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B.Sc. (Physical Science)

Chapter - 1
Vectors and Differential Equations

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PHYSICS-DSC 1 A: MECHANICS

Ch-01

Vectors: Vector algebra. Scalar and vector products. Derivatives of a vector with respect to a parameter. Ordinary Differential Equations: 1^{st} order homogeneous differential equations. 2^{nd} order homogeneous differential equations with constant coefficients.

In this chapter, we will study some of the basic concepts about vectors, various operations on vectors, and their algebraic and geometric properties. These two type of properties, when considered together give a full realisation to the concept of vectors, and lead to their vital applicability in various areas as mentioned above.

Q. What is a Vector? Explain position vector its direction cosine and the types of vector.

Sol.

Let 'l' be any straight line in plane or three dimensional space. This line can be given two directions by means of arrowheads. A line with one of these directions prescribed is called a *directed line* (Fig 10.1 (i), (ii)).

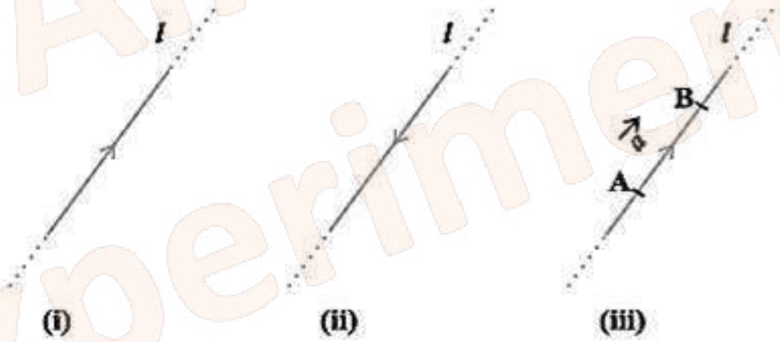


Fig 10.1

Now observe that if we restrict the line l to the line segment AB, then a magnitude is prescribed on the line l with one of the two directions, so that we obtain a *directed line segment* (Fig 10.1(iii)). Thus, a directed line segment has magnitude as well as direction.

Definition 1 A quantity that has magnitude as well as direction is called a vector.

Notice that a directed line segment is a vector (Fig 10.1(iii)), denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$, and read as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.

The point A from where the vector $\overrightarrow{AB}$ starts is called its *initial point*, and the point B where it ends is called its *terminal point*. The distance between initial and terminal points of a vector is called the *magnitude* (or length) of the vector, denoted as $|\overrightarrow{AB}|$, or $|\vec{a}|$, or a. The arrow indicates the direction of the vector.

Note	Since the length is never negative, the notation $|\vec{a}| < 0$ has no meaning.

Position Vector

From Class XI, recall the three dimensional right handed rectangular coordinate system (Fig 10.2(i)). Consider a point P in space, having coordinates (x, y, z) with respect to the origin O (0, 0, 0). Then, the vector $\overrightarrow{OP}$ having O and P as its initial and terminal points, respectively, is called the *position vector* of the point P with respect to O. Using distance formula (from Class XI), the magnitude of $\overrightarrow{OP}$ (or $\vec{r}$) is given by

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$

In practice, the position vectors of points A, B, C, etc., with respect to the origin O are denoted by $\vec{a}$, $\vec{b}$, $\vec{c}$, etc., respectively (Fig 10.2 (ii)).

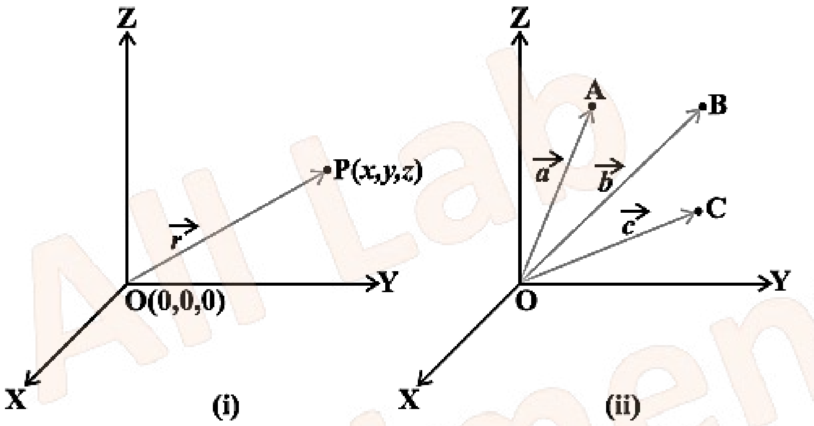


Fig 10.2

Direction Cosines

Consider the position vector $\overrightarrow{OP}$ or $\vec{r}$ of a point P(x, y, z) as in Fig 10.3. The angles α, β, γ made by the vector $\vec{r}$ with the positive directions of x, y and z-axes respectively, are called its *direction angles*. The cosine values of these angles, i.e., cos α, cos β and cos γ are called *direction cosines* of the vector $\vec{r}$, and usually denoted by l, m and n, respectively.

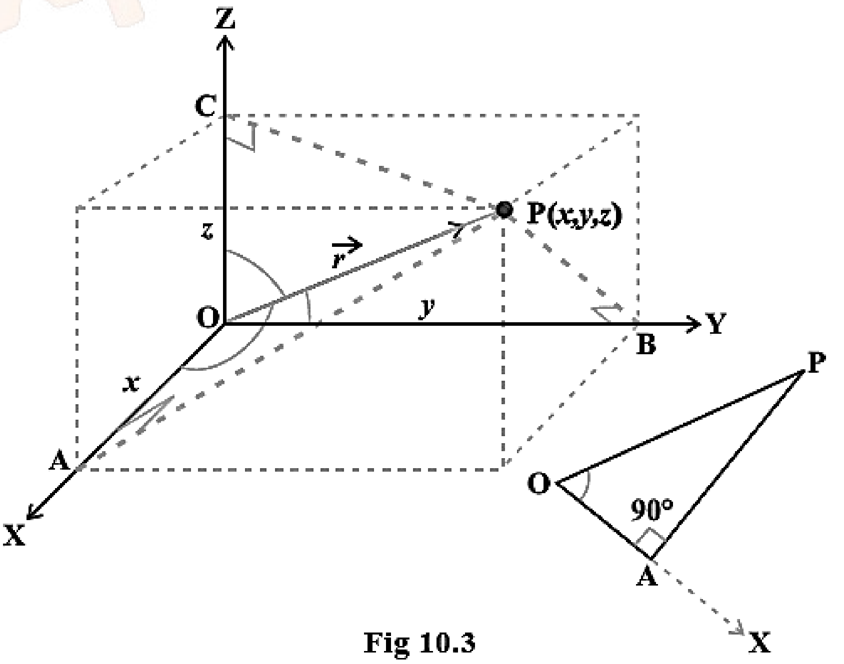


Fig 10.3

From Fig 10.3, one may note that the triangle OAP is right angled, and in it, we have $\cos\alpha = \frac{x}{r}$ (r stands for $|\vec{r}|$). Similarly, from the right angled triangles OBP and OCP, we may write $\cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$. Thus, the coordinates of the point P may also be expressed as (lr, mr, nr). The numbers lr, mr and nr, proportional to the direction cosines are called as *direction ratios* of vector $\vec{r}$, and denoted as a, b and c, respectively.

> **Note** One may note that $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$, in general.

Types of Vectors

Zero Vector A vector whose initial and terminal points coincide, is called a zero vector (or null vector), and denoted as $\vec{0}$. Zero vector can not be assigned a definite direction as it has zero magnitude. Or, alternatively otherwise, it may be regarded as having any direction. The vectors $\overrightarrow{AA}, \overrightarrow{BB}$ represent the zero vector,

Unit Vector A vector whose magnitude is unity (i.e., 1 unit) is called a unit vector. The unit vector in the direction of a given vector $\vec{a}$ is denoted by $\hat{a}$.

Coinitial Vectors Two or more vectors having the same initial point are called coinitial vectors.

Collinear Vectors Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitudes and directions.

Equal Vectors Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, if they have the same magnitude and direction regardless of the positions of their initial points, and written as $\vec{a} = \vec{b}$.

Negative of a Vector A vector whose magnitude is the same as that of a given vector (say, $\overrightarrow{AB}$), but direction is opposite to that of it, is called *negative* of the given vector. For example, vector $\overrightarrow{BA}$ is negative of the vector $\overrightarrow{AB}$, and written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

Remark The vectors defined above are such that any of them may be subject to its parallel displacement without changing its magnitude and direction. Such vectors are called *free vectors*. Throughout this chapter, we will be dealing with free vectors only.

