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UNIT-4
Rotational Dynamics

UNIT-5 Gravitation

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UNIT – 4

Rotational Dynamics

Q.1 Define centre of mass. What is physical significance of defining center of mass of the system.

Ans. The center of mass is a point defined in the system where all the mass of the system can be supposed to concentrate. For two particles of equal masses are moving opposite to each others with the same velocity, then center of mass will lie at the center of line joining the position of two particles.

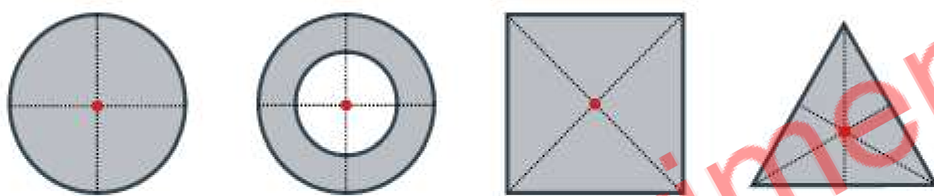


Figure 1: Center of mass for some simple geometric shapes (red dots).

For simple rigid objects with uniform density, the center of mass is located at the centroid. For example, the center of mass of a uniform disc shape would be at its center. It is not necessary that centre of mass should always lie inside the object. Sometimes the center of mass doesn't lie anywhere inside the object. For example the center of mass of a ring is located at its center, which outside the object material.

The center of mass of an object or system is that it is the point where any uniform force on the object acts. This is useful because it makes it easy to solve mechanics problems where we have to describe the motion of oddly-shaped objects and complicated systems. The center of mass greatly simplifies the problem and helps us analyze its rotational motion, linear motion, skidding, spinning, oscillation, periodic motion and most other motions quite easily. This concept is useful in case of rotational motion as well as in those calculations where size of the object does not matter, only mass matters. The center of mass is a useful reference point for calculations in mechanics that involve masses distributed in space, such as the linear and angular momentum of planetary bodies and rigid body dynamics.

If we apply a force on a rigid object at its center of mass, then the object will always move as if it is a point mass. It will not rotate about any axis, regardless of its actual shape. If the object is subjected to an unbalanced forcesthen it will begin rotating about the center of mass.

Q.2 . What is difference between centre of mass and centre of gravity?

Ans. Center of mass is the point in the body or a system of particles where all the mass of the system can be supposed to concentrate. It is found by taking the weighted average position of the mass. Equations used in dynamics are often applicable to the center of mass. For symmetrical objects with uniform density, the center of mass lies at the geometrical center of the objet. In nothing exist at the location of centre of mass. It is only a mathematical concept.

Center of gravity is the point through which all of the weight of the body supposed to act. The sum of torques due to gravitational forces is zero about the center of mass. A body can be balanced by applying a force through the center of gravity. Additionally, if a body is suspended then the center of gravity falls directly below the point of suspension. However centre of gravity of a body coincides with the centre o mass in uniform gravity or gravity free space.

Q.3. **What is radius of gyration and write down its characteristics.**

Ans. The square root of the ratio of the moment of inertia of a rigid body and its mass is called radius of gyration.

If M be the mass ass of the rigid body havingI as the moment of inertia, then radius of gyration of the system is given by :

$$K = \left[\frac{I}{M}\right]^{1/2}$$

$$I = MK^2$$

Thus, radius of gyration may also be defined as the perpendicular distance from the axis of rotation to a point at which whole of the mass of the body may be concentrated so that its moment of inertia is same as with actual distribution of mass.

$$K = \left[\frac{I}{M}\right]^{1/2} = \left[\frac{\sum M_i r_i^2}{M}\right]^{1/2}$$

Characteristics of radius of gyration

Where summation is on all the n particles

1. It is the distance from the axis of rotation at which whole of the mass of the may be concentrated so that moment of **inertia** remains unchanged.

2. Radius of gyration depends on the position and the orientation of the axis rotation.

3. Also $I = MK^2$

4. Radius of gyration does not depend on the mass of tho body but depends on the shape and nature of distribution of the mass of the body.

5. Dimension of $K = M^{\circ}LT^{\circ}.$

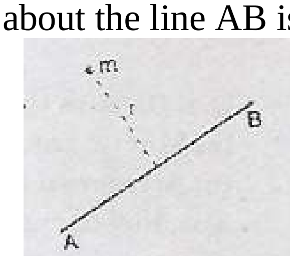
6. S.1. unit of K = Metre.

Q.4. **Q. 4 Define the following —**

- (a) **Rigid Body**
- (b) **Moment of inertia of a particle**
- (c) **Moment of inertia of a system of particles**
- (d) **Moment of inertia of a continuous distribution of mass.**

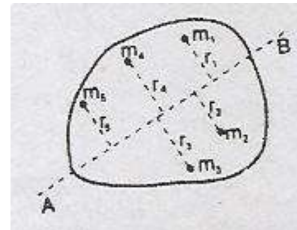
Ans. (a) Rigid Body: A rigid body may be defined as the system of particles such that the mutual distance of every pair of specified particles in it is invariable and the body does not expand or contract or change its shape in any way. i.e. the rigid body has invariable size and shape and the distance between any two particles remains always same.

(b) Moment of inertia of a particle: Consider a particle of mass m and a line a line AB, then the moment of Inertia of the particle of mass m about the line AB is defined as



I = mr2, where r is the perpendicular distance of the particle from the line.

(c) Moment of inertia of a system of it particles: Let there be a number of particles m_l, $m_{2,}$ m_3,…m_p, and let r1, r_2, r_3….. r_p be the perpendicular distances of the masses from the given line AB, then the moment of inertia of system is defined as

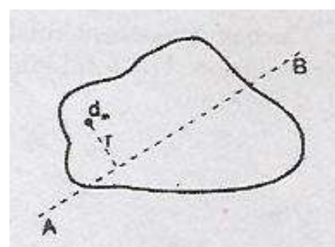


$$\mathbf{I} = m_i r_i^2 + m, r_2^2 + m_3 r_3^2 + \ldots. + m_p r_p^2$$

$$= \sum_{p=1}^{n} m_p r_p^2$$

(d) Moment of inertia of a continuous distribution of mass: Consider a rigid body and let dm be mass of the elementary portion of the body which is at a perpendicular distance r from the given line AB, then the moment of inertia

Of the whole body is defined as $I = \int r^2 dm,$

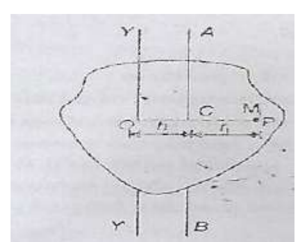


where the integration is taken over the whole body.

Q.5. Explain the theorem of parallel and perpendicular axis.

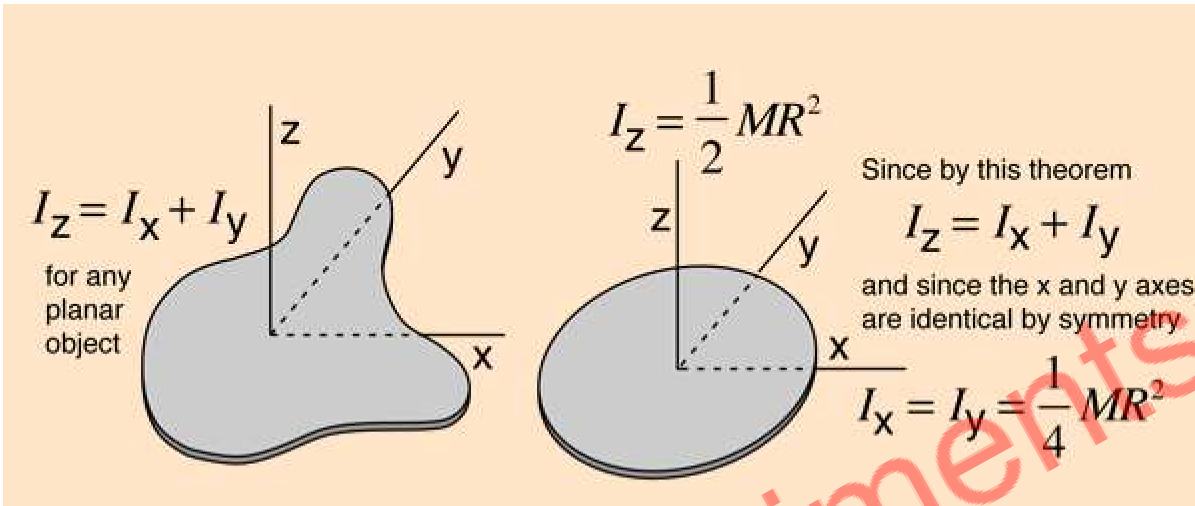
Ans. **Theorem of parallel axis**: It states that moment of inertia of a rigid body about any axis is equal to its moment of inertia a about a parallel axis passing through its centre of mass plus the product of the mass of the body and the square of the perpendicular distance between the two axes.

That is $$1 = l_{cm} + Mh^2$$



Where, I is the moment of inertia about any axis, I_{cm}. is that about the axis passing through the centre of mass, M is the total mass of the body and h is the distance between the two axes. See Fig.

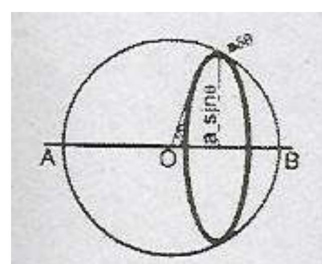
Theorem of perpendicular axis: The moment of inertia about an axis perpendicular to the plane is equal of the sum of the moments of inertia of two mutually perpendicular axes through the same point in the plane of the object.



Q.6. **Find the moment of inertia of-**

(i) **A hollow sphere about a diameter**

(ii) **A hollow sphere about a diameter its external and internal radii being a and b.**



Ans. (i) **Hallow sphere about a diameter**: If a semicircular arc is revolved about its diameter, then the surface so formed is known as hollow sphere. Consider elementary arc a δθ. This arc a δθ will generate a circular ring of radius a sin θ when revolved about the diameter AB.

Now mass of the elementary ring = 2π a sin θ. a δθρ .

M.I. of the elementry ring about AB = (2π a sin θ. a δθρ). $a^2\sin^2\theta$

$= 2\pi a^4 \rho \sin^3\theta\, \delta\theta$

M.I. of the hollow sphere about the diameter AB

$$= 2\pi a4\rho \int_0^{\pi} \sin^3\theta\, d\theta = 2\pi a^4 . \frac{M}{4\pi a^4} \int_0^{\pi} \sin^3\theta\, d\theta \quad (\because M = 4\pi a^2 \rho)$$

$$= 2\pi a4\rho \int_0^{\pi/2} \sin^3\theta\, d\theta = M a^2 \int_0^{\pi} \sin^3\theta\, d\theta = M a^2 . \frac{2}{3} = \frac{2}{3} M a^2 .$$

(ii) Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being a and b.

