B.Sc. (Physical Science)
Mechanics (2017)

Sr. No. of Question Paper : 6871 **HC**

2017

Unique Paper Code : 42221101

Name of the Paper : Physics - I (Mechanics)

Name of the Course : **B.Sc. (Prog.)** Science

Semester : I

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **five** questions in all, including **Q. No. 1** which is compulsory.
3. Use of nonprogrammable calculator is allowed.

1. Attempt any **five** of the following: $(5\times3=15)$

   (a) What are the two postulates of special theory of relativity?

   (b) Differentiate b/w inertial and non-inertial reference frames with one example.

P.T.O.

(c) What do you understand by inertial mass and gravitational mass?

(d) Show that addition of vectors is associative

i.e. $\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}$

(e) If $\vec{r}$ = (t^3+2+) **i**-$3e^{-2t}$ **j**+2 sin5t **k**, find d^2r/dt^2 at t = 0.

(f) Solve differential equation $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0$

(g) A spring of spring constant 'k' is loaded by mass 'm'. If this spring is cut half in length and same mass is loaded on it, what will be new time period?

(h) What is Hook's law in elasticity? What do you mean by elastic limit and breaking stress?

2. (a) What are polar and axial vectors? Give one example of each. (5)

(b) Solve differential equation $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0$,

where, y = 2 and $\frac{dy}{dx}=\frac{d^2y}{dx^2}$, when $x = 0$ (5)

(c) Prove that $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}.\vec{C})-\vec{C}(\vec{A}.\vec{B})$ (5)

3. (a) Find the centre of mass of a uniform solid hemisphere of radius 'R'. (5)

(b) A bomb of mass **'4M'** in flight explodes into two fragments when its velocity is $(10\hat{i}+2\hat{j})$ *m/sec*. If the smaller mass *'M'* files with velocity $(20\hat{i}+50\hat{j})$ *m/sec*, deduce the velocity of the larger mass **'3M'**. (5)

(c) State and prove the 'work-energy' theorem. (5)

4. (a) Describe the principle of a rocket? Why multistage rocket is necessary? Establish the following relation for a rocket, $V = V_0 + v \log_e (M_0/M)$, where v is the exhaust Velocity of gases relative to rocket, M_0, V_0 are initial mass and velocity of rocket respectively, M and V are mass and velocity of rocket at any time 't'. (3,3,4)

(b) For a particle of mass m = 10 gm, position $\vec{r}=(10\hat{i}+6\hat{J})$ cm and velocity $\vec{v}=5\hat{i}$ cm/s, calculate the angular momentum about the z-axis through origin. (5)

5. (a) Show that the ratio of rotational to translational kinetic energy for a solid cylinder rolling down a plane without slipping is 1:2. (5)

P.T.O.

(b) Define the central forces. Show that the path or orbit of the particle under the central force must be a plane curve and its areal velocity is constant. (2,4,4)

6. (a) What do you understand by a damping or a dissipative force? Deduce the differential equation of damped harmonic oscillator and discuss in detail the case of critical and under-damped cases. (10)

(b) A smooth straight tunnel is board through the centre of the earth. A particle of mass 'm' is dropped into tunnel. Prove that the motion is simple harmonic and calculate its time period. (5)

7. (a) If Y, η and K represent Young modulus, coefficient of rigidity and bulk modulus, respectively, then prove that,

$$\frac{9}{y}=\frac{3}{\eta}+\frac{1}{K}$$ (5)

(b) Write down the Lorentz space-time transformation equations. Discuss the time dilation in special theory of relativity. (5)

(c) Two objects are moving in the opposite direction, each with a speed of 0.9c. Find the relative speed of the two objects. (*c=velocity of light in free space*) (5)

Q 1 (a)

Ans. 1.The laws of physics all take the same identical form for all frames of reference in uniform relative motion i.e. For all inertial frames of reference.

2. The velocity of light in free space is same relative to any inertial frame of reference i.e. it is invariant to transformation from one inertial frame to another for all observers irrespective of their state of motion.

Q1 (b) Ans. Inertial Frame of Reference: A reference frame in which Newton's first law of motion holds good, is known as an inertial frame of reference. All frames of reference, moving with a constant velocity with respect to an inertial frame , are also inertial frames of reference.
Ex: A train moving with uniform velocity is an inertial frame.

Non Inertial Frame of Reference: A non inertial frame is one which is accelerated (linearly or due to rotation) with respect to fixed stars. Newton's second law of motion is not valid in such a frame of reference, unless we introduce a force called pseudo - force.
Ex: A freely falling elevator may be taken as a non inertial frame.

Q 1(c) Ans.

1) Inertial mass: This is mainly defined by Newton's law, F = ma, which states that when a force F is applied to an object, it will accelerate proportionally, and that constant of proportion is the mass of that object. In very concrete terms, to determine the inertial mass, you apply a force of F Newtons to an object, measure the acceleration in m/s^2, and F/a will give you the inertial mass m in kilograms.

2) Gravitational mass. This is defined by the force of gravitation, which states that there is a gravitational force between any pair of objects, which is given by

$$F = G\, m_1 m_2/r^2$$

where G is the universal gravitational constant, m_1 and m_2 are the masses of the two objects, and r is the distance between them. This, in effect defines the gravitational mass of an object.