Example 1 Represent graphically a displacement of 40 km, 30° west of south.

Solution The vector $\overrightarrow{OP}$ represents the required displacement (Fig 10.4).

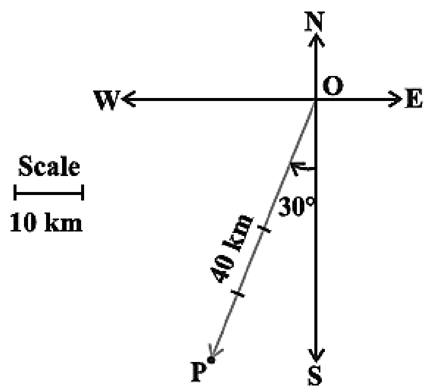


Fig 10.4

Example 2 Classify the following measures as scalars and vectors.

(i) 5 seconds

(ii) 1000 cm^3

(iii) 10 Newton (iv) 30 km/hr (v) 10 g/cm^3

(vi) 20 m/s towards north

Solution

(i) Time-scalar (ii) Volume-scalar (iii) Force-vector

(iv) Speed-scalar (v) Density-scalar (vi) Velocity-vector

Example 3 In Fig 10.5, which of the vectors are:

(i) Collinear (ii) Equal (iii) Coinitial

Solution

(i) Collinear vectors : $\vec{a}$, $\vec{c}$ and $\vec{d}$.

(ii) Equal vectors : $\vec{a}$ and $\vec{c}$.

(iii) Coinitial vectors : $\vec{b}$, $\vec{c}$ and $\vec{d}$.

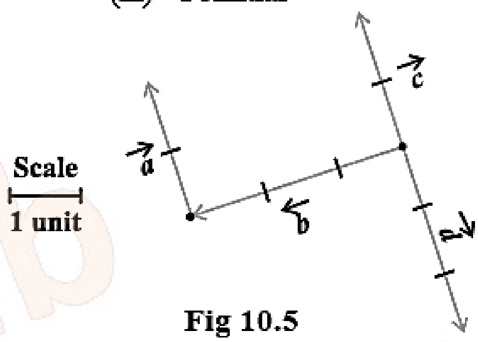


Fig 10.5

For example, in Fig 10.8 (ii), we have shifted vector $\vec{b}$ without changing its magnitude and direction, so that it's initial point coincides with the terminal point of $\vec{a}$. Then, the vector $\vec{a}+\vec{b}$, represented by the third side AC of the triangle ABC, gives us the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$ i.e., in triangle ABC (Fig 10.8 (ii)), we have

$$\overrightarrow{AB}+\overrightarrow{BC} = \overrightarrow{AC}$$

Now again, since $\overrightarrow{AC} = -\overrightarrow{CA}$, from the above equation, we have

$$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA} = \overrightarrow{AA} = \vec{0}$$

This means that when the sides of a triangle are taken in order, it leads to zero resultant as the initial and terminal points get coincided (Fig 10.8(iii)).

Now, construct a vector $\overrightarrow{BC'}$ so that its magnitude is same as the vector $\overrightarrow{BC}$, but the direction opposite to that of it (Fig 10.8 (iii)), i.e.,

$$\overrightarrow{BC'} = -\overrightarrow{BC}$$

Then, on applying triangle law from the Fig 10.8 (iii), we have

$$\overrightarrow{AC'} = \overrightarrow{AB} + \overrightarrow{BC'} = \overrightarrow{AB} + (-\overrightarrow{BC}) = \vec{a} - \vec{b}$$

The vector $\overrightarrow{AC'}$ is said to represent the *difference of* $\vec{a}$ and $\vec{b}$.

Now, consider a boat in a river going from one bank of the river to the other in a direction perpendicular to the flow of the river. Then, it is acted upon by two velocity vectors–one is the velocity imparted to the boat by its engine and other one is the velocity of the flow of river water. Under the simultaneous influence of these two velocities, the boat in actual starts travelling with a different velocity. To have a precise idea about the effective speed and direction (i.e., the resultant velocity) of the boat, we have the following law of vector addition.

If we have two vectors $\vec{a}$ and $\vec{b}$ represented by the two adjacent sides of a parallelogram in magnitude and direction (Fig 10.9), then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point. This is known as the *parallelogram law of vector addition.*

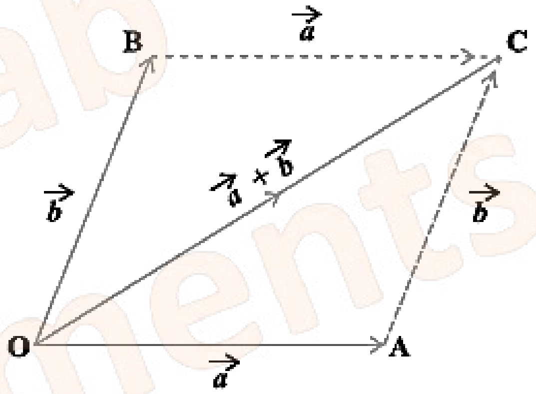


Fig 10.9

> Note From Fig 10.9, using the triangle law, one may note that
>
> $$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$
>
> or $$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC} \quad \text{(since } \overrightarrow{AC} = \overrightarrow{OB}\text{)}$$
>
> which is parallelogram law. Thus, we may say that the two laws of vector addition are equivalent to each other.

Properties of vector addition

Property 1 For any two vectors $\vec{a}$ and $\vec{b}$,

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{(Commutative property)}$$

Hence $$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

Q. Explain Vector Multiplication.

Sol.

Let $\vec{a}$ be a given vector and λ a scalar. Then the product of the vector $\vec{a}$ by the scalar λ, denoted as $\lambda\vec{a}$, is called the multiplication of vector $\vec{a}$ by the scalar λ. Note that, $\lambda\vec{a}$ is also a vector, collinear to the vector $\vec{a}$. The vector $\lambda\vec{a}$ has the direction same (or opposite) to that of vector $\vec{a}$ according as the value of λ is positive (or negative). Also, the magnitude of vector $\lambda\vec{a}$ is |λ| times the magnitude of the vector $\vec{a}$, i.e.,

$$|\lambda\vec{a}| = |\lambda||\vec{a}|$$

A geometric visualisation of multiplication of a vector by a scalar is given in Fig 10.12.

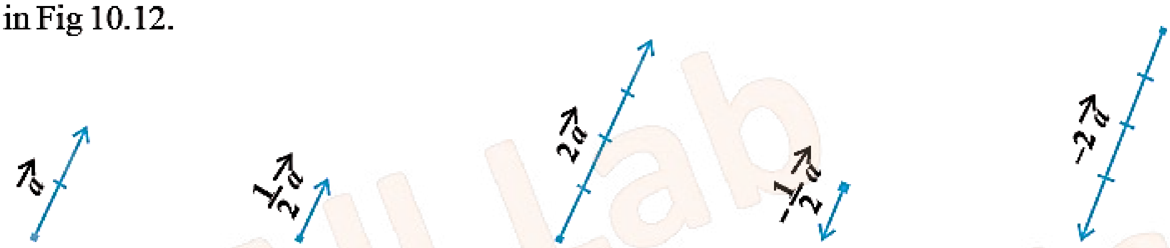


Fig 10.12

When $\lambda = -1$, then $\lambda\vec{a} = -\vec{a}$, which is a vector having magnitude equal to the magnitude of $\vec{a}$ and direction opposite to that of the direction of $\vec{a}$. The vector $-\vec{a}$ is called the *negative* (or *additive inverse*) *of vector* $\vec{a}$ and we always have

$$\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$$

Also, if $\lambda = \frac{1}{|\vec{a}|}$, provided $\vec{a}$ 0, i.e. $\vec{a}$ is not a null vector, then

$$|\lambda\vec{a}| = |\lambda||\vec{a}| = \frac{1}{|\vec{a}|}|\vec{a}| \quad 1$$

So, $\lambda\vec{a}$ *represents the unit vector in the direction of* $\vec{a}$. We write it as

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$$

Note	For any scalar k, $k\vec{0} = \vec{0}$.

Vector componants.

Sol.

Let us take the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1) on the x-axis, y-axis and z-axis, respectively. Then, clearly

$$|\overrightarrow{OA}|=1, |\overrightarrow{OB}| = 1 \text{ and } |\overrightarrow{OC}|=1$$

The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, each having magnitude 1, are called *unit vectors along the axes* OX, OY and OZ, respectively, and denoted by $\hat{i}, \hat{j}$ and $\hat{k}$, respectively (Fig 10.13).