Sol. Consider a hollow sphere, the external and internal radius of which is a and b respectively. Take a spherical shell in it of radius x and of width dx.

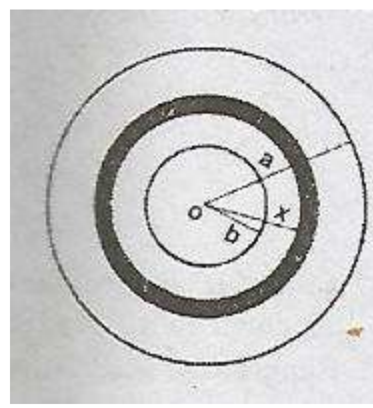
Moment of inertia of this shell about diameter

$$= \frac{2}{3} (\text{Mass of the shell}) \times (\text{radius of the shell})^2$$

$$= \frac{2}{3} . 4\pi x^2 \rho \delta x . x^2 = \frac{8\pi\rho}{3} x^4 dx.$$

Hence M.I. of the given hollow sphere

$$= \int_0^a \frac{8\pi\rho}{3} x^4 dx = \frac{8\pi\rho}{3} \left[\frac{x^5}{5}\right]_0^a = \frac{8}{15} \pi \rho (a^5 - b^5)$$



$$= \frac{8}{15} \pi . \frac{3M}{4\pi(a^3 - b^3)} . (a^5 - b^5) \quad [\because M = \frac{4}{3}\pi(a^3 - b^3)\rho]$$

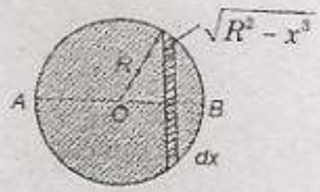
Q. 8 Obtain the expression for the moment of inertia of a solid sphere about its:

(i) **Diameter and**

(ii) **A tangent**

Ans. (i) M.I. of a solid sphere about one of its diameter .

Let AB be a diameter of the sphere of radius R



Vol. of a sphere $= \frac{4}{3}\pi R^3$

Let the mass of sphere $= m$

Then density $= \frac{\text{Mass}}{\text{Volume}}$

$$= \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

If we take a circular section of the sphere of thickness dx at a distance x from the centre, surface area of this portion

$$= \pi\,(\text{rad})^2 = \pi\left(\sqrt{R^2 - x^2}\right)^2 = \pi\,(R^2 - x^2)$$

vol of this section

$$= \pi\,(R^2 - x^2)dx$$

Mass of this section

$$= \pi\,(R^2 - x^2)dx\,\frac{3M}{4\pi R^3}$$

$$= \frac{3M(R^2 - x^2)}{4R^3}$$

M.I. of this section about AB

$$= \frac{1}{2}\,(\text{Mass}) \times (\text{Radius})2$$

$$= \frac{1}{2}\left(\frac{3M(R^2 - x^2)}{4R^3}\right) \times (dx) \times \left(\frac{R^2 - x^2}{2}\right)$$

$$= \frac{3M}{8R^3}\,(R^2 - x^2)^2\,dx$$

To get M.I. of the full sphere, we integrate from O to R

$$\text{M.I.} = 2\int_O^R \frac{3M}{8}\left(\frac{(R^2 - x^2)^2}{R^3}\right)dx$$

$$= 2 \times \frac{3M}{8}\left[\int_O^R\left(\frac{R^4}{R^3} + \frac{x^4}{R^3} - \frac{2R^2x^2}{R^3}\right)dx\right.$$

$$= \frac{3M}{4}\left[\int_O^R\left(R + \frac{1}{R^3}x^4 - 2.\frac{1}{R}x^2\right)dx\right.$$

$$= \frac{3M}{4}\left[\int_O^R\left(Rx + \frac{1}{R^3}.\frac{x^5}{5} - \frac{2}{R}.\frac{x^3}{3}\right)_O^R\right.$$

$$= \frac{3M}{4}\left[R^2 + \frac{R^5}{5R^3} - \frac{2R^3}{3R}\right]$$

$$= \frac{3M}{4}\left[R^2 + \frac{R^5}{5} - \frac{2R^3}{3}\right]$$

$$= \frac{2}{5}MR^2$$

(ii) M.I. of a sphere about a tangent

By tbe principal of parallel axis,

$$\text{M.I.} = \frac{2}{5}MR^2 + MR^2$$

$$= \frac{7}{5}MR^2$$

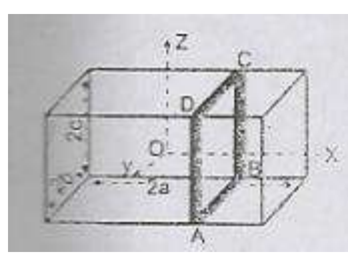
Q. 9 .Calculate moment of inertia for following-

(i) **Rectangular parallelopiped**

(ii) **Cube**

(iii) **Right circular cylinder**

Ans. (i) Rectangular Parallelopiped : Let o be the centre and 2a, 2b, 2c the lengths of the edges of the parallelopiped and further let OX, 0Y, OZ, be the axes of reference, parallel to the edges of lengths 2a, 2b and 2c respectively. Divide the parallelopiped into thin rectangular slices perp. to OX, ABCD being one such slice at a distance x. Let the width of the slice be δx



So Moment of inertia of rectangular slice about OX

$$= \text{Mass} \times \frac{b^2+c^2}{3} = 2b.\ 2c.\ \rho.\ \delta x.\ \frac{b^2+c^2}{3} = 4b\ c\ \rho\ \frac{b^2+c^2}{3}\ \delta x$$

[mass of the slice $ABCD = 2b\ 2c\ \delta x\ \rho$]

$\Rightarrow$ M.I. of the parallelopiped abotu OX

$$= 4bc\ \rho \frac{b^2+c^2}{3} \int_{-a}^{a} dx, = 8abc\ \rho \frac{b^2+c^2}{3}$$

$$= M \frac{b^2+c^2}{3}$$ [$\therefore$ mass of the parallelopiped $= 2a\ 2b.\ 2cr = 8abc\ \rho$]

Similarly, M.I. of the parallelopiped about $OY = M\ \frac{c^2+a^2}{3}$ and M.I. of the parallelopiped about $OZ = M\ \frac{b^2+c^2}{3}$.

(ii) **Cube :** Let each side of cube be 2a since all edges of the cube are identical, M.I. of a cube about any axis

$$= \text{M}\left(\frac{a^2+a^2}{3}\right) = \frac{2Ma^2}{3}$$

(iii) **Right circular cylinder.** Let there be a right circular cylinder of radius a and height h. Consider a circular disc of thickness dx at a distance x from O the centre of base. Mass of the disc $= \pi\ a^2 dx.\ \rho$, where r is mass per unit volume.

$\therefore$ M.I.of the disc about the axes perp. to the plane of the disc

$$= \pi a^2\ dx\rho.\ \frac{1}{2}\ a^2$$

$$\Rightarrow \text{M.I. of the cylinder} = \frac{\pi\ a^4 \rho}{2} \int_0^k dx = \frac{\pi\ a^4 h\ \rho}{2}$$

$$= \frac{1}{2}\ M\ a^2 \qquad [\because M = \pi\ a^2\ h\ \pi]$$

Q. 10. Find moment of inertia of a rectangular lamina for two different axes.

Ans. **Rectangular Lamina:**

(i) Moment of inertia of a rectangular lamina about **a** line through its centre and parallel to one of its edges.

Consider the strip RSPQ of breadth δx of the rectangular lamina ABCD such that AB = 2a and AD = 2b. Let M be the mass of the rectangular lamina. Then

mass per unit area $= \frac{M}{4ab} = \rho$ (say).

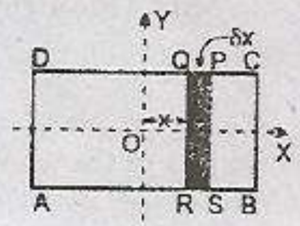
$\therefore$ Mass of the strip $RSPQ = 2b\,\delta x\,\rho = \frac{M}{4ab}(2b\,.\delta x)$

M.I. of the strip about

$$OX = \frac{M}{4ab} 2b\,\delta x\left(\frac{b^2}{3}\right) = \frac{M}{2a}\cdot\frac{b^2}{3}\delta x$$

$\therefore$ M.I. of the rectangular lamina about OX

$$= \int_{-a}^{a} \frac{M}{2a}\cdot\frac{b^2}{3} dx = \frac{1}{3}Mb^2$$



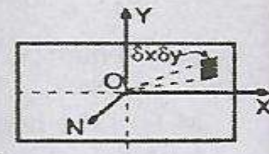
Similarly M.I. of the rectangular lamina about OY is $\frac{1}{3}Ma^2$.

(ii) *Moment of inertia of a rectangular lamina about a line through its centre and perp. to its plane.*

Consider an elementary area $\delta x \delta y$ of the lamina at a distance $\sqrt{(x^2 + y^2)}$ rom

$\therefore$ Mass of the elementary area $= \frac{M}{4ab}\delta x.\,\delta y.$

M.I. of this elementary area about the line ON through O and perpendicular to the plane of the rectangular lamina $= \frac{M}{4ab}\delta x.\,\delta y(x^2+y^2)$



$\therefore$ M.I. of the rectangular lamina about ON is

$$= \frac{M}{4ab}\int_{x=-a}^{a}\int_{y=-b}^{b}(x^2+y^2)dx\,dy = \frac{M}{4ab}\int_{x=0}^{a}\int_{y=0}^{b}(x^2+y^2)dx\,dy$$

$$= \frac{M}{ab}\left[\frac{1}{3}bx^3+\frac{1}{3}b^3x\right]_0^a = \frac{M}{3}(a^2+b^2).$$

Q. 11. Calculate moment of inertia of a rod of length 2a and mass m about two different axes.

Ans. (i) Moment of inertia of a road of length 2a and mass m about a line through one of its extermities perpendicular to its length.

Consider an element RS of breadth δx of the road AB at distance x from the line AN, where AN is perpendicular to AB, M.I.of the elements RS about

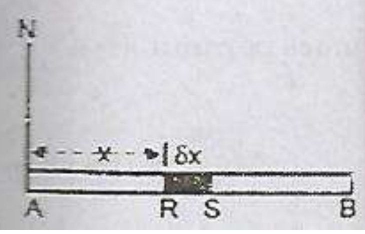
AN = (M/2a) δx.x^2 , where (M/2a) δx is mass of the element.

So Moment of Inertia of whole rod

Fig.

$\therefore$ M.I. of the whole rod

$$= \int_0^{2a}\frac{M}{2a}x^2\,dx = \frac{M}{2a}\left[\frac{x^3}{2}\right]_0^{2a} = M.\frac{4a^2}{3}.$$



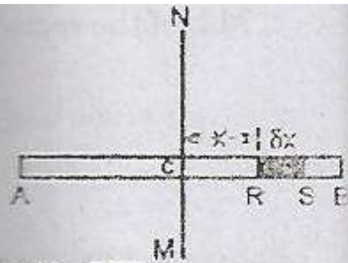
(ii) Moment of inertia of a rod of length 2a and of mass m about a line through its centre and perpendicular to its length.