Q 1(d) Ans.

The law states that the sum of vectors remains same irrespective of their order or grouping in which they are arranged.

Consider three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$

Applying "head to tail rule" to obtain the resultant of $(\vec{A} + \vec{B})$ and $(\vec{B} + \vec{C})$

Then finally again find the resultant of these three vectors :

$$\overline{OR} = \overline{OP} + \overline{PR}$$

or

$$\vec{R} = \vec{A} + (\vec{B} + \vec{C}) \longrightarrow \text{(i)}$$

and

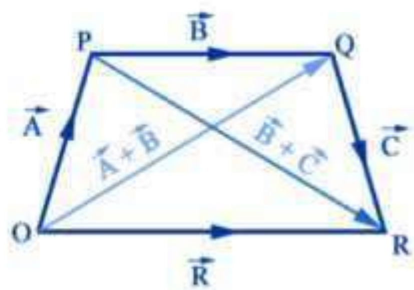
$$\overline{OR} = \overline{OQ} + \overline{QR}$$

or

$$\vec{R} = (\vec{A} + \vec{B}) + \vec{C} \longrightarrow \text{(ii)}$$

thus, from (i) and (ii)

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$



This fact is known as the **ASSOCIATIVE LAW OF VECTOR ADDITION**.

Q 1 (e)Ans

c) Given, $\vec{r} = (t^3 + 2t)\hat{i} - 3e^{-2t}\hat{j} + 2\sin 5t\,\hat{k}$

$$\frac{dr}{dt} = (3t^2 + 2)\hat{i} - 3(-2)e^{-2t}\hat{j} + 2(5)\cos 5t\,\hat{k}$$

$$\frac{d^2r}{dt^2} = 6t\,\hat{i} + 6(-2)e^{-2t}\hat{j} - 10(5)\sin 5t\,\hat{k}$$

At t=0

$$\frac{d^2r}{dt^2} = 0\hat{i} - 12\hat{j} + 0\hat{k}$$

Ans: $\frac{d^2r}{dt^2} = -12\hat{j}$

Q1 (f)Ans

(f) $$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

$$(D^2 - 6D + 9)y = 0$$

$$(D-3)^2 y = 0 \Rightarrow D = 3, 3$$

Then $y = (c_1 + c_2 x)\,e^{3x}$

Q1 (g)Ans.

g) Time period of oscillation is given by

$$T = 2\pi\sqrt{\frac{m}{K}}$$

If spring is cut into 2 ~~hal~~ halves, then,

$$K' = 2K$$

$$T' = 2\pi\sqrt{\frac{m}{2K}} = \frac{T}{\sqrt{2}}$$

Q 1 (h) Ans.

Hooke's law **states that, for relatively small deformations of an object, the displacement or size of the deformation is directly proportional to the deforming force or load. Under these conditions the object returns to its original shape and size upon removal of the load.**

Mathematically, Hooke's law states that the applied force *F* equals a constant *k* times the displacement or change in length *x*, or *F* = *kx*. The value of *k* depends not only on the kind of elastic material under consideration but also on its dimensions and shape.

Elastic limit: the maximum extent to which a solid may be stretched without permanent alteration of size or shape.

Breaking stress: Breaking stress is the maximum force that can be applied on a cross sectional area of a material in such a way that the material is unable to withstand any additional amount of stress before breaking.
Breaking Stress = Force / Area

Q 2(a) Ans.

Axes: Now we shall choose a set of axes. The simplest set of axes are known as the Cartesian axes, x-axis, y-axis, and the z-axis. In Figure 2.1.1, we draw these axes.

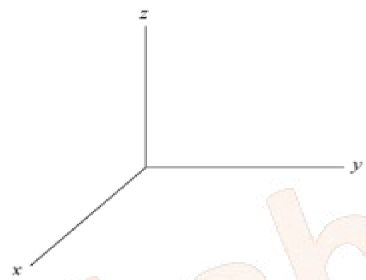
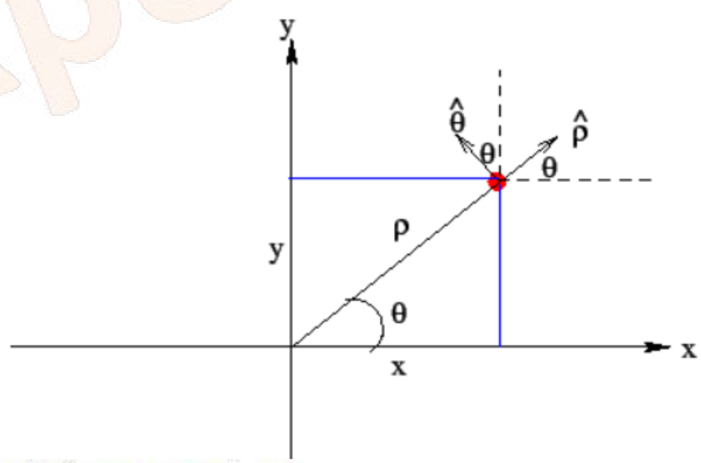


Figure 2.1.1 Cartesian coordinates

Then each point P in space our S can be assigned a triplet of values (x_P, y_P, z_P), the coordinates of the point P. The ranges of values of the coordinates are: $-\infty < x_P < +\infty$, $-\infty < y_P < +\infty$, $-\infty < z_P < +\infty$.