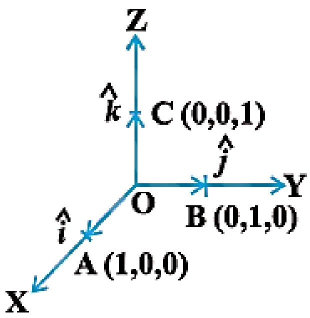


Fig 10.13

Now, consider the position vector $\overrightarrow{OP}$ of a point P (x, y, z) as in Fig 10.14. Let P_1 be the foot of the perpendicular from P on the plane XOY. We, thus, see that P_1 P is

This form of any vector is called its *component form*. Here, x, y and z are called as the *scalar components* of $\vec{r}$, and $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the *vector components* of $\vec{r}$ along the respective axes. Sometimes x, y and z are also termed as *rectangular components*.

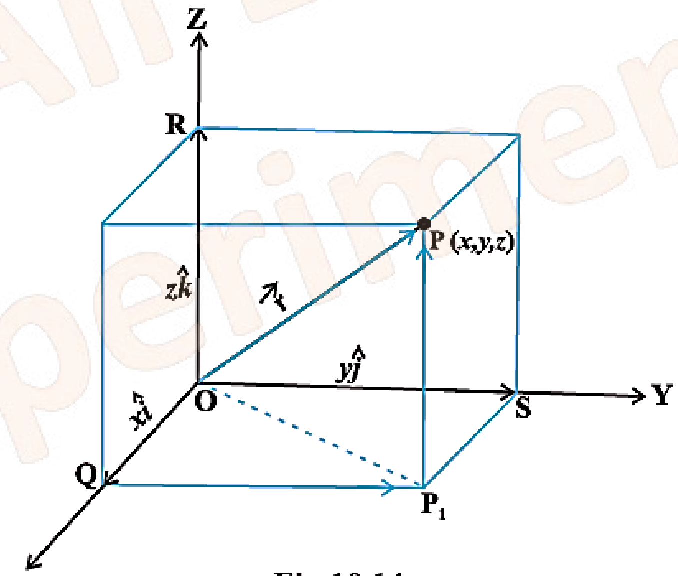


Fig 10.14

parallel to z-axis. As $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors along the x, y and z-axes, respectively, and by the definition of the coordinates of P, we have $\overrightarrow{P_1P} = \overrightarrow{OR} = z\hat{k}$. Similarly, $\overrightarrow{QP_1} = \overrightarrow{OS} = y\hat{j}$ and $\overrightarrow{OQ} = x\hat{i}$.

Therefore, it follows that $\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{QP_1} = x\hat{i} + y\hat{j}$

and $\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = x\hat{i} + y\hat{j} + z\hat{k}$

Hence, the position vector of P with reference to O is given by

$$\overrightarrow{OP} \text{ (or } \vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$$

Q. If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, respectively, then how to sum, subtract and multiply them?

(i) the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

(ii) the difference of the vector $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$$

(iii) the vectors $\vec{a}$ and $\vec{b}$ are equal if and only if

$$a_1 = b_1,\ a_2 = b_2 \quad \text{and} \quad a_3 = b_3$$

(iv) the multiplication of vector $\vec{a}$ by any scalar λ is given by

$$\lambda\vec{a} = (\ a_1)\hat{i} \quad (\ a_2)\hat{j} \quad (\ a_3)\hat{k}$$

Sol.

Also the distributive laws can be given by

Example 4 Find the values of x, y and z so that the vectors $\vec{a} = x\hat{i} + 2\hat{j} + z\hat{k}$ and $\vec{b} = 2\hat{i} + y\hat{j} + \hat{k}$ are equal.

Solution Note that two vectors are equal if and only if their corresponding components are equal. Thus, the given vectors $\vec{a}$ and $\vec{b}$ will be equal if and only if

$$x = 2,\ y = 2,\ z = 1$$

Let $\vec{a}$ and $\vec{b}$ be any two vectors, and k and m be any scalars. Then

(i) $k\vec{a} + m\vec{a} = (k + m)\vec{a}$

(ii) $k(m\vec{a}) = (km)\vec{a}$

(iii) $k(\vec{a} \quad \vec{b}) \quad k\vec{a} \quad k\vec{b}$

Example 5 Let $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j}$. Is $|\vec{a}| = |\vec{b}|$? Are the vectors $\vec{a}$ and $\vec{b}$ equal?

Solution We have $|\vec{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$ and $|\vec{b}| \quad \sqrt{2^2 \quad 1^2} \quad \sqrt{5}$

So, $|\vec{a}| = |\vec{b}|$. But, the two vectors are not equal since their corresponding components are distinct.

Example 6 Find unit vector in the direction of vector $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$

Solution The unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$.

Now $$|\vec{a}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

ıagnitude

Therefore $$\hat{a} = \frac{1}{\sqrt{14}}(2\hat{i} + 3\hat{j} + \hat{k}) = \frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} + \frac{1}{\sqrt{14}}\hat{k}$$

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a} = \frac{1}{\sqrt{5}}(\hat{i} - 2\hat{j}) = \frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}$$

Therefore, the vector having magnitude equal to 7 and in the direction of $\vec{a}$ is

$$7\hat{a} = 7\left(\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}\right) = \frac{7}{\sqrt{5}}\hat{i} - \frac{14}{\sqrt{5}}\hat{j}$$

Example 8 Find the unit vector in the direction of the sum of the vectors, $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$.

Solution The sum of the given vectors is

$$\vec{a} \quad \vec{b}\,(\quad \vec{c}, \text{say}) = 4\hat{i} \quad 3\hat{j} \quad 2\hat{k}$$

and $$|\vec{c}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$$

Thus, the required unit vector is

$$\hat{c} = \frac{1}{|\vec{c}|}\vec{c} = \frac{1}{\sqrt{29}}(4\hat{i} + 3\hat{j} - 2\hat{k}) = \frac{4}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} - \frac{2}{\sqrt{29}}\hat{k}$$

Example 9 Write the direction ratio's of the vector $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ and hence calculate its direction cosines.

Solution Note that the direction ratio's a, b, c of a vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ are just the respective components x, y and z of the vector. So, for the given vector, we have $a = 1$, $b = 1$ and $c = -2$. Further, if l, m and n are the direction cosines of the given vector, then

$$l = \frac{a}{|\vec{r}|} = \frac{1}{\sqrt{6}}, \quad m = \frac{b}{|\vec{r}|} = \frac{1}{\sqrt{6}}, \quad n = \frac{c}{|\vec{r}|} = \frac{-2}{\sqrt{6}} \text{ as } |\vec{r}| = \sqrt{6}$$

Thus, the direction cosines are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$.

Q. Prove that the vectors joining two points have the magnitude $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Sol.

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, then the vector joining P_1 and P_2 is the vector $\overrightarrow{P_1P_2}$ (Fig 10.15).

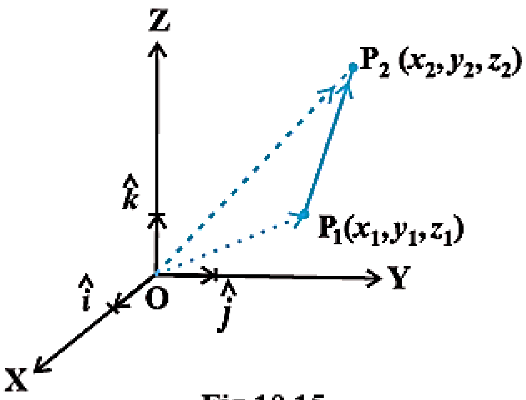


Fig 10.15

Joining the points P_1 and P_2 with the origin O, and applying triangle law, from the triangle OP_1P_2, we have

$$\overrightarrow{OP_1} + \overrightarrow{P_1P_2} = \overrightarrow{OP_2}.$$

Using the properties of vector addition, the above equation becomes

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

i.e. $$\overrightarrow{P_1P_2} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

The magnitude of vector $\overrightarrow{P_1P_2}$ is given by

$$\overrightarrow{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 10 Find the vector joining the points P(2, 3, 0) and Q(– 1, – 2, – 4) directed from P to Q.