Consider an element RS of breadth δx at distance x form the centre C.

M.I. of the element RS about NCM is given by

$$= \frac{M}{2a}\,\delta x\, x^2$$

$\Rightarrow$ M.I. of the whole rod about



$$MN = \int_{-a}^{a} \frac{M}{2a} x^2\, dx = \frac{M}{2a}\left[\frac{x^3}{2}\right]_{-a}^{a} = M.\ \frac{\quad}{3}$$

Q.12. **Discuss kinetic energy of rotation**.

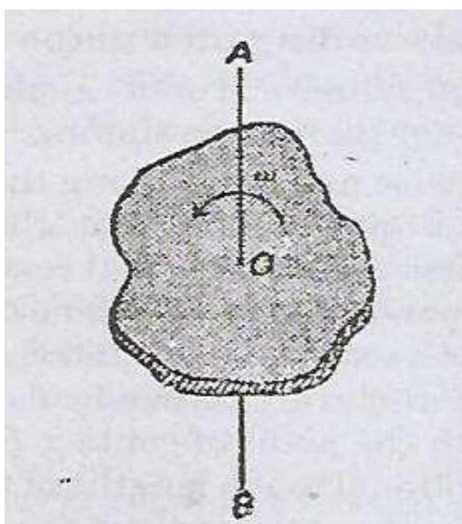
Ans. Kinetic energy of rotation can be explain different parts.

(i) **Pure Rotation:** There is no translatory motion involved. The linear velocity of the centre of mass is zero. When a body rotates about an axis through its centre of mass, we get such a situation.

(ii)Rotatory plus translatorymotion : When toration is accompanied by a linear velocity of its centre of mass, we get such a situation. An example can be the rolling of a body. This can be along a plane. The plane can be horizontal or inclined.

Kinetic energy — Pure Rotation

Let a body of mass M have pure rotatory motion due to rotation about an axis AB through its centre of mass.



The body has no translatory motion. The linear velocity of the centre of mass will be zero.

Let the angular velocity of the body be W

The body is made up of different particles. Let these particles have masses

m_1,m_2 and m_3 and these be situated at distance r_1, r_2, r_3... from the centre of mass, 0respectively.

Now v = rw, or $u^2 = r^2w^2$.

Therefore linear velocities of these particles will be $m_1\, r_1^2\, w^2$, $m_2\, r_2^2\, w^2$ and so on

$$\therefore \quad \text{Total K.E.} = \frac{1}{2}\,[m_1r_1^2w^2 + m_2r_2^2w^2 + \ldots\ldots]$$

$$= \frac{1}{2}\,w^2\,[m_1r_1^2 + m_2r_2^2 + \ldots\ldots]$$

$$\text{or K.E.} = \frac{1}{2}\,w^2 \sum mr^2$$

$$= \frac{1}{2}\,w^2 Mk^2$$

where K is the radius of gyration of the body about the axis of rotation AB. But $Mk^2 = I$, where I is the moment of inertia of the body about axis AB.

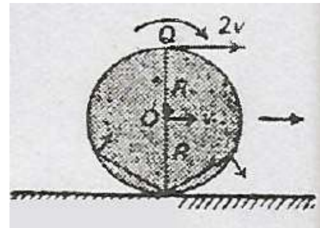
Hence $\qquad \text{K.E.} = \frac{1}{2}\,\text{Iw}^2$

The units of K.E. are joule in Mks (or S.I.) system and erg in cgs system. Kinetic energy rotatory plus translatory motion.

This can also be divided in following two parts

(a) Body rolling on a plane surface

(b) Body rolling down an inclined plane.



Kinetic energy of a body rolling on a plane surface

Let the body be of spherical, cylindrical or of circular disc shape. Let its mass be m and radius r. We assume that there is no slipping.

Let the body rotate clockwise and move along the positive direction of x-axis, at any given instant, the point P, where the body touches the surface, is at rest, so that an axis through P, perpendicular to the plane of the paper is its instantaneous axis of rotation and the linear velocities of its various particles are perpendicular to the lines joining them with the point of contact P, their magnitudes being proportional to the lengths of these lines, as shown by the directions and lengths of the arrows at the various point. Thus, if the linear velocity of the centre of mass 0 (where PO = R) be v, that of the particle at Q (where PQ = 2R) is 2v.

Let w be the angular velocity of rotation of the body. All the particles will have the same angular velocity. The total K.E. of the body will be equal to the sum of its K.E. due to rotation and K.E. due to linear motion.

Let v be the linear speed of centre of mass with respect to P. The moment of inertia about an axis through centre of mass (I_{cin}) will be mK^2. Using the theorem of parallel axis, M.I. of body through an axis at P, I_p can be expressed as:

$$I_p = I_{c.m.} + mr^2$$

$$= mk^2 + mr^2$$

Since $$\text{K.E.} = \frac{1}{2} Iw^2,$$

Total $$\text{K.E.} = \frac{1}{2}(mk^2 + mr^2)w^2 + \frac{1}{2}mv^2$$

$$= \frac{1}{2}mk^2w^2 + \frac{1}{2}mr^2w^2 + \frac{1}{2}mv^2$$

$$= \frac{1}{2} mk^2 \frac{v^2}{R^2} + \frac{1}{2} mr^2w^2 + \frac{1}{2} mv^2$$

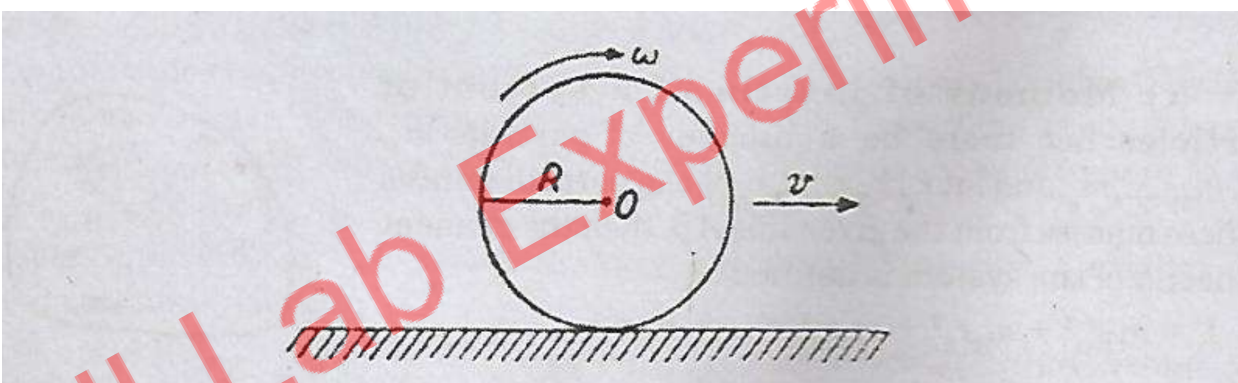
$$= \frac{1}{2} \text{mv}^2 \frac{k^2}{R^2} +$$

Q. 6 Explain the following —

(a). kinetic energy of a body rolling on a horizontal palne.

(b). Acceleration of a body rolling down an inclined plane.

Ans. Kinetic energy a body rolling on a Horizontal plane



Consider body (circular disc or sphere) of mass M, radius R and moment of Inertia I rolling on a horizontal plane. It has (i) Rotational motion and (ii) Translational motion. Its angular velocity is w and linear velocity is v

$$v = Rw$$

The body possessed kinetic energy due to (i) Rotational and (ii) translational motion. The total kinetic energy of the body at any instant *is given by*

$$E = \frac{1}{2}Iw^2 + \frac{1}{2}Mv^2 = \frac{1}{2}mK^2w^2 + \frac{1}{2}Mv^2$$

$$E = \frac{\frac{1}{2}MK^2v^2}{R^2} + \frac{1}{2}Mv^2 = \frac{1}{2}Mv^2\left[\frac{K2}{R^2}+1\right] \quad \text{...(i)}$$

Special cases

(1) for a circular disc

$$K^2 = \frac{R^2}{2}, \quad \frac{K^2}{R^2} = \frac{1}{2}$$

$$\therefore \quad E = \frac{1}{2}Mv^2\left(\frac{1}{2}+1\right)$$

$$E = \frac{3}{4}Mv^2 \quad \text{...(ii)}$$

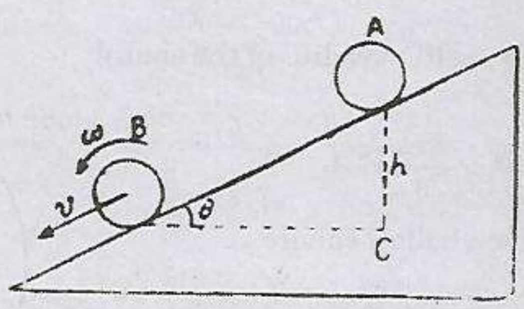
(2) for a sphere,

$$K^2 = \frac{2}{5}R^2 \quad \text{or} \quad \frac{K^2}{R^2} = \frac{2}{5}$$

$$\therefore \quad E = \frac{1}{2}Mv^2\left[\frac{2}{5}+1\right]$$

$$E = \frac{7}{10}Mv^2 \quad \text{...(iii)}$$

Acceleration of a body rolling down an Inclined Plane



Consider a body (disc or sphere) of mass M, radius R and moment of inertia I rolling down an inclined plane.

Suppose the body starts at A and reaches B after covering a distance AB =*l*. At B, it angular velocity is w and linear velocity is v. The vertical distance through which the body has moved = h.

Loss in potential energy $= Mgh = Mgl \sin\theta$

Gain in kinetic energy $= \frac{1}{2} I\omega^2 + \frac{1}{2} Mv^2$

Q.13. Discuss the following

(a) Principle of conservation of liner momentum (under finite forces)

(b) principle of conservation of angular momentum. (under finite forces).

Ans. Principle of conservation of Linear Momentum I (under finite forces)

(a) If the forces acting on a system be such that they have no component along a certain fixed straight line, throughout the motion, then the motion is such that the linear momentum resolved along this line is constant. **Let (x,** y, z) be the coordinates of any particle in of a body at time t, referred to fixed axes where the fixed straight line has been taken as the **axis** of x. Let X be the resolved part parallel to fixed line of the external forces acting on the particle, then

$$\sum m \frac{d^2x}{dt^2} = \sum X \quad \text{or} \quad \frac{d}{dt}\sum m \frac{dx}{dt} = \sum X.$$

But from the given conditions åX. = 0, hence $\frac{d}{dt}\Sigma m \frac{dx}{dt} = 0$

i.e., $\Sigma m \frac{dx}{dt} = \text{constant} \Rightarrow M\frac{d\bar{x}}{dt} = \text{constant} \left(\because \bar{x} = \frac{\sum mx}{M}\right)$ ...(1)

Where $\bar{x}$ is the x-coordinate of the centre of gravity of body and M its total mass.