In two dimensions one defines the polar coordinate (ρ, θ) of a point by defining ρ as the radial distance from the origin O and θ as the angle made by the radial vector with a reference line (usually chosen to coincide with the x-axis of the cartesian system). The radial unit vector $\hat{\rho}$ and the tangential (or angular) unit vector $\hat{\theta}$ are taken respectively along the direction of increasing distance ρ and that of increasing angle θ respectively, as shown in the figure.



Relationship with the cartesian components are

$$x = \rho\cos\theta$$
$$y = \rho\sin\theta$$

By definition, the distance $\rho > 0$. we will take the range of angles θ to be $0 \le \theta < 2\pi$ (It is possible to define the range to be $-\pi \le \theta < +\pi$). One has to be careful in using the inverse tangent as the arc-tan function is defined in $0 \le \theta < \pi$. If y is negative, one has to add π to the principal value of θ calculated by the arc - tan function so that the point is in proper quadrant.

Q 2(b) Ans.

2(b) Given equation is

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$$

A.E : $m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i$$

$$\therefore \quad y = e^{-2x}(C_1 \cos x + C_2 \sin x)$$

Boundary conditions : ① $y = 2 \quad x = 0$

$$2 = e^{-2(0)}(C_1 \cos 0 + C_2 \sin 0)$$

$$2 = C_1$$

② $\frac{dy}{dx} = \frac{d^2y}{dx^2}$ at $x = 0$

$$\frac{dy}{dx} = (-2)e^{-2x}(C_1 \cos x + C_2 \sin x) + e^{-2x}(C_1(-\sin x) + C_2 \cos x)$$

$$\left.\frac{dy}{dx}\right|_{x=0} = -2C_1 + C_2$$

$$C_1 = 2.$$

$$\left.\frac{dy}{dx}\right|_{x=0} = -4 + C_2.$$

$$\frac{d^2y}{dx^2} = \left[(-2)(-2)e^{-2x}(C_1 \cos x + C_2 \sin x) + (-2)e^{-2x}(C_1(-\sin x) + C_2 \cos x)\right] + \left[(-2)e^{-2x}(C_1(-\sin x) + C_2 \cos x) + (-e^{-2x})(C_1 \cos x + C_2 \sin x)\right]$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = [4c_1 - 2c_2 - 2c_2 - c_1]$$

$$= 3c_1 - 4c_2$$

$$c_1 = 2$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 6 - 4c_2$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}$$

$$6 - 4c_2 = -4 + c_2$$

$$10 = 5c_2$$

$$c_2 = 2$$

$$\therefore \quad y = 2e^{-2x}(\cos x + \sin x)$$

Q 2(c) Ans.

Note that the vector $\vec{G} = \vec{B} \times \vec{C}$ is perpendicular to the plane on which vectors $\vec{B}$ and $\vec{C}$ lie. Thus, taking the cross product of vector $\vec{G}$ with an arbitrary third vector, say $\vec{A}$, the result will be a vector perpendicular to $\vec{G}$ and thus lying in the plane of vectors $\vec{B}$ and $\vec{C}$. Therefore, one can express the vector $\vec{F} = \vec{A} \times \vec{G}$ as a linear combination of the vectors $\vec{B}$ and $\vec{C}$, i.e.,

$$\vec{F} = m\vec{B} + n\vec{C}$$

Taking the scalar product of the both sides of this expression with vector $\vec{A}$, and noting that $\vec{A} \cdot \vec{F} = 0$ one obtains

$$m(\vec{A} \cdot \vec{B}) + n(\vec{A} \cdot \vec{C}) = 0$$

For this equality to be valid for any $\vec{A}$, $\vec{B}$ and $\vec{C}$, one is tempted to write

$$m = \lambda(\vec{A} \cdot \vec{C}), \quad n = -\lambda(\vec{A} \cdot \vec{B})$$

in which the unknown proportionality constant λ has been introduced so as serve for the above solutions to hold true with no loss in generality.

Thus, one has

$$\begin{aligned} \vec{F} &= \vec{A} \times (\vec{B} \times \vec{C}) \\ &= \lambda\left\{(\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}\right\} \end{aligned}$$

Selecting arbitrarily $\vec{A} = \hat{k}$, $\vec{B} = \hat{j}$, and $\vec{C} = \hat{k}$, for instance, and substituting in the above equality, one obtains $\lambda = 1$.

Hence, one eventually obtains the vector identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

Q 3(a) Ans.

The center of mass formula is for a figure with density $\rho(\mathbf{r})$ is given by

$$\mathbf{R} = \frac{\iiint_V \rho(\mathbf{r})\mathbf{r}\, dV}{\iiint_V \rho(\mathbf{r})\, dV}$$

In this case, because the density is constant, we can just set it equal to 1 and then it effectively drops out of the equation.