Solution Since the vector is to be directed from P to Q, clearly P is the initial point and Q is the terminal point. So, the required vector joining P and Q is the vector $\overrightarrow{PQ}$, given by

$$\overrightarrow{PQ} = (-1-2)\hat{i} + (-2-3)\hat{j} + (-4-0)\hat{k}$$

i.e.

$$\overrightarrow{PQ} = -3\hat{i} - 5\hat{j} - 4\hat{k}.$$

Example 11 Consider two points P and Q with position vectors $\overrightarrow{OP} = 3\vec{a} - 2\vec{b}$ and $\overrightarrow{OQ}\ \ \vec{a}\ \ \vec{b}$. Find the position vector of a point R which divides the line joining P and Q in the ratio 2:1, (i) internally, and (ii) externally.

Solution

(i) The position vector of the point R dividing the join of P and Q internally in the ratio 2:1 is

$$\overrightarrow{OR} = \frac{2(\vec{a}+\vec{b}) + (3\vec{a}-2\vec{b})}{2+1} = \frac{5\vec{a}}{3}$$

(ii) The position vector of the point R dividing the join of P and Q externally in the ratio 2:1 is

$$\overrightarrow{OR} = \frac{2(\vec{a}+\vec{b}) - (3\vec{a}-2\vec{b})}{2-1} = 4\vec{b} - \vec{a}$$

Example 12 Show that the points $A(2\hat{i}\ \ \hat{j}\ \ \hat{k})$, $B(\hat{i}\ \ 3\hat{j}\ \ 5\hat{k})$, $C(3\hat{i}\ \ 4\hat{j}\ \ 4\hat{k})$ are the vertices of a right angled triangle.

Solution We have

$$\overrightarrow{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} \quad \hat{i}\ \ 2\hat{j}\ \ 6\hat{k}$$

$$\overrightarrow{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

and

$$\overrightarrow{CA} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$

Further, note that

$$|\overrightarrow{AB}|^2 = 41 = 6 + 35 = |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2$$

Hence, the triangle is a right angled triangle.

Q. Write a Shortnote on Scalar Product.

Sol.

1. **Scaler or Dot Product:**

The scalar product of two nonzero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$, is

defined as $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta,$

where, θ is the angle between $\vec{a}$ and $\vec{b}$, 0 (Fig 10.19).

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined, and in this case,

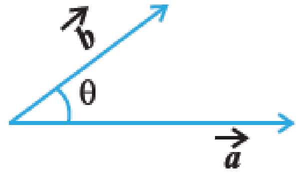


Fig 10.19

Observations

1. $\vec{a} \cdot \vec{b}$ is a real number.
2. Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors, then $\vec{a} \cdot \vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other. i.e.

   $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$
3. If θ = 0, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$

   In particular, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$, as θ in this case is 0.
4. If θ = π, then $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$

   In particular, $\vec{a}\ (\ \vec{a})\quad |\vec{a}|^2$, as θ in this case is π.
5. In view of the Observations 2 and 3, for mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1,$$

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\ \hat{i}\quad 0$$

$$a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

Q. What is the Scalar product of a = and b = $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

$$\begin{aligned}
\vec{a}\cdot\vec{b} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\
&= a_1\hat{i}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_2\hat{j}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_3\hat{k}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\
&= a_1b_1(\hat{i}\cdot\hat{i}) + a_1b_2(\hat{i}\cdot\hat{j}) + a_1b_3(\hat{i}\cdot\hat{k}) + a_2b_1(\hat{j}\cdot\hat{i}) + a_2b_2(\hat{j}\cdot\hat{j}) + a_2b_3(\hat{j}\cdot\hat{k}) \\
&\quad + a_3b_1(\hat{k}\cdot\hat{i}) + a_3b_2(\hat{k}\cdot\hat{j}) + a_3b_3(\hat{k}\cdot\hat{k}) \text{ (Using the above Properties 1 and 2)} \\
&= a_1b_1 + a_2b_2 + a_3b_3 \qquad \text{(Using Observation 5)}
\end{aligned}$$

Thus $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Sol.

Q. What is a Projection of a vector on a line? Explain with example.

Sol.

Suppose a vector $\overline{AB}$ makes an angle θ with a given directed line l (say), in the *anticlockwise direction* (Fig 10.20). Then the projection of $\overrightarrow{AB}$ on l is a vector $\vec{p}$ (say) with magnitude $|\overrightarrow{AB}|\cos\theta$, and the direction of $\vec{p}$ being the same (or opposite) to that of the line l, depending upon whether $\cos\theta$ is positive or negative. The vector $\vec{p}$ of AB along the line l is vector AC.

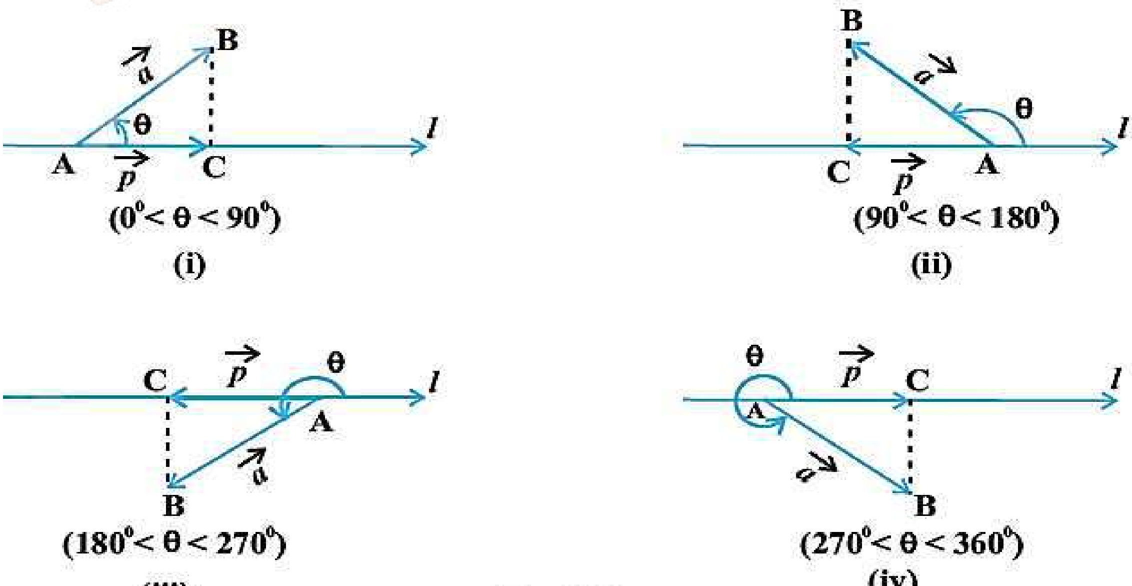


Fig 10.20

Observations

1. If $\hat{p}$ is the unit vector along a line l, then the projection of a vector $\vec{a}$ on the line l is given by $\vec{a} \quad \hat{p}$.
2. Projection of a vector $\vec{a}$ on other vector $\vec{b}$, is given by

$$\vec{a}\cdot\hat{b}, \quad \text{or} \quad \vec{a}\cdot\left(\frac{\vec{b}}{|\vec{b}|}\right), \quad \text{or} \quad \frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b})$$

3. If $\theta = 0$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{AB}$ itself and if $\theta = \pi$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{BA}$.
4. If $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, then the projection vector of $\overrightarrow{AB}$ will be zero vector.

Remark If α, β and γ are the direction angles of vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then its direction cosines may be given as

$$\cos \quad \frac{\vec{a} \quad \hat{i}}{|\vec{a}||\hat{i}|} \quad \frac{a_1}{|\vec{a}|}, \quad \cos \quad \frac{a_2}{|\vec{a}|}, \quad \text{and} \quad \cos \quad \frac{a_3}{|\vec{a}|}$$

Also, note that $|\vec{a}|\cos\alpha$, $|\vec{a}|\cos\beta$ and $|\vec{a}|\cos\gamma$ are respectively the projections of $\vec{a}$ along OX, OY and OZ. i.e., the scalar components a_1, a_2 and a_3 of the vector $\vec{a}$, are precisely the projections of $\vec{a}$ along x-axis, y-axis and z-axis, respectively. Further, if $\vec{a}$ is a unit vector, then it may be expressed in terms of its direction cosines as

$$\vec{a} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$$

Example 13 Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and when $\vec{a}\cdot\vec{b} = 1$.