From (1), we observe that the total momentum of the body parallel to the axis of x is constant throughout the motion. Principle of conservation of angular Momentum.

(Moment of Momentum under finite forces)

(b) if the forces acting on a system be such that they have no moment about a certain fixed straight line, throughout the motion then the angular momentum about this straight line is constant.

Let the co-ordinates of a particles of mass m at time t referred to fixed axes be (x, y, z) where the fixed straight line has been taken as the axis of x. Let X, Y, Z be the resolved parts parallel to the axis of the external forces acting on particle m. The sum of the moments of the external force about the given fixed line is Σ(yZ-zY). By D-Alembert"s principle we have

$$\Sigma m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)=\Sigma(yZ-zY)$$

But from the given condition that throughout the motion, the sum of the moments of external forces about the axis of x is zero

i.e., $$\Sigma(yZ-zY)=0$$

$\therefore$ $$\Sigma m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)= 0 \text{ throughout the motion}$$

i.e., $$\frac{d}{dt}\Sigma m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)=0$$

i.e., $$\Sigma m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)=\text{constant.} \qquad ...(2)$$

From (2), we observe that total moment of momentum of the body about x-axis Is constant throughout the motion.

Q. 14. Show that in a central force field, the angular momentum of particle remains conserved.

Ans. A Central force is a force which always acts towards or away from a fixed point Let "F" be the central force acting on a particle. Then it is represented as :

F= **r** f(r) where r is a unit vector along the direction of r and is equal to **r**/r and f(r) is a scalar function of the distance r. When we apply the central forces the torque acting on a particle is given by

$$\vec{\tau} = \frac{d\vec{L}}{dt} \quad \text{Where } \vec{L} \text{ is the angular momentum.}$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$= \vec{r} \times \vec{r} f(r)$$

$$= f(r)\, \vec{r} \times \frac{\vec{r}}{r}$$

But $\vec{r} \times \vec{r} = 0$

$\therefore \quad \vec{\tau} = 0$

$\therefore\ \vec{\tau}$ for central force field about origin is zero

showing that $\vec{L} = m\vec{v} \times \vec{r} =$ constant

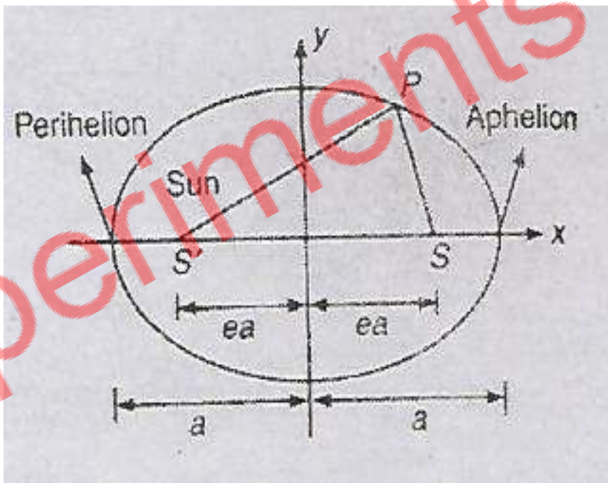
The angular momentum of a particle moving under the influence of a central force always remains constant.

UNIT – 5
Gravitation and Central Force Motion

Q. 1 Explain Kepler's law of planetary motion.

Ans. Kepler reduced the large numbers of the observation of planetary motion made by the Tycho Brahe to three laws, known as Kepler's law of planetary motion. These laws can be described as

1. **Law of orbit**. Each planet revolves in an elliptical orbit, with the sun at one focus of ellipse. This law is also known as the law of elliptical orbit.
2. **Law of area**. The radius vector, drawn from the sun to a planet, sweeps out equal in equal interval of time, *i.e.*, the areal velocity of planet around Sun is constant.
3. **Law of periods.** The square of the planet's time period is proportional to the cube of the semi-major axis of its elliptical orbit. This is known as the harmonic law and gives the relationship between the size of the orbit of a planet and its time of revolution.



$$T^2 \alpha\, r^3$$

Smaller the orbit of planet around the sun shorter will be time period to complete one revolution around the sun.

These three laws are consequences of Newton's laws of motion and laws of gravitation.

Let us first consider the elliptical orbits described in Kepler's first law. Figure shows the geometry of the ellipse. The longest dimension is the major axis with half length a, this half length is called the semi-major axis of ellipse.

$$SP + S'P = \text{constant.}$$

Here, S and S' are the foci and P any point on the ellipse. The sun is at San planet at P. The distance of each focus from the centre of ellipse is ea, where e the eccentricity a dimensionless quantity varies between 0 to1.

If $e = o$ *then* ellipse is a circle. The actual orbits of the planets are nearly circular; there eccentricities range from 0.007 for Venus to 0.248 for Pluto. For earth e = 0.017 the point in the planet's orbit closest to the sun is the perihelion and the point outermost distant from the sun is aphelion.

Explanation of First Law

Newton was able to show that for a body acted on by an attractive force proportional to $1/r^2$ the only possible closed orbits are a circle or an ellipse.

The open orbits must be parabolas or hyperbolas. He also showed that if total energy E is negatives the orbit is an ellipse (or circle), if it is zero the orbit is a parabola and if E is positive the orbit is a hyperbola. Further, it was also shown that orbits under the attractive force

$$F = K/r^n \text{ are stable for } n < 3.$$

Therefore, follows that circular orbits will be stable for a force varying inversely as the distance or the square of the distance and will be unstable for the inverse cube (or a higher power) law.

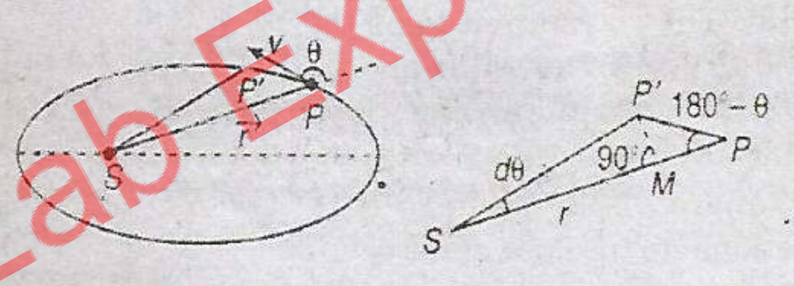
Explanation of Second Law

$$PP' = v\,dt$$

$$P'M = (PP) \sin(180-\theta) = PP' \sin\theta$$

$$= (v \sin\theta)\,dt$$

Kepler's second law is shown in figure. In a small time interval dt, the line from the sun S to the planet P turns through an angle $d\theta$. The area swept out in this time interval is,



$$dA = \text{area of triangle shown in figure} = \frac{1}{2}(\text{base})(\text{height})$$

$$= \frac{1}{2}(SP)(P'M)$$

$$= \frac{1}{2}(r)(v \sin\theta)\, dt$$

$\therefore$ Areal velocity $\qquad \frac{dA}{dt} = \frac{1}{2} rv \sin\theta \qquad$ ...(i)

Now, $rv \sin\theta$ is the magnitude of the vector product $\vec{r} \times \vec{v}$ which in turn is $\frac{1}{m}$ times the angular momentum $\vec{L} = \vec{r} \times m\vec{v}$ of the planet with respect to the sun. So we have,

$$\frac{dA}{dt} = \frac{1}{2m}|\vec{r} \times m\vec{v}| = \frac{L}{2m} \qquad \text{... (ii)}$$

or $\qquad \frac{dA}{dt} = \frac{L}{2m}$

Thus, Kepler's second law, that areal velocity is constant, means that angular momentum is constant. According to Newton's law the rate of change of angular momentum **L** is equal to torque of the gravitational force **F** acting on the planet,

$$\frac{d\bar{L}}{dt} = \bar{r} \times \bar{F} = \bar{\tau}$$

Here r is the radius vector of planet from the sun and the force F is directed the planet to the sun. So, these vectors always lie along the same line and vector product **r** x **F** zero. Hence, d**L**/dt is zero if **L**, = contant. Thus the second law actually the law of conservation of angular momentum.

Explanation third law

Newton was able to derive $\frac{d\bar{L}}{dt}$ is zero if

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM_s}}$$

where m_s is the mass of the sun.

Q. 2 Deduce Kepler's 2nd and 3rd law from Newton's law gravitation.

Ans. Consider a planet of mass m which revolves around the sun having mass M. Let us take the orbit to be a circular path of radius, *r*.

Gravitational force, F = GMm/r^2

If ω is the angular velocity of the planet and T is time period of its revolution , then the centrifugal force, **F**' can be written as

$$\mathbf{F}^! = m\omega^2 r$$

$$= m\,(2\pi/T)^2 r$$

The centrifugal and gravitational force balances each other.

$$F^! = F$$

$$GMm/r^2 = m\,(2\pi/T)^2 r$$

$$\Rightarrow \quad T^2 = \frac{4\pi^2}{Gm} r^3$$

$$= Kr^3, \text{ Where}$$

$$K = \frac{4\pi^2}{Gm} \text{ and is constant.}$$

or ∴ $T^2 \;\alpha\; r^3$

This is Kepler's third law.

The equation $T^2 = \dfrac{4\pi^2}{GM} r^3$ can be

Written as :

$$\frac{4\pi^2}{T^2} = \frac{GM}{r^3}$$

or $$\frac{2\pi}{T} = \sqrt{\frac{GM}{r^3}}$$

or $$w = \sqrt{\frac{GM}{r^3}}$$

The area of circle swept when an angle ofw is covered is:

$$A = \frac{w}{2\pi} \times \pi r^2$$

or $$A = \frac{r^2}{2} w$$

$$= \frac{1}{2}\sqrt{GMr}$$

or $$A = \text{constant}$$

This is kepler's second law.

Q.3. Explain Newton's law of gravitation and context of Einstein's theory of relativity.

Ans. Newton's Law of gravitation: Every object of the universe attracts every other object with a force that is directly proportional to the product of the masses of the objects and inversely proportional to the square of the distance between them."

Thus, the magnitude of the gravitational force *F* between two objects m_1, and m_2,placed at a distance r is

$$F \propto \frac{m_1 m_2}{r^2}$$

$$\mathbf{F} = G\frac{m_1 m_2}{r^2}$$

Here, *G* is a universal constant called gravitational constant whose 'magnitude is,

$G = 6.67 \times 10^{-11}$ N-m^2/kg^2

The direction of the force *F* is along the line joining the two objects following three points are important regarding the gravitational force:

(i) Unlike the electrostatic force, it is independent of the medium between objects.

(ii) It is conservative in nature.

(iii) It expresses the force between two point masses (of negligible volume), however for external points of spherical bodies the whole mass can be assumed be concentrated at its centre of mass.

The force of attraction between any two bodies is called the gravitation. If one of the objects is earth, this force is called gravity. We can define gravity as *a force* with which earth attracts a body towards it centre.

Newton's law of gravitation holds good for only weak gravitational fields. It assumes that the masses (both the objects) are at rest.