In spherical coordinates, the upper hemisphere of a ball of radius R is given by $0 \le r \le R,\ 0 \le \theta \le \frac{\pi}{2}$,

Now we need to plug these into the two integrals in our center of mass equation:

$$\begin{aligned}\iiint_V dV &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R r^2 \sin(\theta)\, dr d\theta d\varphi \\ &= \frac{R^3}{3} \int_0^{2\pi} \int_0^{\pi/2} \sin(\theta)\, d\theta d\varphi \\ &= \frac{R^3}{3} \left[-\cos(\theta)\right]\Big|_0^{\pi/2} \int_0^{2\pi} d\varphi \\ &= \frac{R^3}{3}(2\pi) \\ &= \frac{2\pi R^3}{3}\end{aligned}$$

Note that this part is just the volume of a hemisphere, which you could have also obtained by halving the known volume of a sphere. Confirm that this produces the same result.

Now for the other integral:

$$\begin{aligned}\iiint_V \mathbf{r}\, dV &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R r^2 \sin(\theta) \big[r \sin(\theta)\cos(\varphi)\mathbf{x} + r\sin(\theta)\sin(\varphi)\mathbf{y} \\ &\qquad + r\cos(\theta)\mathbf{z}\big]\, dr d\theta d\varphi \\ &= \frac{R^4}{4}\Bigg[\mathbf{x}\int_0^{2\pi}\int_0^{\pi/2} \sin^2(\theta)\cos(\varphi)\, d\theta d\varphi \\ &\qquad + \mathbf{y}\int_0^{2\pi}\int_0^{\pi/2} \sin^2(\theta)\sin(\varphi)\, d\theta d\varphi \\ &\qquad + \mathbf{z}\int_0^{2\pi}\int_0^{\pi/2} \sin(\theta)\cos(\theta)\, d\theta d\varphi\Bigg] \\ &= \frac{R^4}{4}\left[0\mathbf{x} + 0\mathbf{y} + \pi\mathbf{z}\right] \\ &= \frac{\pi R^4}{4}\mathbf{z}\end{aligned}$$

Putting these together, we thus find that the center of mass is at

$$\mathbf{R} = \frac{\frac{\pi R^4}{4}\mathbf{z}}{\frac{2\pi R^3}{3}} = \frac{3}{8}R\,\mathbf{z}$$

So the center of mass of a hemisphere is $\frac{3}{8}$ of the radius along the axis of symmetry from the flat side of the hemisphere.

Q 3(b) Ans.

(b):

mass $= 4M$

velocity $= 10\hat{i} + 2\hat{j}$ m/sec

smaller mass $= M$

its velocity $= (20\hat{i} + 50\hat{j})$ m/s

Acc. to conservation of momentum,

$$4M(10\hat{i} + 2\hat{j}) = M(20\hat{i} + 50\hat{j}) + 3M(x\hat{i} + y\hat{j})$$

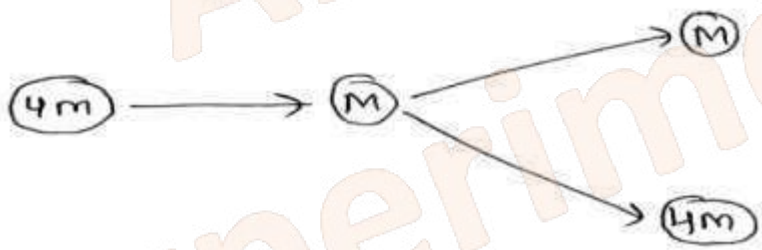
$$40\hat{i} + 8\hat{j} = 20\hat{i} + 50\hat{j} + 3x\hat{i} + 3y\hat{j}$$

Equating $\hat{i}$ and $\hat{j}$ components separately,

$40 = 20 + 3x \Rightarrow x = 20/3$

$8 = 50 + 3y \Rightarrow y = -14$

$$v = \frac{20}{3}\hat{i} - 14\hat{j}$$



Q 3(c) Ans.

The work-energy theorem states that the work done by all forces acting on a particle equals the change in the particle's kinetic energy.

Now for infinitesimal displacement calculations and forces whether variable or constant, we can write from the definition of work,

$$dW = Fdx$$

Now let the total work done is **W**, and the displacement is from x_0 to x

Integrating the both sides we get,

$$\int_0^W dW = \int_{x_0}^{x} F dx$$

$$W = \int_{x_0}^{x} F dx$$

$$W = m \int_{x_0}^{x} \frac{dv}{dt} dx$$

$$W = m \int_{v_0}^{v} v dv$$

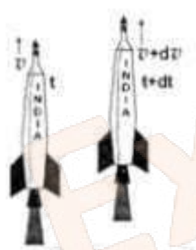
$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

$$W = \Delta E_k$$

(Proved)

Q 4(a) Ans. A rocket in its simplest form is a chamber enclosing a gas under pressure. A small opening at one end of the chamber allows the gas to escape, and in doing so provides a thrust that propels the rocket in the opposite direction. As the rocket moves its mass decreases which gives an additional thrust by conservation of momentum.

Consider, a rocket of mass m_0 take off with velocity v_0 from ground.



Let the fuel burn at the rate of dm/dt and burnt gases eject with velocity u w.r.t. rocket.

Let at any instant t, v be the velocity of rocket and m be the mass of rocket.

Therefore velocity of burnt gas ejected w.r.t. ground is (v − u).