Solution Given $\vec{a} \quad \vec{b} \quad 1, |\vec{a}| \quad 1$ and $|\vec{b}| \quad 2$. We have

$$\cos^{1} \frac{\vec{a} \quad \vec{b}}{|\vec{a}||\vec{b}|} \quad \cos^{1} \frac{1}{2} \quad \frac{}{3}$$

Example 14 Find angle 'θ' between the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

Solution The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Now $\quad \vec{a}\cdot\vec{b} = (\hat{i} + \hat{j} - \hat{k})\cdot(\hat{i} - \hat{j} + \hat{k}) = 1 - 1 - 1 = -1.$

Therefore, we have $\quad \cos\theta = \frac{-1}{3}$

hence the required angle is $\quad \theta = \cos^{1}\ \frac{1}{3}$

Example 15 If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular.

Solution We know that two nonzero vectors are perpendicular if their scalar product is zero.

Here $\quad \vec{a} + \vec{b} = (5\hat{i} - \hat{j} - 3\hat{k}) + (\hat{i} + 3\hat{j} - 5\hat{k}) = 6\hat{i} + 2\hat{j} - 8\hat{k}$

and $\quad \vec{a} - \vec{b} = (5\hat{i} - \hat{j} - 3\hat{k}) - (\hat{i} + 3\hat{j} - 5\hat{k}) = 4\hat{i} - 4\hat{j} + 2\hat{k}$

So $\quad (\vec{a} + \vec{b})\cdot(\vec{a} - \vec{b}) = (6\hat{i} + 2\hat{j} - 8\hat{k})\cdot(4\hat{i} - 4\hat{j} + 2\hat{k}) = 24 - 8 - 16 = 0.$

Hence $\quad \vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular vectors.

Example 16 Find the projection of the vector $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on the vector $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

Solution The projection of vector $\vec{a}$ on the vector $\vec{b}$ is given by

$$\frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b}) = \frac{(2\times1 + 3\times2 + 2\times1)}{\sqrt{(1)^2 + (2)^2 + (1)^2}} = \frac{10}{\sqrt{6}} = \frac{5}{3}\sqrt{6}$$

Example 17 Find $|\vec{a} - \vec{b}|$, if two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|\ \ 2$, $|\vec{b}|\ \ 3$ and $\vec{a}\cdot\vec{b} = 4$.

Solution We have

$$|\vec{a}\ \ \vec{b}|^2 = (\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b})$$

$$= \vec{a}.\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 - 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2$$

$$= (2)^2 - 2(4) + (3)^2$$

Therefore $\quad |\vec{a} - \vec{b}| = \sqrt{5}$

Example 18 If $\vec{a}$ is a unit vector and $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=8$, then find $|\vec{x}|$.

Solution Since $\vec{a}$ is a unit vector, $|\vec{a}|=1$. Also,

$$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 8$$

or $$\vec{x}\cdot\vec{x}+\vec{x}\cdot\vec{a}-\vec{a}\cdot\vec{x}-\vec{a}\cdot\vec{a} = 8$$

or $$|\vec{x}|^2 \quad 1 = 8 \text{ i.e. } |\vec{x}|^2 = 9$$

Therefore $$|\vec{x}| = 3$$ (as magnitude of a vector is non negative).

Example 19 For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a}\cdot\vec{b}|\le|\vec{a}||\vec{b}|$ (Cauchy-Schwartz inequality).

Solution The inequality holds trivially when either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$. Actually, in such a situation we have $|\vec{a}\cdot\vec{b}|=0=|\vec{a}||\vec{b}|$. So, let us assume that $|\vec{a}|\ne 0\ne|\vec{b}|$. Then, we have

$$\frac{|\vec{a}\cdot\vec{b}|}{|\vec{a}||\vec{b}|} = |\cos\theta|\le 1$$

Therefore $$|\vec{a}\cdot\vec{b}|\le|\vec{a}||\vec{b}|$$

Example 20 For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$ (triangle inequality).

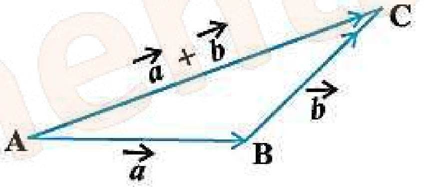


Fig 10.21

Solution The inequality holds trivially in case either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$ (How?). So, let $|\vec{a}| \quad \vec{0} \quad |\vec{b}|$. Then,

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

$$= \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2$$ (scalar product is commutative)

$$\le |\vec{a}|^2+2|\vec{a}\cdot\vec{b}|+|\vec{b}|^2$$ (since $x\le|x|\ \forall x\in\mathbf{R}$)

$$\le |\vec{a}|^2+2|\vec{a}||\vec{b}|+|\vec{b}|^2$$ (from Example 19)

$$= (|\vec{a}| \quad |\vec{b}|)^2$$

Hence $$|\vec{a} \quad \vec{b}|\le|\vec{a}| \quad |\vec{b}|$$

Remark If the equality holds in triangle inequality (in the above Example 20), i.e.

$$|\vec{a}+\vec{b}| = |\vec{a}|+|\vec{b}|,$$

then $$|\overrightarrow{AC}| = |\overrightarrow{AB}|+|\overrightarrow{BC}|$$

showing that the points A, B and C are collinear.

Example 21 Show that the points $A(-2\hat{i}+3\hat{j}+5\hat{k})$, $B(\hat{i}+2\hat{j}+3\hat{k})$ and $C(7\hat{i}-\hat{k})$ are collinear.

Solution We have

$$\overrightarrow{AB} = (1 \quad 2)\hat{i} \quad (2 \quad 3)\hat{j} \quad (3 \quad 5)\hat{k} \quad 3\hat{i} \quad \hat{j} \quad 2\hat{k},$$

$$\overrightarrow{BC} = (7 \quad 1)\hat{i} \quad (0 \quad 2)\hat{j} \quad (\; 1 \quad 3)\hat{k} \quad 6\hat{i} \quad 2\hat{j} \quad 4\hat{k},$$

$$\overrightarrow{AC} = (7 \quad 2)\hat{i} \quad (0 \quad 3)\hat{j} \quad (\; 1 \quad 5)\hat{k} \quad 9\hat{i} \quad 3\hat{j} \quad 6\hat{k}$$

$$|\overrightarrow{AB}| = \sqrt{14}, \; |\overrightarrow{BC}| \quad 2\sqrt{14} \text{ and } |\overrightarrow{AC}| \quad 3\sqrt{14}$$

Therefore $$|\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$$

Hence the points A, B and C are collinear.

> **Note** In Example 21, one may note that although $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$ but the points A, B and C do not form the vertices of a triangle.

Q. What will be the vector product of two nonzero vectors $\vec{a}$ and $\vec{b}$

Sol.

The vector product of two nonzero vectors $\vec{a}$ and $\vec{b}$, is denoted by $\vec{a} \quad \vec{b}$ and defined as

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n},$$

where, θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system (Fig 10.23). i.e., the right handed system rotated from $\vec{a}$ to $\vec{b}$ moves in the direction of $\hat{n}$.

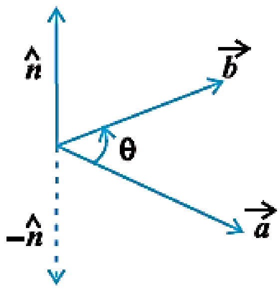


Fig 10.23

If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then θ is not defined and in this case, we define $\vec{a}\times\vec{b}=\vec{0}$.

Observations

1. $\vec{a}\times\vec{b}$ is a vector.
2. Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors. Then $\vec{a}\times\vec{b}=\vec{0}$ if and only if $\vec{a}$ and $\vec{b}$ are parallel (or collinear) to each other, i.e.,

$$\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a} \,\square\, \vec{b}$$

In particular, $\vec{a}\times\vec{a}=\vec{0}$ and $\vec{a}\times(-\vec{a})=\vec{0}$, since in the first situation, θ = 0 and in the second one, θ = π, making the value of sin θ to be 0.

3. If $\theta=\frac{\pi}{2}$ then $\vec{a}$ $\vec{b}$ $|\vec{a}||\vec{b}|$.

4. In view of the Observations 2 and 3, for mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ (Fig 10.24), we have

$$\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0}$$

$$\hat{i}\times\hat{j}=\hat{k},\quad \hat{j}\times\hat{k}=\hat{i},\quad \hat{k}\times\hat{i}=\hat{j}$$

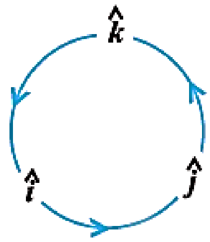


Fig 10.24

5. In terms of vector product, the angle between two vectors $\vec{a}$ and $\vec{b}$ may be given as

$$\sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$

Q. Prove that If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle then its area is given as $\frac{1}{2}|\vec{a}\quad\vec{b}|$.

