Apply the formula of principal of simultaneity and get the answer of this question.

Q. 7. (*i*) Derive the expression for relativistic Dopplar's effect. **(7)**

Ans. The sound of a train's horn shifts in frequency as the train passes by due to the relative motion of the train and the one who hears it. Similarly, the frequency of light shifts due to relative motion of the source and observer, even without relativity. Relativity modifies the **Doppler Effect** due to time dilation.

Consider a source at rest at the origin with an observer moving the x direction. We will consider the possibility that the observer is at some distance in y. The beginning of one wavelength is at $t_1 = 0$ and $x_1 = y_1 = 0$. The end of the wave is emitted at $t_2 = \tau$ and still at $x = 2 = y_2 = 0$. This transforms to the observers frame to be at

$$
\begin{aligned}
ct'_1 &= \gamma(ct_1 - \beta x) = 0 \\
x'_1 &= \gamma(x - \beta ct) = 0 \\
y'_1 &= y_1 = 0 \\
ct'_2 &= \gamma(c\tau - \beta x) = \gamma c\tau \\
x'_2 &= \gamma(x - \beta c\tau) = -\beta\gamma c\tau \\
y'_2 &= y_2 = 0 \\
\tau' &= \gamma\tau
\end{aligned}
$$

The time to emit the wave in the observers frame is dilated which decreases the frequency. if the wave travels to the observer in the y direction, the travel time is essentially the same for the beginning and the end of the wave so the frequency is not affected. That is the **Transverse Doppler Effect** gives a red-shifts

$$v'_{\perp} = \frac{v}{\gamma}$$

which is entirely a relativistic effect.

If the observer is moving directly away from the source we have the additional effect of the distance to the observer increasing with time which gives rise to the **parallel Doppler Effect**. The time at which the beginning and end of the wave arrive at the observer is

$$t'_{10} = t'_1 - \frac{x'_1}{c} = 0$$

$$t'_{20} = t'_2 - \frac{x'_2}{c} = \gamma\tau + \beta\gamma\tau = \gamma(1+\beta)\tau$$

$$\tau'_0 = \gamma(1+\beta)\tau = \frac{(1+\beta)}{\sqrt{(1+\beta)(1-\beta)}}\tau = \sqrt{\frac{1+\beta}{1-\beta}}\tau$$

$$v'_{\parallel} = \sqrt{\frac{1-\beta}{1+\beta}}v$$

β is positive for the observer moving away from the source and negative if the observer is moving toward the source.

(*ii*) A particle with a rest mass m_0 and kinetic energy $3m_0c^2$ makes a completely inelastic collision with a stationary particle of rest mass $2m_0$, without any radiation loss and two particles forming a composite particle. What is the rest mass of the composite particle and its speed ? (7)

Ans. Please check your text book for the solution to this equation.

(*iii*) (*a*) Suppose that a particle move relative to O' with a constant velocity of $c/2$ in the $x'y'$-plane such that its trajectory makes an angle of 60° with x-axis. If the velocity of O' with respect to O is 0.6c along the x-x'-axis, find the equations of motion of the particle as determined by O.

(*b*) Define proper time. What is time dilation? With what velocity should a rocket move so that as observed from Earth every year spent on the rocket corresponds to 4 years on Earth ? (4 + 3)

Ans. (*a*) Check your text book for its solution.

(*b*) Just apply dilation formula to get its solution.

Proper time is called clock time or process time, and it is a measure of the amount of physical process that a system undergoes. For example, proper time for an ordinary mecanical clock is recorded by the number of rotations of the hands of the clock. Alternatively, we might take a gyroscope, or a freely spinning wheel, and measure the number of rotations in a given period.