Here, fuel is burning at the rate of dm/dt.

Therefore in time dt, dm mass of the fuel will burn and velocity of rocket increases by dv.

Momentum of rocket at any instant is,

$$p_1 = mu$$

Momentum of (rocket + burnt fuel) at t + dt is,

$$p_2 = (m - dm)(v + dv) + dm(v + dv - u)$$

$$= mv + mdv - udm$$

$\therefore$ *Change in momentum in time dt is,*

$$dp = p_2 - p_1$$

$$= (mv + mdv - udm) - mv$$

$$= mdv - udm$$

We know,

Impulse = Change in momentum

Here, impulse is due to gravitaional force.

$$\therefore \quad -mg\,dt = m\,dv - udm$$

$$\Rightarrow \quad dv = u\frac{dm}{m} - g\,dt$$

Here, we have

'm' is the mass of rocket, and

dm is the mass of gas ejected.

As the fuel burns the mass of rocket decreases.

Therefore, to calculate the velocity of rocket at any instant in terms of mass of rocket, m must be replaced by 'dm'.

$$\therefore dv = -u\frac{dm}{m} - g\,dt$$

Integrating both sides,

$$\int_{v_o}^{v} dv = -u\int_{m_o}^{m}\frac{dm}{m} - g\int_0^t dt$$

$$\Rightarrow \quad v\Big|_{v_o}^{v} = -u\,(\text{Ln}\,m)\Big|_{m_o}^{m} - g\,t\Big|_0^t$$

$$\Rightarrow \quad v - v_0 = -u\,[\text{Ln}\,m - \text{Ln}\,m_o] - gt$$

$$\Rightarrow \quad v = v_0 + \text{Ln}\left(\frac{m_o}{m}\right) - gt$$

Rockets have multiple stages because the effectiveness of a rocket is inversely proportional to its mass and using stages allows us to reduce the mass of the rockets as it operates.

Tsiolkovsky's rocket equation tells us:

$$\Delta v = v_e \ln \frac{m_i}{m_f}$$

In words that means that the change in velocity achievable is equal to the effective exhaust velocity times the natural log of the initial mass divided by the final mass. So we can see that the greater the ratio between the initial and final mass of the rocket, the more effective the rocket can be.

In the below picture are depictions of two rockets. The one on the left is a single stage rocket. The one on the left is a multi-stage rocket.

Q 4(b)

4(b) $m = 10\,gm$. $\vec{r} = (10\hat{i} + 6\hat{j})$ cm

$\vec{v} = 5\hat{i}$ cm/s

angular momentum, $\vec{L} = -m(\vec{r} \times \vec{v})$

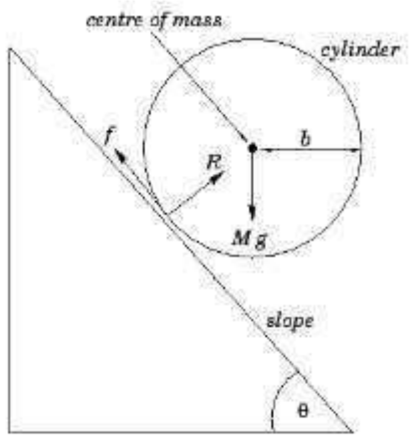
$$\vec{r} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10 & 6 & 0 \\ 5 & 0 & 0 \end{vmatrix}$$

$$= -30\hat{k}$$

$$\vec{L} = -10 \times -30\hat{k} = 300\ g\ cm/sec.$$

Q 5(a) Ans. In a pure translatory motion, all the particles in the body, at any instant of time, have equal velocity and acceleration. Kinetic energy is a scalar quantity with no direction associated with it.

$$\begin{aligned} KE_{\text{translation}} &= \frac{1}{2} m_1 v^2 + \frac{1}{2} m_2 v^2 + \cdots + \frac{1}{2} m_N v^2 \\ &= \frac{1}{2} (m_1 + m_2 + \cdots + m_N) v^2 \\ &= \frac{1}{2} M_{\text{body}} v^2. \end{aligned}$$



In case of a rigid body in pure rotation, all the particles on the body rotates in circular motion with their centers lying on the same axis

$$KE_{PureRotation} = \frac{1}{2} I_{\rm rot} \omega^2,$$

KE = 1/2. $(1/2.MR^2)(v/R)^2$ = 1/4. Mv^2

Hence the ratio of these K.E. is 1:2.

Q 5(b) Ans.

Let a body be moving under a central force law. By Newton's Second Law its motion is given as follows:

$$m\underline{A} = \underline{F}.$$

Here, *m* is the mass (assumed constant), the position vector of the particle is $\underline{X}$, the velocity vector is $\underline{V} = \frac{d}{dt}\underline{X}$ and we have:

$$\underline{A} = \frac{d^2}{dt^2}\underline{X},$$

$$\underline{F} = g(\underline{X}.\underline{X})\underline{X}.$$

Here *g*(*u*) is a function representing the strength of the force which is assumed to depend only on the distance to the center and the force $\underline{F}$ is in the radial direction, so $\underline{F}$ is proportional to the position vector $\underline{X}$.