Sol.

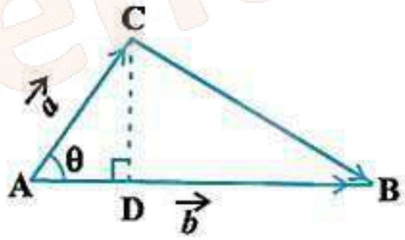


Fig 10.26

By definition of the area of a triangle, we have from Fig 10.26,

$$\text{Area of triangle ABC}=\frac{1}{2}\text{AB}\cdot\text{CD}.$$

But $\text{AB}=|\vec{b}|$ (as given), and $\text{CD}=|\vec{a}|\sin\theta$.

$$\text{Thus, Area of triangle ABC}=\frac{1}{2}|\vec{b}||\vec{a}|\sin\theta=\frac{1}{2}|\vec{a}\times\vec{b}|.$$

Q. Prove that if $\vec{a}$ and $\vec{b}$ be two vectors given in component form as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, respectively. Then their cross product may be given by

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Explanation We have

$$\begin{aligned}
\vec{a}\times\vec{b} &= (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}) \\
&= a_1b_1(\hat{i}\times\hat{i})+a_1b_2(\hat{i}\times\hat{j})+a_1b_3(\hat{i}\times\hat{k})+a_2b_1(\hat{j}\times\hat{i}) \\
&\quad + a_2b_2(\hat{j}\times\hat{j})+a_2b_3(\hat{j}\times\hat{k}) \\
&\quad + a_3b_1(\hat{k}\times\hat{i})+a_3b_2(\hat{k}\times\hat{j})+a_3b_3(\hat{k}\times\hat{k}) \qquad \text{(by Property 1)} \\
&= a_1b_2(\hat{i}\times\hat{j})-a_1b_3(\hat{k}\times\hat{i})-a_2b_1(\hat{i}\times\hat{j}) \\
&\quad + a_2b_3(\hat{j}\times\hat{k})+a_3b_1(\hat{k}\times\hat{i})-a_3b_2(\hat{j}\times\hat{k})
\end{aligned}$$

(as $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$ and $\hat{i}\times\hat{k}=-\hat{k}\times\hat{i}$, $\hat{j}\times\hat{i}=-\hat{i}\times\hat{j}$ and $\hat{k}\times\hat{j}=-\hat{j}\times\hat{k}$)

$$= a_1b_2\hat{k}-a_1b_3\hat{j}-a_2b_1\hat{k}+a_2b_3\hat{i}+a_3b_1\hat{j}-a_3b_2\hat{i}$$

(as $\hat{i}\times\hat{j}=\hat{k}$, $\hat{j}\times\hat{k}=\hat{i}$ and $\hat{k}\times\hat{i}=\hat{j}$)

$$= (a_2b_3-a_3b_2)\hat{i}-(a_1b_3-a_3b_1)\hat{j}+(a_1b_2-a_2b_1)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 22 Find $|\vec{a}\times\vec{b}|$, if $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+5\hat{j}-2\hat{k}$

Solution We have

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$$

$$= \hat{i}(-2-15)-(-4-9)\hat{j}+(10-3)\hat{k} = -17\hat{i}+13\hat{j}+7\hat{k}$$

Hence $\quad |\vec{a}\ \ \vec{b}| = \sqrt{(-17)^2+(13)^2+(7)^2} = \sqrt{507}$

Example 23 Find a unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$, where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$.

Solution We have $\vec{a}+\vec{b}=2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{a}-\vec{b}=-\hat{j}-2\hat{k}$

A vector which is perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is given by

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = -2\hat{i}+4\hat{j}-2\hat{k} \quad (=\vec{c}, \text{ say})$$

Now $\quad |\vec{c}| = \sqrt{4+16+4}=\sqrt{24}=2\sqrt{6}$

Therefore, the required unit vector is

$$\frac{\vec{c}}{|\vec{c}|} = \frac{-1}{\sqrt{6}}\hat{i}+\frac{2}{\sqrt{6}}\hat{j}-\frac{1}{\sqrt{6}}\hat{k}$$

> **Note** There are two perpendicular directions to any plane. Thus, another unit vector perpendicular to $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ will be $\frac{1}{\sqrt{6}}\hat{i}-\frac{2}{\sqrt{6}}\hat{j}+\frac{1}{\sqrt{6}}\hat{k}$. But that will be a consequence of $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$.

Example 24 Find the area of a triangle having the points A(1, 1, 1), B(1, 2, 3) and C(2, 3, 1) as its vertices.

Solution We have $\overrightarrow{AB}=\hat{j}+2\hat{k}$ and $\overrightarrow{AC}=\hat{i}+2\hat{j}$. The area of the given triangle is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$.

Now, $\quad \overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4\hat{i}+2\hat{j}-\hat{k}$

Therefore $\quad |\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{16+4+1}=\sqrt{21}$

Thus, the required area is $\frac{1}{2}\sqrt{21}$

Example 25 Find the area of a parallelogram whose adjacent sides are given by the vectors $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$

Solution The area of a parallelogram with $\vec{a}$ and $\vec{b}$ as its adjacent sides is given by $|\vec{a} \times \vec{b}|$.

Now $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = 5\hat{i} + \hat{j} - 4\hat{k}$$

Therefore $$|\vec{a} \times \vec{b}| = \sqrt{25+1+16} = \sqrt{42}$$

and hence, the required area is $\sqrt{42}$.

Example 26 Write all the unit vectors in XY-plane.

Solution Let $\vec{r} = x\hat{i} + y\hat{j}$ be a unit vector in XY-plane (Fig 10.28). Then, from the figure, we have $x = \cos\theta$ and $y = \sin\theta$ (since $|\vec{r}| = 1$). So, we may write the vector $\vec{r}$ as

$$\vec{r}\,(=\overrightarrow{OP}) = \cos \quad \hat{i} \quad \sin \quad \hat{j} \qquad \text{... (1)}$$

Clearly, $$|\vec{r}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

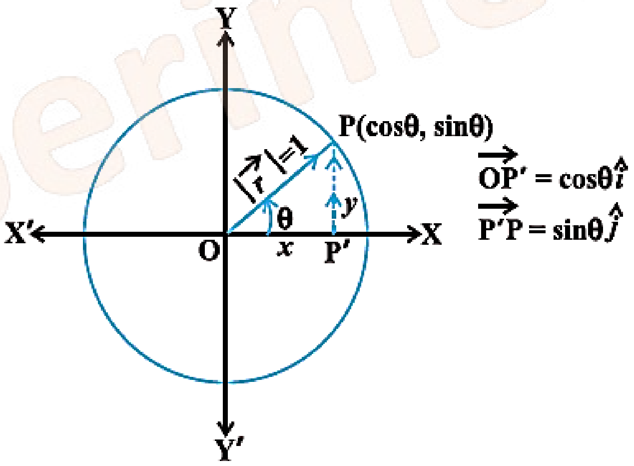


Fig 10.28

Also, as θ varies from 0 to 2π, the point P (Fig 10.28) traces the circle $x^2 + y^2 = 1$ counterclockwise, and this covers all possible directions. So, (1) gives every unit vector in the XY-plane.

Example 27 If $\hat{i}\quad \hat{j}\quad \hat{k}$, $2\hat{i}\quad 5\hat{j}$, $3\hat{i}\quad 2\hat{j}\quad 3\hat{k}$ and $\hat{i}\quad 6\hat{j}\quad \hat{k}$ are the position vectors of points A, B, C and D respectively, then find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Deduce that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Solution Note that if θ is the angle between AB and CD, then θ is also the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

Now $\overrightarrow{AB}$ = Position vector of B – Position vector of A

$$= (2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k}) = \hat{i}+4\hat{j}-\hat{k}$$

Therefore $$|\overrightarrow{AB}| = \sqrt{(1)^2+(4)^2+(-1)^2} = 3\sqrt{2}$$

Similarly $$\overrightarrow{CD} = -2\hat{i}-8\hat{j}+2\hat{k} \text{ and } |\overrightarrow{CD}| = 6\sqrt{2}$$

Thus $$\cos\theta = \frac{\overrightarrow{AB}\quad \overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|}$$

$$= \frac{1(-2)+4(-8)+(-1)(2)}{(3\sqrt{2})(6\sqrt{2})} = \frac{-36}{36} = -1$$

Since 0 ≤ θ ≤ π, it follows that θ = π. This shows that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Alternatively, $\overrightarrow{AB}\quad \frac{1}{2}\overrightarrow{CD}$ which implies that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear vectors.