Consider Einstein's theory of relativity, we consider the moving mass if the object /objects are moving.

According to this theory,

, $m = \dfrac{m_0}{\sqrt{1 - \dfrac{u^2}{c^2}}}$, where m is the moving mass and m_0

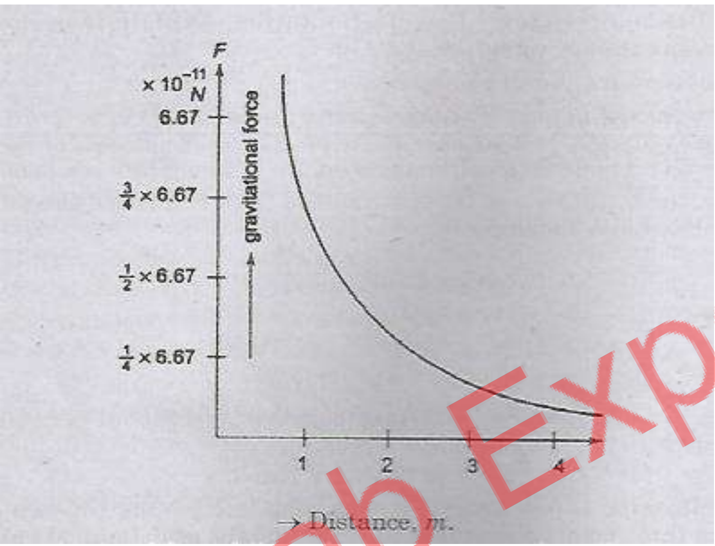
Is the rest mass u is the velocity of the moving body and c is the velocity of light.

Gravitational forces obey Newton's third law. Both the bodies attract each other with an equal force and form an action-reaction pair.

This is a universal law and operates from very small to very large distance. But it does not hold good for molecular distances or the order of 10^{-9} m.

The plot below depicts the gravitational force vs distance between the two objects. The gravitational force follows the inverse square law. The force decreases as the distance increases but it never becomes zero.

All planetary motion is due to gravitational forces. all heavenly bodies are assumed as point like objects.



Q.4 what is acceleration due to gravity? How it varies with height and depth on earth surface.

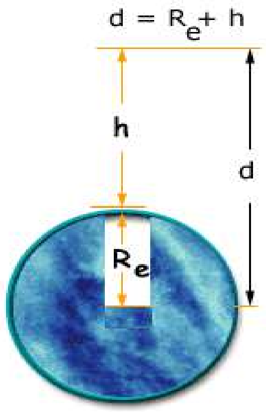
Ans. Accelerating due to gravity: The acceleration which is gained by an object because of gravitational force is called its acceleration due to gravity. Its SI unit is m/s^2. The acceleration due to gravity at the surface of Earth is represented as g. It has a standard value defined as 9.8 m/s^2.

1. **Variation with height**: Consider the variation of g when a body moves distance upward or downward from the surface of earth. Let g be the value of acceleration due to gravity at the surface of earth and g' at a height h above the surface of earth. If the earth is considered as a sphere of homogeneous composition, then g at any point on the surface of the earth is given by:

$$g = \frac{G M_e}{R_e^2} \longrightarrow (i)$$

Now we will determine the value of **g** at a distance (R_e+ h) from the centre of the earth.
In this condition distance is (R_e+ h)
The acceleration due to gravity is:

$$g' = \frac{G M_e}{(R_e + h)^2} \longrightarrow (ii)$$



dividing equation (ii) by (i)

$$\frac{g'}{g} = \frac{\dfrac{\cancel{G M_e}}{(R_e + h)^2}}{\dfrac{\cancel{G M_e}}{R_e^2}}$$

$$\frac{g'}{g} = \frac{R_e^2}{(R_e + h)^2}$$

$$\frac{g'}{g} = R_e^2 (R_e + h)^{-2}$$

Taking R_e^{-2} common

$$\frac{g'}{g} = \cancel{R_e^2}\ \cancel{R_e^{-2}} \left(1 + \frac{h}{R_e}\right)^{-2}$$

$$\frac{g'}{g} = \left(1 + \frac{h}{R_e}\right)^{-2}$$

Expanding by binomial theorem and neglecting higher powers of h/R_e

$$\boxed{\frac{g'}{g} = \left(1 - 2\frac{h}{R_e}\right)}$$

from above equation, we see that acceleration due to gravity decreases with the height.

2. **Variation with depth:** Consider earth to be a homogeneous sphere of mass M and radius R with centre O.

Let g be the value of acceleration due to gravity at a point A on the surface of earth.

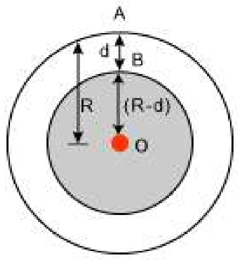
Therefore, $g = \dfrac{GM}{R^2}$

If ρ is uniform density of material of the earth, then,

$$M = \frac{4}{3}\pi R^3 \rho$$

or,

$$g = \frac{G \times \frac{4}{3}\pi R^3 \rho}{R^2} = \frac{4}{3}\pi G R \rho \quad \ldots$$



Variation in 'g' with depth

Let g be the acceleration due to gravity at point B at a depth of'd' below the surface of the earth. The body at B will experience a gravity pull due to earth whose radius is (R - d) and mass is M'.

Therefore, $g' = \dfrac{GM'}{(R-d)^2}$

or,

$$M' = \frac{4}{3}\pi (R-d)^3 \rho$$

Therefore,

$$g' = \frac{G \times \frac{4}{3}\pi (R-d)^3 \rho}{(R-d)^2}$$

$$= \frac{4}{3}\pi G (R-d)\rho$$

Dividing equation (v) by equation (iv), we have,

$$\frac{g'}{g} = \frac{\frac{4}{3}\pi G(R-d)\rho}{\frac{4}{3}\pi G R \rho}$$

$$= \frac{R-d}{R}$$

$$= \frac{R}{R} - \frac{d}{R}$$

$$g' = g\left(1 - \frac{d}{R}\right)$$

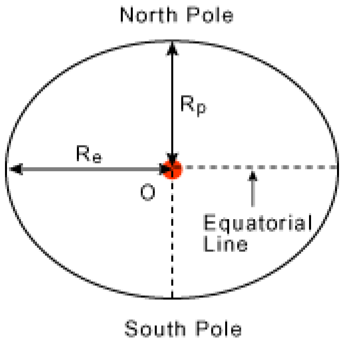
We see that value of acceleration due to gravity decreases with depth.

Q.5. Explain the effect of shape of Earth and latitude on the value of acceleration due to gravity.

Ans. Effect of Shape of Earth: Earth is not a perfect sphere. It is flattened at the poles and bulges out at the equator. Equatorial radius Re of the earth, is greater than the polar radius Rp.

We know, $g = \dfrac{GM}{R^2}$

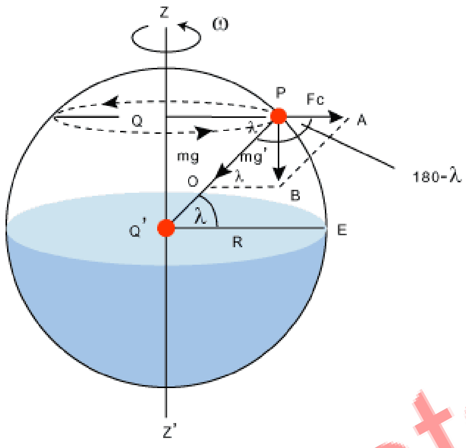
Since G and M are constants. Therefore, $g \propto \dfrac{1}{R^2}$



Variation of 'g' due to Shape of Earth

Thus, we conclude that the value of g is least at the equator and maximum at the poles. It means the value of acceleration due to gravity increases as we go from equator to the poles.

2. Effect of latitude (due to rotation of earth about its axis: Latitude at a place is defined as the angle which the line joining the place to the centre of the earth makes with the equatorial plane.



Effect of rotation on 'g'

Let earth be a sphere of mass M and radius R with centre Q'. The whole mass of earth can be supposed to be concentrated at the center Q'. Since, Earth rotates about its polar axis from west to east, every particle lying on its surface moves along a horizontal circle with same angular velocity as that of Earth. The centre of each circle lies on the polar axis.

Consider a particle of mass m at a place P of latitude (= Q'PQ). If earth is rotating about its polar axis with constant angular velocity, then particle at P also rotates and describes a horizontal circle of radius 'r'. Where, Centrifugal force (Fc) mr ^{2} acting on particle at P, is directed along PA away from centre of the circle of rotation.

Let 'g' be the acceleration due to gravity, when earth is at rest. Then the gravity pull on the

Particle (mg) acts along verticle direction PQ'.

Let g' be the acceleration due to gravity, when earth rotation is taken into consideration

The apparent weight of particle at P = mg'. This is the resultant of the true weight and centrifugal force acting on the particle at P and hence must be represented by diagonal PB of parallelogram, PABO. Using parallelogram Law of forces

$$mg' = \sqrt{(mg)^2 + \left(mr\omega^2\right)^2 + 2(mg)\left(mr\omega^2\right)\cos\left(180^0 - \lambda\right)}$$

$$g' = \sqrt{g^2 + r^2\omega^4 - 2gr\omega^2 \cos\lambda}$$

From equation (vi)

$$g' = \sqrt{g^2 + R^2 \cos^2 \lambda\omega^4 - 2gR\cos\lambda\omega^2 \cos\lambda}$$

$$= g\sqrt{1 + \frac{R^2\omega^4}{g^2}\cos^2\lambda - 2\frac{R\omega^2}{g}\cos^2\lambda}$$

$= \frac{1}{289}$. Since the value of the term $\frac{R\omega^2}{g}$ is very small, therefore we can neglect the higher terms.

$$g' = g\left(1 - \frac{2R\omega^2 \cos^2\lambda}{g}\right)^{\frac{1}{2}}$$

Expanding by Binomial theorem, we have

$$g' = g\left(1 - \frac{1}{2} \times \frac{2R\omega^2 \cos^2\lambda}{g}\right)$$

$$= g\left(1 - \frac{R\omega^2 \cos^2\lambda}{g}\right)$$

$$g' = g - R\omega^2 \cos^2\lambda$$

As $\cos\lambda$ and ω are positive, $g' < g$.
Equation shows that acceleration due to gravity:

- Decreases on account of rotation of earth.
- Increases with the increase in latitude of the place.

Q. 6 Explain inertial and gravitational mass. Why do we take these to be equal?

Ans. Inertial mass

(i) Inertia is the property of a body by virtue of which body resist any change in the position of rest form motion along a straight line. Some force has to be applied 'for achieving this. The greater the mass, greater will be the effort needed.

Obviously, gravity does not come into the picture in this case (the motion being perpendicular to the direction in which the gravity acts). In fact, the effort or the force required to set the block in motion or to stop it would still be the same if the experiment were performed in gravity-free space.