Now put $\underline{J} = m\underline{X} \times \underline{V}$, so $\underline{J}$ is the "torque" of the momentum vector $m\underline{V}$. Then we have:

$$\frac{d}{dt}(\underline{J}) = m\frac{d}{dt}(\underline{X} \times \underline{V}) =$$

$$= m(\frac{d}{dt}(\underline{X})) \times \underline{V} + m\underline{X} \times \frac{d}{dt}(\underline{V})$$

$$= m\underline{V} \times \underline{V} + m\underline{X} \times \underline{A} = \underline{X} \times \underline{F} = g(\underline{X}.\underline{X})\underline{X} \times \underline{X} = 0.$$

So $\underline{J}$ is constant for a central force law.
The quantity $\underline{J}$ is called the angular momentum of the motion and we have shown that the angular momentum is constant for a central force law. But from its definition, we have $\underline{X}.\underline{J} = 0$.

But this is the equation of a plane through the origin, with normal $\underline{J}$. So we have proved that under a central force law, the motion lies in a plane. The only exception might be the case that $\underline{J}$ is zero. Then we have $V = sX$, for some scalar *s*. From this it is not difficult to show that the motion is along a straight line, so is planar in this case also. Henceforth we assume that $\underline{J}$ is non-zero. Then the motion is in a plane, which we may take to be the (*x*,*y*)-plane.

$$m\left[r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt}\right] = 0$$

This implies that

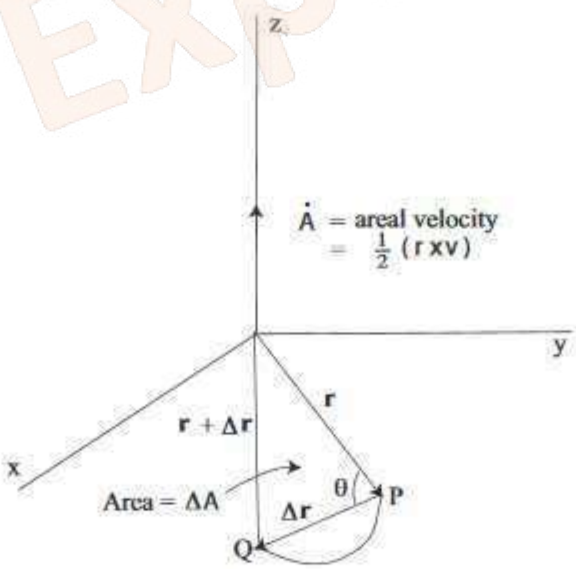
$$\frac{d}{dt}\left(mr^2\frac{d\theta}{dt}\right) = 0$$

or,

$$\frac{dJ}{dt} = 0$$

where,

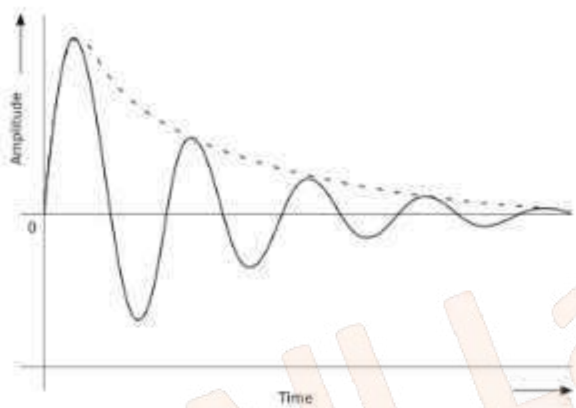
$$J = \left(mr^2\frac{d\theta}{dt}\right) = constant$$



Q 6(a) Ans.

When the motion of an oscillator reduces due to an external force, the oscillator and its motion are **damped**. These periodic motions of gradually decreasing amplitude are damped simple harmonic motion. An example of damped simple harmonic motion is a simple pendulum.

In the damped simple harmonic motion, the energy of the oscillator dissipates continuously. But for small damping, the oscillations remain approximately periodic. The forces which dissipate the energy are generally **frictional forces**.



Let's drive our damped spring-object system by a sinusoidal force. Suppose that the x-component of the driving force is given by

$$F_x(t) = F_0 \cos(\omega t) \ , \qquad (23.6.1)$$

where F_0 is called the *amplitude* (maximum value) and ω is the *driving angular frequency*. The force varies between F_0 and $-F_0$ because the cosine function varies between $+1$ and -1. Define $x(t)$ to be the position of the object with respect to the equilibrium position. The x-component of the force acting on the object is now the sum

$$F_x = F_0 \cos(\omega t) - kx - b\frac{dx}{dt} \ . \qquad (23.6.2)$$

Newton's Second law in the x -direction becomes

$$F_0 \cos(\omega t) - kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \ . \qquad (23.6.3)$$

We can rewrite Eq. (23.6.3) as

$$F_0 \cos(\omega t) = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx \ . \qquad (23.6.4)$$

We shall now use complex numbers to solve the differential equation

$$F_0 \cos(\omega t) = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx \ . \qquad (23.D.1)$$