Example 28 Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$ and each one of them being perpendicular to the sum of the other two, find $|\vec{a}+\vec{b}+\vec{c}|$.

Solution Given $\vec{a}\cdot(\vec{b}+\vec{c}) = 0,\ \vec{b}\cdot(\vec{c}+\vec{a}) = 0,\ \vec{c}\cdot(\vec{a}+\vec{b}) = 0.$

Now $$|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})^2 = (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$$

$$= \vec{a}\cdot\vec{a}+\vec{a}\cdot(\vec{b}+\vec{c})+\vec{b}\cdot\vec{b}+\vec{b}\cdot(\vec{a}+\vec{c})$$

$$+\ \vec{c}.(\vec{a}+\vec{b})+\vec{c}.\vec{c}$$

$$= |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2$$

$$= 9+16+25 = 50$$

Therefore $$|\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}$$

Example 29 Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Evaluate the quantity $\mu=\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$, if $|\vec{a}|=1$, $|\vec{b}|=4$ and $|\vec{c}|=2$.

Solution Since $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, we have

$$\vec{a}\ (\vec{a}\quad \vec{b}\quad \vec{c}) = 0$$

or $$\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} = 0$$

Therefore $$\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} = -|\vec{a}|^2 = -1 \qquad \text{... (1)}$$

Again, $$\vec{b}\cdot(\vec{a}+\vec{b}+\vec{c}) = 0$$

or $$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c} = -|\vec{b}|^2 = -16 \qquad \text{... (2)}$$

Similarly $$\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c} = -4. \qquad \text{... (3)}$$

Adding (1), (2) and (3), we have

$$2\,(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{c}) = -21$$

or $$2\mu = -21, \text{ i.e., } \mu = \frac{-21}{2}$$

Q. Explain various derivatives of Vectors.

Sol.

Recall that if **u**, **v**, **w** are vectors and α is a scalar, there are a number of different products that can be made;

Name of product	Formula	Type of result
Scalar multiplication	$\alpha\mathbf{u}$	Vector
Scalar or dot product	$\mathbf{u}\cdot\mathbf{v}$	Scalar
Vector or cross product	$\mathbf{u}\times\mathbf{v}$	Vector

Now consider the vector differential operator

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

This is read as *del* or *nabla* and is not to be confused with Δ, the capital Greek letter delta. One can form "products" of this vector with other vectors and scalars, but because it is an operator, it always has to be the first term if the product is to make sense. For example, if f is a scalar field, we can form the scalar "multiple" with ∇ as the first term

$$\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$

the result being a vector.

Below we will introduce the "derivatives" corresponding to the product of vectors given in the above table.

The derivatives of vectors are :

1. Gradient ("multiplication by a scalar")

This is just the example given above. We define the *gradient* of a scalar field f to be

$$\boxed{\operatorname{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)}.$$

We will use both of the notation $\operatorname{grad} f$ and ∇f interchangably.

Remark Note that f must be a *scalar* field for $\operatorname{grad} f$ to be defined and $\operatorname{grad} f$ itself is a *vector* field.

2 Divergence of a vector field ("scalar product")

The *divergence* of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is the *scalar* obtained as the "scalar product" of ∇ and $\mathbf{F}$,

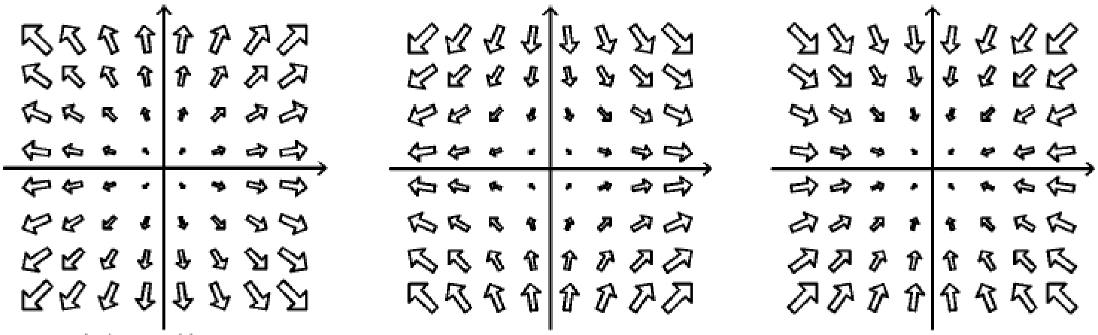
$$\boxed{\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}.$$

It is so called, because it measures the tendency of a vector field to diverge (positive divergence) or converge (negative divergence). In particular, a vector field is said to be *incompressible* (or *solenoidal*) if its divergence is zero.

Figure 4.3 shows the vector fields $\mathbf{F} = (x, y, 0)$, $\mathbf{G} = (x, -y, 0)$ and $\mathbf{H} = (-x, -y, 0)$ in the xy-plane. We have

$$\operatorname{div}\mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 2 > 0$$

and similarly, $\operatorname{div}\mathbf{G} = 0$ and $\operatorname{div}\mathbf{H} = -2 < 0$. Notice how the arrows on the plot of **F** diverge and on the plot of **H** converge.



F, positive divergence **G**, incompressible **H**, negative divergence

Figure 4.3: Positive and negative divergence

A particular example of divergence is the *Laplacian* of a scalar field. Given a scalar field f, grad $f = \nabla f$ is a vector field and the divergence of ∇f is the Laplacian of f, written $\nabla^2 f$. This means that

$$\boxed{\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}}$$

This definition may be extended in a natural way to the Laplacian of a vector field $\mathbf{F} = (F_1, F_2, F_3)$,

$$\boxed{\nabla^2 \mathbf{F} = (\nabla^2 F_1, \nabla^2 F_2, \nabla^2 F_3)}$$

.3 Curl of a vector field ("vector product")

The *curl* of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is the *vector* obtained as the "vector product" of ∇ and $\mathbf{F}$

$$\boxed{\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}}\ .$$

Like any other vector product, curl $\mathbf{F}$ can be calculated using a 3×3 determinant,

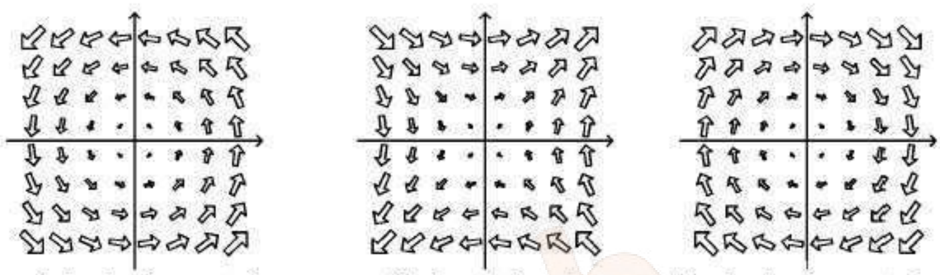
$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}\ .$$

The curl of a vector field measures its tendency to rotate. In particular, a vector field is said to be *irrotational* if its curl is the zero vector. Figure 4.4 shows the vector fields $\mathbf{F} = (-y, x, 0)$, $\mathbf{G} = (y, x, 0)$ and

$\mathbf{H} = (y, -x, 0)$. We have

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\mathbf{k}$$

and similarly, $\operatorname{curl}\mathbf{G} = \mathbf{0}$ and $\operatorname{curl}\mathbf{H} = -2\mathbf{k} < 0$. The coefficient of $\mathbf{k}$ in $\operatorname{curl}\mathbf{F}$ being positive indicates anticlockwise rotation.



F, anticlockwise rotation   **G**, irrotational   **H**, clockwise rotation

Figure 4.4: Clockwise and anticlockwise rotation

Q. write the Identities for vector derivatives.

sol.

There are analogues involving div, grad and curl of the elementary rules of differentiation such as linearity $(f+g)'(x) = f'(x) + g'(x)$ the product rule $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$.