This inertness or reluctance of the block, or a material body, in general, to change its state of rest or uniform motion along a straight line is thus an inherent property of the body, in virtue of its mass which, since it represents the inertia of the body, is called its inertial mass. It can obviously be measured in terms of the ratio between the force (F) applied and the acceleration (a) produced in the body. For, in accordance with Newton's second law of motion, we have

F = ma and, therefore m = F/a.

Here, m represents the *inertial mass* of the body and May, therefore, more appropriately, be denoted by the symbol *m*.

If we apply the same force F to two bodies of inertial masses m_1 and m_2, and if the accelerations produced in them be a and *a' respectively*, we have

$$F = m_1 a = m'_1 a', \text{ whence, } m_1/m'_1 = a'/a.$$

Thus, the inertial masses of two bodies are inversely proportional to the accelerations produced in them by a given force. This holds good irrespective of the magnitude of the force applied.

We can, therefore, measure the inertial mass (m_1) of a body by comparing the acceleration produced in it with that produced in standard mass (M) such as the standard kilogram (preserved at Sevres) by the application of the same force F to both. If the accele-rations produced in the two masses be a and A, respectively, we have

$F = m_i a = MA$, whence, $m_i/M = A/a$. Or, $m_i = (A/a)M$.
So that, m_i can be easily evaluated.

(ii) **Gravitational mass**: mass of a body measured in terms of the gravitational force of attraction exerted on it by another body, such as the earth called gravitational mass. For, as we know, we have to exert an upward force to prevent a body from falling under the gravitational force of the earth and again, the bulkier is the body, the greater the force required to hold it. Here, clearly, the property of inertia plays no part but only the action of gravity. This mass of a body, measured in terms of the gravitational force of attraction on it due to another body, usually the earth, is called its gravitational mass. Thus if F and F^l be the forces of attraction exerted by a given body A, of mass M, on two bodies B and C, of masses 371 and A' respectively, at the same distance R from it, we have

$F = mMG/R^2$ and $F' = m'MG/R^2$, whence, $F/F' = m/m'$.
Here, m and m' are the *gravitational masses* of the two bodies and may be denoted by m_g and m'_g, respectively to distinguish them from their inertial masses m_i and m'_i. So that, $F/F' = m_g/m_g'$.

Thus, we see that the gravitational masses of the two bodies B and C, are directly proportional to the forces of attraction exerted on them by the body A.

If the body A happens to be the earth, clearly F= ω and F^1= ω where ω and ω 'are the weights of bodies B and C respectively at the given place. We, therefore, have

$$w = \frac{m_g M}{R^2} G \text{ and } w' = \frac{m_g M}{R^2} G, \text{ whence, } \frac{w}{w'} = \frac{m_g}{m_g'}$$

Or, the ratio between the gravitational masses of two bodies is equal to the between their weights at a given place.

It follows, therefore, that the mass of a body, as measured by a spring balance balancing it in one pan against a standard mass in the other, of a physical balance, is its gravitational mass.

Equality of inertial and gravitational masses : The question naturally arises whether the inertial mass of a body is different from or the same as, its gravitational mass. As a first step to enquire into the problem, let us try to see whether the inertial masses of two bodies too are proportional to their weights a given place. For this, we first determine the inertial masses m_i and $m_i^!$ of two bodies in the manner discussed under (i) above. Then, since in free space (or vacuum) all bodies fall with the same acceleration (g) at a given place, under the action of gravity, we have forces acting on the two bodies due to attraction by the earth, i.e., the weights w and w^1, equal to m_ig and m_i^1g respectively And, if m_g . And m_g^1.e the gravitational masses of the two bodies, theirs weights w and w^1 equal to

m_a MG/R^2 and m $'_g$ MG/R^2 respectively,

So that, $m_i'g = mg\ MG/R^2$ and $rn_i'\ g = m^1{}_g\ MG/R^2$, whence, $m_i/m_i = m_g/m_g$ ring that the ratio of the inertial masses of two bodies is the same as that of the gravitational masses.

The inertial and gravitational masses of a body are thus clearly proportional each other. Newton, Eotvos and others have conclusively found that both inertial and gravitational masses agree to be equal within 1 part in 10^{10}. These can be taken to be equal.

Q. 7. Define gravitational field and give its expression for different situations.

Ans. **Gravitational Field**: It is defined as the space around a body in which any other body experiences a force of attraction is called the gravitational field of the first body.

The force experienced (both in magnitude and direction) by a unit mass placed at a point in a gravitational field is called the gravitational field strength or intensity of gravitational field at that point. Usually it is denoted by **E** Thus,

$$\vec{E} = \frac{\vec{F}}{m}$$

Acceleration due to gravity $\vec{g}$ is also $\frac{\vec{F}}{m}$. Hence, for the earth's gravitational field $\vec{g}$ and $\vec{E}$ are same. The *E versus r* (the distance from the centre of earth) graph are same as that of *g' versus r* graph.

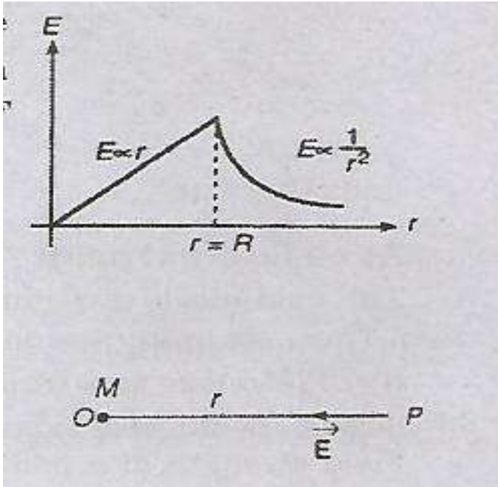
(I) Field due to a point mass

Suppose, a point mass M is placed at point o. We want to find the intensity of gravitational field **E** at a P, a distance r from o. Magnitude of force F on a particle of mass m placed at P is given by

$$F = \frac{GMm}{r^2}$$

$$E = \frac{F}{m} = \frac{GM}{r^2}$$

$$E = \frac{GM}{r^2}$$



The direction of the force **F** and hence of **E** is from P to o as shown.

(ii) Gravitational field due to a uniform solid sphere

(a). field at External point: A uniform sphere may be treated as a single particle of same mass placed at centre are for calculating the gravitational field at an external point. Thus,

$$E(r) = \frac{GM}{r^2} \quad \text{for} \quad r \geq R$$

$$E(r) \propto \frac{1}{r^2}$$

Here, r is the distance of the point from the centre of the sphere and R the radius of sphere.

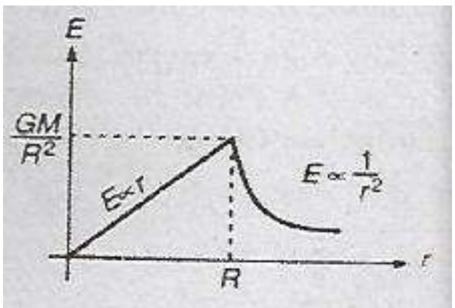
(b). Field at an internal point

The gravitational field due to a uniform sphere at an internal point is proportional to the distance of the point from the centre of the sphere. At the centre itself, it is zero and at surface it is GM/r^2 Where R is radius of the sphere. Thus

$$E(r) = \frac{GM}{R^3}r \quad \text{for} \quad r \leq R$$

$$E(r) = \propto r$$

Hence, *E versus r* graph is as shown



(iii) Field due to a uniform spherical shell

(a). **At an external point**

For an external point the shell may be treated as a single particle of same mass placed at its centre. Thus, at an external point the gravitational field is given by,

$$E(r) = \frac{GM}{r^2} \quad \text{for} \quad r \geq R$$

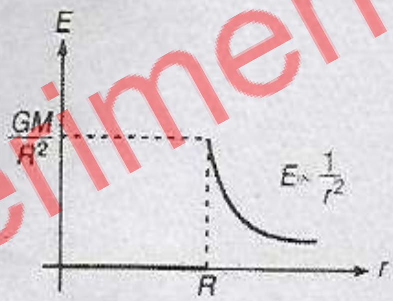
at $r = R$ (the surface of shell)

$$E = \frac{GM}{r^2}$$

and otherwise $E \propto \frac{1}{r^2}$

At an internal point

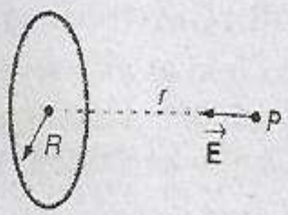
The field inside a uniform spherical shell is zero. Thus, E *versus* r graph is as shown.



(iv) Field due to a uniform circular ring at a point on its axis

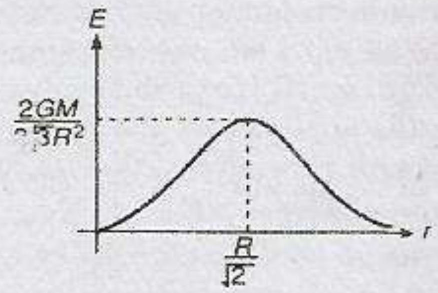
Field strength at a point P on the axis of a circular ring of radius R and mass M is given by,

$$E(r) = \frac{GMr}{(R^2 + r^2)^{3/2}}$$



This is directed towards the centre of the ring. It is zero at the centre of the ring and maximum at $r = \frac{R}{\sqrt{2}}$

can be obtained by putting $\frac{dE}{dr} = 0$). Thus, E-r graph is as Shown in fig.



The, maximum value is $E_{max} = \frac{2GM}{3\sqrt{3}R^2}$.

Q. 8. Discuss gravitional field due to a spherical shell at different location.

Ans. **Gravitational field**

Ans. Gravitational field

(a) **At a point outside the shell.** Since the potential at a point *outside the shell*, distant r from its centre *i.e.,* when $r > R$, is given by $V = -MG/r$, we have *intensity of the gravitational Field at the point,* $E = -dV/dr = -\frac{d}{dr}\left(-\frac{M}{r}G\right) = -\frac{M}{r^2}G$,

i.e., the same as though the whole mass of the shell were concentrated at its centre.

Force on a point mass m. If instead of a unit mass at P we have a point-mass m there, its potential energy will be given by $U = mV = -mMG/r$ and the force acting on it by $F = -dU/dr = -(mM/r^2)G$, the same as due to a mass M at O.

Thus, *for all points lying outside it (i.e., for all values of $r > R$) a spherical shell behaves as though its whole mass were concentrated at its centre.*

(a) **At a point on the outer, surface of the shell.** As we know, the gravitational potential at a point on the outer surface of the shell is given by $V = -MG/R$.

Therefore, *intensity of the gravitational field at the point, i.e,*

$$E = -\frac{d}{dr}\left(-\frac{MG}{R}\right) = -\frac{M}{R^2}G,$$

again, as though the mass of the shell were concentrated at its centre O.

(b) **At a point inside the shell.** We have seen under 1 (c) above that the gravitational potential at all points inside a spherical shell is the same.