We begin by assuming a solution of the form

$$x(t) = x_0 \cos(\omega t + \phi) \ . \qquad (23.D.2)$$

where the amplitude x_0 and the phase constant ϕ need to be determined. We begin by defining the complex function

$$z(t) = x_0 e^{i(\omega t + \phi)} \ . \qquad (23.D.3)$$

Our desired solution can be found by taking the real projection

$$x(t) = \mathrm{Re}(z(t)) = x_0 \cos(\omega t + \phi) \ .$$

Our differential equation can now be written as

$$F_0 e^{i\omega t} = m\frac{d^2z}{dt^2} + b\frac{dz}{dt} + kz \ .$$

We take the first and second derivatives of Eq. (23.D.3),

$$\frac{dz}{dt}(t) = i\omega x_0 e^{i(\omega t+\phi)} = i\omega z \ . \tag{23.D.6}$$

$$\frac{d^2z}{dt^2}(t) = -\omega^2 x_0 e^{i(\omega t+\phi)} = -\omega^2 z \ . \tag{23.D.7}$$

We substitute Eqs. (23.D.3), (23.D.6), and (23.D.7) into Eq. (23.D.5) yielding

$$F_0 e^{i\omega t} = (-\omega^2 m + bi\omega + k)z = (-\omega^2 m + bi\omega + k)x_0 e^{i(\omega t+\phi)} \ . \tag{23.D.8}$$

We divide Eq. (23.D.8) through by $e^{i\omega t}$ and collect terms using yielding

$$x_0 e^{i\phi} = \frac{F_0 / m}{((\omega_0{}^2 - \omega^2) + i(b/m)\omega)} \ . \tag{23.D.9}$$

where we have used $\omega_0{}^2 = k/m$. Introduce the complex number

$$z_1 = (\omega_0{}^2 - \omega^2) + i(b/m)\omega \ . \tag{23.D.10}$$

Then Eq. (23.D.9) can be written as

$$x_0 e^{i\phi} = \frac{F_0}{my} \ . \tag{23.D.11}$$

Multiply the numerator and denominator of Eq. (23.D.11) by the complex conjugate $\overline{z}_1 = (\omega_0{}^2 - \omega^2) - i(b/m)\omega$ yielding

$$x_0 e^{i\phi} = \frac{F_0 \overline{z}_1}{m z_1 \overline{z}_1} = \frac{F_0}{m} \frac{((\omega_0{}^2 - \omega^2) - i(b/m)\omega)}{((\omega_0{}^2 - \omega^2)^2 + (b/m)^2 \omega^2)} \equiv u + iv \ . \tag{23.D.12}$$

where

$$u = \frac{F_0}{m} \frac{(\omega_0{}^2 - \omega^2)}{((\omega_0{}^2 - \omega^2)^2 + (b/m)^2 \omega^2)} \ , \tag{23.D.13}$$

$$v = -\frac{F_0}{m}\frac{(b/m)\omega}{((\omega_0{}^2 - \omega^2)^2 + (b/m)^2\omega^2)} \quad . \tag{23.D.14}$$

Therefore the modulus x_0 is given by

$$x_0 = (u^2 + v^2)^{1/2} = \frac{F_0/m}{((\omega_0{}^2 - \omega^2)^2 + (b/m)^2\omega^2)} \quad , \tag{23.D.15}$$

and the phase is given by

$$\phi = \tan^{-1}(v/u) = \frac{-(b/m)\omega}{(\omega_0{}^2 - \omega^2)} \quad . \tag{23.D.16}$$

Oscillator Equation. The solution to is given by the function

$$x(t) = x_0 \cos(\omega t + \phi) \quad , \tag{23.6.5}$$

where the amplitude x_0 is a function of the driving angular frequency ω and is given by

$$x_0(\omega) = \frac{F_0/m}{\left((b/m)^2\omega^2 + (\omega_0{}^2 - \omega^2)^2\right)^{1/2}} \quad . \tag{23.6.6}$$

The phase constant ϕ is also a function of the driving angular frequency ω and is given by

$$\phi(\omega) = \tan^{-1}\left(\frac{(b/m)\omega}{\omega^2 - \omega_0{}^2}\right) \quad . \tag{23.6.7}$$

In Eqs. (23.6.6) and (23.6.7)

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{23.6.8}$$

is the natural angular frequency associated with the undriven undamped oscillator. The x -component of the velocity can be found by differentiating Eq. (23.6.5),

$$v_x(t) = \frac{dx}{dt}(t) = -\omega x_0 \sin(\omega t + \phi) \quad , \tag{23.6.9}$$

where the amplitude $x_0(\omega)$ is given by Eq. (23.6.6) and the phase constant $\phi(\omega)$ is given by Eq. (23.6.7).

Q 6(b) Ans

② For the Falling particle,

$$F_{grav.} = G.\frac{M_{enc.}.m}{r^2}$$

Since density is uniform,

$$\frac{M_{enc.}}{M} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}$$

$$\Rightarrow M_{enc.} = \frac{M.r^3}{R^3}$$

$$\Rightarrow F_{grav.} = G.\frac{Mr^3.m}{R^3.r^2} = \frac{GMmr}{R^3}$$

Using $M = d \times V$ (d: density, V: volume)

$$F_{grav.} = G.d.\frac{4}{3}\pi\frac{R^3.m.r}{R^3} \Rightarrow F_{grav.} = \frac{4\pi}{3}mrGd$$ –

These 2 are the same

Particle will execute SHM like a spring, $F = Kx$ – ②

Comparing ① & ②, we get

$$K = \frac{4\pi}{3}Gdm$$

$$T = 2\pi\sqrt{\frac{m}{K}} \Rightarrow T^2 = 4\pi^2.\frac{m}{\frac{4\pi}{3}Gdm} \Rightarrow \boxed{T^2 = \frac{3\pi}{Gd}}$$

Q 7(a) Ans.