Now, the gravitational field at a point is given by the space rate of change of

Potential there, Therefore field at the point is given by E = -dV/dr

Although V is constant for all points existing inside the shell, dV/dr = 0, i.e., the field in the interior of the shell is zero at all points. In other words, there is no gravitational field inside a spherical shell.

Since the gravitational field inside a spherical shell is zero, the force acting on a unit mass or any mass m at any point inside the shell is zero. This may also be seen from the following:

Considering a mass m at P, we have its P.E. = U = mV = -mMG / R, where M is the mass of the shell and R, its radius. And, therefore, force acting on mass in, i.e., F = -dU/dR =0.

Q. 9 Find the gravitational field due to a solid sphere at different locations.

Ans. (a) **Gravitational field at a point outside the solid sphere**. We know that the gravitational potential at a point P outside a solid sphere distant r from its centre (i.e., with r > R) is given by V = - MG/r, and since intensity of the gravitational field at a point is equal to the potential gradient there, we have gravitational field due to a solid sphere at a point P distant r from its centre (r > R), i.e., E = -dV/dr =

$$-\frac{d}{dr}\left(-\frac{MG}{r}\right) = -\frac{MG}{r},$$ *the same as though the whole mass (M) of the sphere were concentrated at its centre.*

Force on a point-mass m. if we replace the unit mass at P by a point-mass m, we have P.E. *of the mass, $U = mV = -mMG/r$.*

And, *therefore force acting on the point-mass m, i.e.,*

$F = dU/dr = -mMG/r^2$.

Force on a point-mass m. if we replace the unit mass at P by a point-mass m, we have P.E. of the mass, U=mV= -mMG/r.

And, therefore force acting on the point-mass m, i.e.,

$F = dU / dr = -mMG/r^2$

(b). At a point on the surface of the solid sphere. For a point on the surface of the solid sphere, obviously, r = R, the rdius of the sphere. We, therefore, have gravitational field at a point on the surface of the solid sphere, i.e.,

$$E = -\frac{MG}{R^2}$$

(c) At a point inside the solid sphere. As we know, the gravitational potential at a point inside a solid sphere distant r from its centre (i.e., with r > R) is given by

$$V= E = -MG\,(3R^2—r^2)/2R^3.$$

Since the gravitational field at a point is given by the potential gradient there, we have

Gravitational field due to a solid sphere at a point P inside it, at distant r from its centre i.e.,

its centre, *i.e.*, $E = \frac{dV}{dt} = -\frac{d}{dr}\left[-MG\frac{(mR^2 - r^2)}{2R^3}\right]$

$= -MG\frac{2r}{2R^3} = -\frac{MG}{R^3}r$, showing that $E \propto r$.

Q. 10. Write a short note on gravitational potential.,

Ans. Gravitational Potential is defined as the amount of work done in bringing a unit mass from infinity to a point in the gravitational field is called the 'gravitational potential' at that point. The gravitational potential is denoted by V. So, let W joule of work is obtained in bringing a test mass in from infinity to some point then gravitational potential at that point will be

$$V=W/m$$

Since, work is obtained, it is negative. Hence, gravitational potential is always negative.

(a) Potential due to a point mass

Consider a point mass M is situated at a point o. We want to find the gravitational potential due to this mass at a point P at a distance r from 0. For let us find work done in taking the unit mass from P to infinity. This will be

be,

$$W = \int_r^{\infty} F\, dr = \int_r^{\infty} \frac{GM}{r^2} . dr = \frac{GM}{R}$$

Hence, the work done in bringing unit mass from infinity to P will be $-\frac{GM}{r}$

Thus, the gravitational potential at P will be,

$$V = -\frac{GM}{r}$$

(b). Potential due to a uniform solid sphere

Potential at an external point

The gravitational potential due to a uniform sphere at an external point is same as that due to a single particle of same mass placed at its centre. Thus,

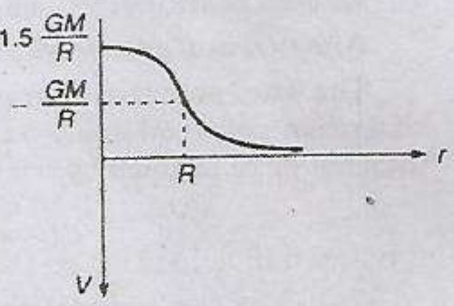
$$V(r) = -\frac{GM}{r} \qquad r \geq R$$

At the surface, $r = R$ and $V = -\frac{GM}{r}$

Potential at internal point

At some internal point, potential at a distance r from the centre is given by,

$$V(r) = -\frac{GM}{R^3}(1.5R^2 - 0.5r^2) \qquad r \leq R$$



At $r = R$, $V = -\frac{GM}{R}$

while at $r = 0$, $V = -\frac{1.5\, GM}{R}$

i.e., at the centre of the sphere the potential is 1.5 times the potential at surface. The variation of V *versus* r graph is as shown

(c). Potential due to a uniform thin spherical shell

Potential at an external point

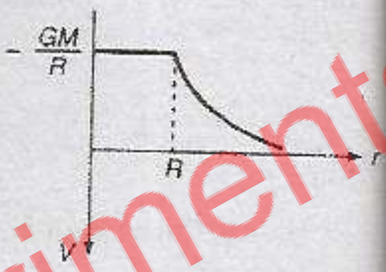
To calculate the potential at an external point, a uniform spherical shell be treated as a point mass of same magnitude at its centre. Thus, potential distance r is given by,

$$V(r) = -\frac{GM}{r} \qquad r \geq R$$

At, $r = R$ $\qquad V = -\frac{GM}{R}$

Potential at an internal point

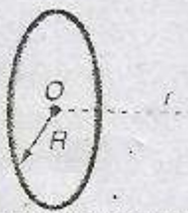
The potential due to a uniform spherical shell is constant throughout at any point inside the shell and this is equal to $-\frac{GM}{R}$. Thus, V-r graph for a spherical shell is as shown.



(iv) Potential due to a uniform ring at a point on its axis

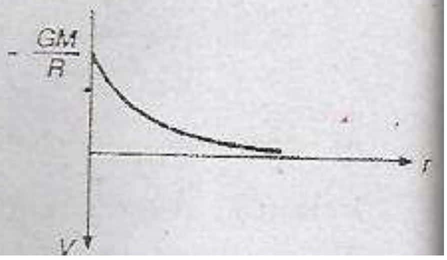
The gravitational potential at a distance r from the centre on the axis of a ring of mass M and radius R is given by,

$$V(r) = -\frac{GM}{\sqrt{R^2 + r^2}} \qquad 0 \leq r < \infty$$



At $r = 0$, $V = -\frac{GM}{R}$, *i.e.*, at the centre of the ring gravitational potential is $-\frac{GM}{R}$.

The V-r graph is as shown in Fig.

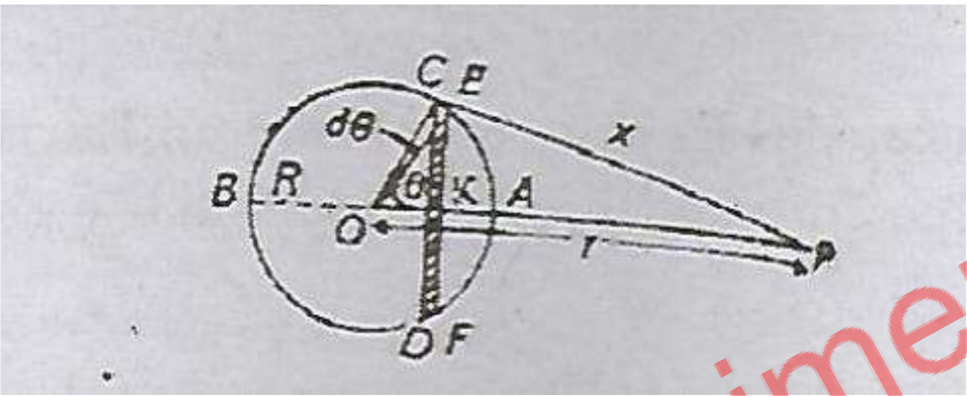


Q. 11 write down an expression for gravitational potential due to a spherical shell at different locations.

Ans. As gravitational potential is the amount of work done in bringing a unit mass from infinity to a point in the gravitational field is called the 'gravitational potential' at that point. The gravitational potential is denoted by V

(a) At a point outside the shell. Let we have to calculate gravitational potential at point P, at a distant r from the centre O of a spherical shell of radius R and surface density σ.

Join OP and cut out a slice CEFD in the form of a ring by two planes CD and EP' close to each other and perpendicular to the radius OA of the shell, and let angle EOP be q and the small angle COE = dθ.



Clearly, radius of the ring, EK = OE sin θ = R sin θ, so that its circumference = 2πR sin q and its width = CE = R dθ.

Therefore,

Surface area of the ring = circumference x width = 2πR sin θ. R dθ

Hence it's mass = 2π R sinθ R dθ σ = $2\pi R^2 \sin\theta\, \delta\theta\sigma$.

If EP = x, every point of the slice or the ring is at a distance x from P and therefore, potential at P due to the ring, say,

$$dV = -\frac{mass\ of\ slice}{x}G = -\frac{2\pi R^2 \sin\theta d\theta\sigma}{x}G. \qquad ...(i)$$

Now, in Δ *OEP*, $EP^2 = OE^2 + OP^2 - 2OE.OP\cos\theta$. or, $x^2 = R^2 + r^2 - 2R\,r\cos\theta$, which on differentiation, gives $2xdx = 0 + 0 + 2Rr\sin\theta d\theta$, whence, $x = R\,r\sin\theta\, d\theta/dx$.
[*R* and *r* being constants]

Substituting this value of *x* in expression (i) above, we have

$$dV = -\frac{2\pi R^2 \sin\theta d\theta\sigma}{Rr\ \sin\theta\ d\theta}Gdx = -\frac{2\pi R\sigma G}{r}dx.$$

The integral of this between the limits x = AP = (r -R) and x = BP = (r + R) clearly gives the potential V at P due to the whole shell thus,

The integral of this between the limits $x = AP = (r - R)$ and $x = BP = (r + R)$ then clearly gives the potential V at P due to the whole shell. thus,

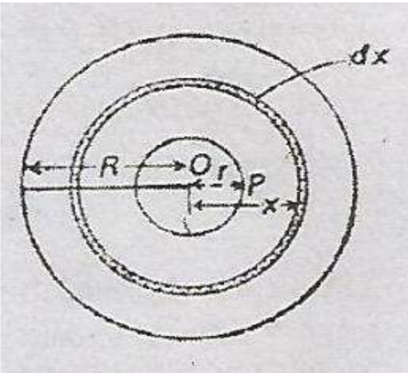
$$V = \int_{(r-R)}^{(r+R)} -\frac{2\pi R\sigma G}{r} dx = -\frac{2\pi R\sigma G}{r}\int_{(r-R)}^{(r+R)} dx$$

$$= -\frac{2\pi R\sigma G}{r}[x]_{(r-R)}^{(r+R)} = -\frac{2\pi R\sigma G}{r}2R = -\frac{4\pi R^2\sigma G}{r}$$

Clearly, $4\pi R^2$ is the surface area of the shell and, therefore $4\pi R^2\sigma$ is mass M. Therefore, gravitational potential at P due to the whole shell is $V = -MG/r$ i.e. the same as due to a mass M at O.