Relation between modulus of rigidity (G) and modulus of elasticity (E):

E	= 2G (1 + μ)
where E	= modulus of elasticity
μ	= Poisson's ratio
G	= modulus of rigidity

Relation between modulus of elasticity(E) and bulk modulus(K):

We known that when body is subjected to a tri-axial stress system, its volumetric strain is given by

$e_V = {}^{\delta V}/_{V}$	$= (^{(\sigma_X + \sigma_Y + \sigma_Z)}/_{E})(1 - 2\mu)$
Here σ_X	$= \sigma_Y = \sigma_Z = \sigma$
$\therefore e_V$	$= (^{3\sigma}/_{E})(1 - 2\mu)$
but K	$= \sigma / e_V = \sigma / (^{3\sigma}/_{E})(1 - 2\mu)$
$\therefore K$	$= E / 3(1 - 2\mu)$

$$\boxed{\therefore E = 3K(1 - 2\mu)}$$

Relation between modulus of elasticity (E), modulus of rigidity (G) and bulk modulus(K):

We know that, $E = 2G(1+\mu)$ ______ (i)

And $E = 3K(1-2\mu)$ ______ (ii)

Form (i) $1 + \mu = \frac{E}{2G}$

$$\therefore \mu = \frac{E}{2G} - 1$$

Equating this value in (i)

$$E = 3K\left[1 - 2\left(\frac{E}{2G} - 1\right)\right] = 3K\left[1 - \frac{E}{G} + 2\right]$$

$$E = 3K\left[3 - \frac{E}{G}\right] = 3K\left[\frac{3G-E}{G}\right]$$

$$EG = 3K(3G - E) = 9KG - 3KE$$

$$\therefore EG + 3KE = 9KG$$

$$\therefore E(G + 3K) = 9KG$$

$$\boxed{\therefore E = \frac{9KG}{G + 3K}}$$

Q 7(b) Ans.

The Lorentz transformations (or transformation) are coordinate transformations between two coordinate frames that move at constant velocity relative to each other. A "stationary" observer in frame F defines events with coordinates t, x, y, z. Another frame F' moves with velocity v relative to F, and an observer in this "moving" frame F' defines events using the coordinates t', x', y', z'.

The coordinate axes in each frame are parallel (the x and x' axes are parallel, the y and y' axes are parallel, and the z and z' axes are parallel), remain mutually perpendicular, and relative motion is along the coincident xx' axes. At $t = t' = 0$, the origins of both coordinate systems are the same, $(x, y, z) = (x', y', z') = (0, 0, 0)$

If an observer in F records an event t, x, y, z, then an observer in F' records the *same* event with coordinates[

Lorentz boost (x *direction*)

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$
$$x' = \gamma\left(x - vt\right)$$
$$y' = y$$
$$z' = z$$

where v is the relative velocity between frames in the x-direction, c is the speed of light, and

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

(lowercase gamma) is the Lorentz factor.

According to the theory of relativity, time dilation is a difference in the elapsed time measured by two observers, either due to a velocity difference relative to each other, or by being differently situated relative to a gravitational field. As a result of the nature of spacetime,[2] a clock that is moving relative to an observer will be measured to tick slower than a clock that is at rest in the observer's own frame of reference. A clock that is under the influence of a stronger gravitational field than an observer's will also be measured to tick slower than the observer's own clock.

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}},$$

This expresses the fact that the moving observer's period Δt' of the clock is longer than the period Δt in the frame of the clock itself.

Q 7(c) Ans.

Classically we would add the speeds to get $1.8c$, which is obviously not allowed. In relativity you simply use the relativistic velocity addition formula:

$$V = \frac{u+v}{1+uv/c^2}$$

Where u and v are the velocities of the particles as seen from some reference frame, and V is the velocity of one particle in the rest frame of the other, i.e., the relative velocity when we consider one of the particles to be stationary. Plugging in $u = v = 0.9c$, we get $V = \frac{180}{181}c \approx 0.9945c$.

Edit: As pointed out by Alfred Centauri, the above explanation is perhaps too simplistic. A more rigorous version would be the following:

Let's take our particles to be moving in the x direction, with particle 1 moving in the positive direction and particle 2 moving in the negative direction. As seen by particle 1, our velocity is $-0.9c$ (note the sign!). As seen by us (that is, the lab frame), particle 2 is moving with a velocity equal to $-0.9c$. The velocity addition formula tells us how to find the velocity of particle 2 with respect to particle 1. If V is this velocity that we are trying to find, u is our velocity with respect to particle 1 and v is particle 2's velocity with respect to us, then:

$$V = \frac{u+v}{1+uv/c^2} \approx -0.9945c$$

This time we get the correct sign, since, relative to particle 1, particle 2 is moving in the negative x direction.