Q. 12 Prove that the gravitational potential due to a solid sphere has maximum (negative) value at its centre.

Ans..The solid sphere may be imagined to be made up of an inner solid sphere of radius r surrounded by a number of spherical shells, concentric with it and with their radii ranging from r to R. The potential at P due to the whole sphere is then clearly equal to the sum of the potentials at P due to the inner solid sphere and all the cal shells outside it.



Since the point P lies on the surface of the inner solid sphere of radius r and inside all the spherical shells of radii greater than r. So that, potential at P due inner solid sphere of radius r

$$= -\frac{\text{mass of the sphere}}{r} G = -\frac{4}{3}\pi r^3 \rho G/r = -\frac{4}{3}\pi r^2 \rho G$$

Mass of the inner solid sphere $=4\pi r^3\rho/3$, where ρ is the density of sphere.

To determine the potential at P due to all the outer shells, let us consider one such shell of radius x and thickness dx, i.e., of volume = area X thickness = $4\pi x^2 dx$ and hence of mass = $4\pi x^2 dx\rho$.

Since potential at a point inside a shell is the same as that at a point on its surface we have

potential at P due to this shell $= \frac{-4\pi x^2 dx\rho}{x} G = 4\pi x dx r G.$

$\therefore$ *potential at P due to all the shells*

$$= \int_r^R -4\pi\rho G x dx = -4\pi\rho G \int_r^R x dx = -4\pi\rho G \left[\frac{x^2}{2}\right]_r^R$$

$$= -4\pi\rho G\left(\frac{R^2 - r^2}{2}\right) = -\frac{4}{3}\pi\rho G.\frac{3(R^2 - r^2)}{2} = -\frac{4}{3}\pi\rho G.\frac{(3R^2 - 3r^2)}{2}$$

$\therefore$ *potential at P due to the whole solid sphere=potential at P due to inner solid sphere + potential at P due all the outer spherical shells*

$$= -\frac{4}{3}\pi r^2\rho G. -\frac{4}{3}\pi\rho G\left(\frac{3(R^2 - r^2)}{2}\right) = -\frac{4}{3}\pi\rho G\left(r^2 + \frac{3R^2}{2} - \frac{3r^2}{2}\right)$$

$$= -\frac{4}{3}\pi\rho G\left(\frac{3(R^2 - r^2)}{2}\right) = -\frac{4}{3}\pi R^3\rho G\left(\frac{3(R^2 - r^2)}{2R^3}\right)$$

[Multiplying and dividing by R^3.

Clearly, $-\frac{4}{3}\pi R^3\rho$ is the mass of the whole solid sphere, *i.e.*, *M*.

gravitational potential at P due to the solid sphere, i.e.

$$V = \frac{M(3R^2 - r^2)}{2R^2} G.$$

It follows at once, therefore, that if the point P lies *at the centre of the sphere,* we have $r = 0$. So that,

gravitational potential at the centre of the solid sphere

$$= -M\left(\frac{3R^2}{2R^3}\right)G = -\frac{3}{2}\cdot\frac{M}{R}G.$$

But $-\frac{M}{R}G$, as we know, is the gravitational potential on the *surface* of the sphere.

We thus have *gravitational potential at the centre of solid sphere* $=\frac{3}{2}$ *times*

The gravitational potential on its surface Or,

The gravitational potential at the centre of the solid sphere: gravitational potential on the surface of the sphere: : 3 : 2. So the gravitational potential due to a solid sphere has its maximum value at its centre.

Q. 13 find out the relation between gravitational field and potential.

Ans. Gravitational field and the gravitational potential are related by the following relation.

$$\vec{E} = \text{gradient } V = -\text{gradient } V$$

$$= -\left[\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right]$$

or $$\vec{E} = -\left[\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right]$$

Here, $\frac{\partial V}{\partial x}$ = partial derivative of potential function V w.r.t. x, *i.e.* differentiate w.r.t. x asuming y and z to be constant.

Eq. (i) Can be written in following different forms.

(i) $E = -\frac{dV}{dx}$, if gravitational field is along x-direction only.

(ii) $dV = -\vec{E}\cdot d\vec{r}$,

Here, $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ and $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$

Q. 14 what is gravitational potential energy. Explain in brief.

Ans. **Gravitational Potential Energy:** Gravitational potential energy is the energy an object has due to its position above Earth. The word potential energy is defined only for a conservative force field. The change in potential energy (dU) of a system corresponding to a conservative internal force is given by

$$dU = -\vec{F}\cdot d\vec{r}$$

$$\int_i^f dU = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}\cdot d\vec{r}$$

$$U_f - U_i = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}\cdot d\vec{r}$$

We generally choose the reference point at infinity and assume potential energy to be zero there, i.e., if we take r_i= infinite and U_i= 0 then we can write

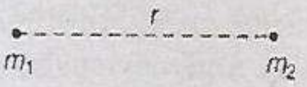
$$U = -\int_{\infty}^{\vec{r}} \vec{F}\cdot d\vec{r} = -W$$

or potential energy of a body or system is negative of work done by the conservative forces in bringing it from infinity to the present position.

Gravitational Potential Energy of a two Particle System

The gravitational potential energy of two particles of masses m_1 and m_2 separated by a distance r is given by,

$$U = \frac{-G\, m_1 m_2}{r}$$



This is actually the negative of work done in bringing those masses from infinity to a distance r by the gravitational forces between them.

Gravitational Potential Energy for a System of Particles

The gravitational potential energy for a system of particles (say m_1, m_2, m_3 and m_4) is given by

$$U = -G\left[\frac{m_4 m_3}{r_{43}} + \frac{m_4 m_2}{r_{42}} + \frac{m_4 m_1}{r_{41}} + \frac{m_3 m_2}{r_{32}} + \frac{m_3 m_1}{r_{31}} + \frac{m_2 m_1}{r_{21}}\right]$$

Thus, for a n particle system there are n (n-1)/2 pairs and the potential energy

calculated for each pair and added to get the total potential energy of the system .

Gravitational Potential Energy of a Body on Earth's Surface

The gravitational potential energy of a object of mass m in the gravitational field of M at a distance r is given by the relation

$$U = - GMm/r$$

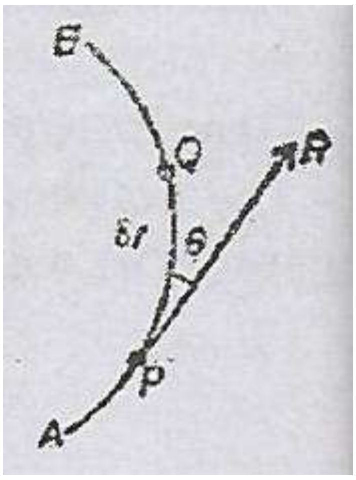
Since the mass m near earth's surface so r can be replace with R. Therefore above relation can be written as

$$U = - GMm/R$$

Q.15 Write a short note on equipotential surface in the context of tional field.

***surfac*e.**

Ans Equipotential surface is defined as the surface in which all points have the same gravitational potential .There is no difference of potential on any two points of this surface. Hence no work will be done in moving any mass against gravitational force on this surface. There is no component of the gravitational field along an equipotential surface. The gravitational field will be perpendicular to the surface at all points.

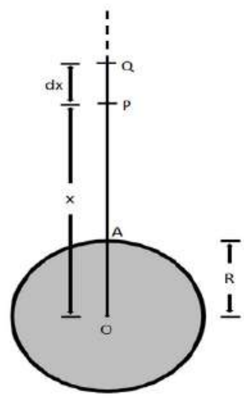


Consider two points P and Q, a small distance dr apart each other, on an equipotential surface AB, and let the intensity of the gravitational field at P is E, directed along PR at an angel θ with PQ. Then, clearly, component of the field along PQ = E cos θ and, therefore work done in moving a unit mass from P to Q= E cosθ δr.

Since P and Q lie on an equipotential surface, the work done, i.e.,

Ecosθ δr = 0 and since neither E δr is zero, we have cosθ = 0 or θ = 90°, i.e., this is directed along the perpendicular to the surface and this is also true for all other points on the surface AB. Thus, the direction of the field at every point an equipotential surface is perpendicular to the surface at that point

Q.16. Define escape velocity and derive expression for it.



Ans. Escape velocity is defined as the minimum initial velocity that will take a body away above the surface of a planet when it's projected vertically upwards.

Suppose that the planet be a perfect sphere of radius R having mass M. Let a body of mass m is to be projected from point A on the earth's surface as shown in the figure. Join OA and produce it further. Let us take two points P and Q which are at distances x and (x + dx) from the center of the earth.

To calculate the escape velocity of the earth, let the minimum velocity to escape from the earth's surface be ve. Then, kinetic energy of the object of mass m is

$$K.E. = \frac{1}{2} m v_e^2$$

When the projected object is at point **P** which is at a distance **x** from the center of the earth, the force of gravity between the object and earth is

$$F = \frac{GMm}{x^2}$$

Work done in taking the body against gravitational attraction from **P** to **Q** is given by

$$dW = F\,dx = \frac{GMm}{x^2}\,dx$$

The total amount of work done in taking the body against gravitational attraction from surface of the earth to infinity can be calculated by integrating the above equation within the limits **x** = **R** to **x** = ∞. Hence, total work done is

$$W = \int_R^\infty dW = \int_R^\infty \frac{GMm}{x^2} dx$$

$$= GMm \int_R^\infty x^{-2} dx = GMm \left[\frac{x^{-1}}{-1}\right]_R^\infty = - GMm \left[\frac{1}{x}\right]_R^\infty = -GMm \left[\frac{1}{\infty} - \frac{1}{R}\right]$$

$$\text{or, } W = \frac{GMm}{R}$$

For the object to escape from the earth's surface, kinetic energy given must be equal to the work done against gravity going from the earth's surface to infinity, hence

K.E. of Object should be Equal to Magnitude of P.E.

$$K.E. = W$$

$$\text{or, } \frac{1}{2} mv_e^2 = \frac{GMm}{R}$$

$$v_e = \sqrt{\frac{2GM}{R}}$$

Since,

$$g = \frac{GM}{R^2}$$

$$v_e = \sqrt{2gR}$$

The relation shows that the escape velocity of an object does not depend on the mass of the projected object but only on the mass and radius of the planet from which it is projected.

Escape Velocity of Earth

For the earth, **g** = **9.8 m/s2** and **R** = **6.4** X **106 m**, then

$$\text{escape velocity of the earth, } v_e = \sqrt{2 \times 9.8 \times 6.4 \times 10^6}$$

$$= 11.2 \times 10^3 \text{ m/s} = 11.2 \text{ km/s}$$