Let f and g be smooth scalar fields and $\mathbf{F}$ and $\mathbf{G}$ smooth vector fields. Then all of the following are straightforward to prove (as illustrated in Example 4.12) just using definitions

$$\begin{array}{ll} \operatorname{grad}(f+g) = \operatorname{grad} f + \operatorname{grad} g & \operatorname{grad}(fg) = f(\operatorname{grad} g) + (\operatorname{grad} f)g, \\ \operatorname{div}(\mathbf{F}+\mathbf{G}) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G} & \operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + (\operatorname{grad} f)\cdot\mathbf{F}, \\ \operatorname{curl}(\mathbf{F}+\mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G} & \operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + \operatorname{grad} f \times \mathbf{F}, \\ \operatorname{curl}\operatorname{grad} f = \mathbf{0}, & \operatorname{div}\operatorname{curl}\mathbf{F} = 0. \end{array}$$

In particular, note the special cases

$$\operatorname{grad}(cf) = c\operatorname{grad} f, \quad \operatorname{div}(c\mathbf{F}) = c\operatorname{div}\mathbf{F}, \quad \operatorname{curl}(c\mathbf{F}) = c\operatorname{curl}\mathbf{F},$$

when c is a (scalar) constant.

All of the identities are easier to remember if written using ∇. For example,

$$\begin{aligned} \operatorname{curl}(f\mathbf{F}) &= \nabla \times (f\mathbf{F}) \\ &= f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F} \\ &= f\operatorname{curl}\mathbf{F} + \operatorname{grad} f \times \mathbf{F}. \end{aligned}$$

Problem Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution

Using (11), it can be seen that the vectors

$n_1 = 3i - 6j - 2k, \quad n_2 = 2i + j - 2k$

are normals to the given planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$, respectively. The angle between them (using the definition of dot product) is

$$\theta = cos^{-1}\left(\frac{n_1 . n_2}{|n_1||n_2|}\right)$$

$$= cos^{-1}\left(\frac{4}{21}\right)$$

Example 1 Determine if $\vec{F} = x^2 y\vec{i} + xyz\,\vec{j} - x^2 y^2 \vec{k}$ is a conservative vector field.

Solution

So all that we need to do is compute the curl and see if we get the zero vector or not.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & xyz & -x^2 y^2 \end{vmatrix}$$

$$= -2x^2 y\vec{i} + yz\,\vec{k} - \left(-2xy^2\,\vec{j}\right) - xy\vec{i} - x^2\vec{k}$$

$$= -\left(2x^2 y + xy\right)\vec{i} + 2xy^2\,\vec{j} + \left(yz - x^2\right)\vec{k}$$

$$\neq \vec{0}$$

So, the curl isn't the zero vector and so this vector field is not conservative.

Example 1: Determine if the vector field $\mathbf{F} = yz^2\mathbf{i} + (xz^2 + 2)\,\mathbf{j} + (2xyz - 1)\,\mathbf{k}$ is conservative.

Solution:

$$curlF = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz^2 & xz^2 + 2 & 2xyz - 1 \end{vmatrix}$$

$$= (2xz - 2xz)i - (2yz - 2yz)j + \left(z^2 - z^2\right)k$$

$$= 0$$

Therefore the given vector field F is conservative.

Example 2: Find the curl of $F(x, y, z) = 3x^2\mathbf{i} + 2z\mathbf{j} - x\mathbf{k}$.

Solution:

$$curlF = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & 2z & -x \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-x) - \frac{\partial}{\partial z}(2z)\right)i - \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial z}(3x^2)\right)j + \left(\frac{\partial}{\partial x}(2z) - \frac{\partial}{\partial y}(3x^2)\right)k$$

$$= (0-2)i - (-1-0)j + (0-0)k$$

$$= -2i + j$$

Example 3: What is the curl of the vector field $\mathbf{F} = (x + y + z, x - y - z, x^2 + y^2 + z^2)$?

Solution:

$$curlF = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+z & x-y-z & x^2+y^2+z^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(x^2 + y^2 + z^2) - \frac{\partial}{\partial z}(x - y - z)\right)i - \left(\frac{\partial}{\partial x}(x^2 + y^2 + z^2) - \frac{\partial}{\partial z}(x + y + z)\right)j +$$

$$\left(\frac{\partial}{\partial x}(x - y - z) - \frac{\partial}{\partial y}(x + y + z)\right)k$$

$$= (2y+1)i - (2x-1)j + (1-1)k$$

$$= (2y+1)i + (1-2x)j + 0k$$

$$= (2y+1, 1-2x, 0)$$

Example 4: Find the curl of $F = (x^2 - y)i + 4zj + x^2k$.

Solution:

$$curlF = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - y) & 4z & x^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial z}(4z)\right)i - \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial z}(x^2 - y)\right)j +$$

$$\left(\frac{\partial}{\partial x}(4z) - \frac{\partial}{\partial y}(x^2 - y)\right)k$$

$$= (0-4)i - (2x-0)j + (0+1)k$$

$$= (-4)i - (2x)j + 1k$$

$$= (-4,-2x,1)$$

Q.1 $Ex) \quad y' + y\sec x = \cos^2 x \quad , \quad y' = \frac{dy}{dx}$

solution

$$\frac{dy}{dx} + p(x)y = Q(x) \Rightarrow p(x) = \sec x \quad , \quad Q(x) = \cos^2 x$$

$$\mu = \exp\int \sec x dx = \exp \ln \left|\sec x + \tan x\right|$$

$$\mu = \sec x + \tan x$$

general solution is

$$\mu\, y = \int \mu\, Q\, dx + c$$

$$\therefore (\sec x + \tan x)\, y = \int (\sec x + \tan x).\cos^2 dx$$

$$= \int (\cos x + \sin x \cos x) dx$$

$$= \sin x + \frac{1}{2}\sin x + c$$

Ex) $y\dfrac{dy}{dx}-\dfrac{y^2}{x}=3\sin x$

solution

$$\frac{dy}{dx}-\frac{y}{x}=3\sin x . y^{-1}$$

$$put \;\; z=y \quad and then \quad \frac{dz}{dx}=2y\frac{dy}{dx}$$

$\dfrac{dz}{dx}+2\dfrac{z}{x}=6\sin x$ thisequation is linear

$$\mu=\exp\int p dx=\exp\int \tfrac{2}{x}dx=\exp 2\ln x=\exp\ln x^2$$

$$\therefore \mu=x^2$$

Ex) $x^2p^2+3xpy+2y^2=0$

Sol

$$(xp+y)(xp+2y)=0$$

$$xp+y=0 \qquad or \qquad xp+2y=0$$

$$x\frac{dy}{dx}+y=0 \qquad or \qquad x\frac{dy}{dx}+2y=0$$

$$\int\frac{dy}{y}=\int-\frac{dx}{x} \qquad or \qquad \int\frac{dy}{y}=\int-2\frac{dx}{x}$$

$$\ln y+\ln x=\ln c_1 \qquad or \qquad \ln y+2\ln x=\ln c_2$$

$$xy-c_1=0 \qquad or \qquad x^2y-c_2=0$$

$$\therefore (xy-c_1)(x^2y-c_2)=0$$

and this is the generalsolution of theequation.

$$Ex) \quad 3y = 2px - 2\frac{p^2}{x} \qquad , \quad p = \frac{dy}{dx}$$

Solution

$$y = \frac{2}{3}px - \frac{2}{3}\frac{p^2}{x}$$ by differentiation withrespect tox

$$\frac{dy}{dx} = p = \frac{2}{3}x\frac{dp}{dx} - \frac{2}{3}2p\frac{1}{x}\frac{dp}{dx} + \frac{2}{3}\frac{p^2}{x^2}$$

$$\frac{1}{3}p - \frac{2}{3}\frac{p^2}{x^2} = \left(\frac{2}{3}x - \frac{4}{3}\frac{p}{x}\right)\frac{dp}{dx}$$, multiplying by 3

$$p - 2\frac{p^2}{x^2} = 2\left(x - 2\frac{p}{x}\right)\frac{dp}{dx}$$, multiplying by x^2

$$px^2 - 2p^2 = 2(x^3 - 2px)\frac{dp}{dx}$$

$$p(x^2 - 2p) = 2x(x^2 - 2p)\frac{dp}{dx}$$

$$(x^2 - 2p)\left(p - 2x\frac{dp}{dx}\right) = 0$$

$$x^2 - 2p = 0 \qquad or \qquad p - 2x\frac{dp}{dx} = 0$$

$$2\frac{dp}{dx} = x^2 \qquad\qquad p = 2x\frac{dp}{dx}$$

$$\int 2dy = \int x^2 dx \qquad\qquad \int\frac{dp}{p} = \int\frac{dx}{2x}$$

$$2y = \frac{x3}{3} + c \qquad\qquad \ln p = \frac{1}{2}\ln x \Rightarrow p = \sqrt{x}$$

to delete p from twoequation substituting about p on origin equation

$$y = \frac{1}{6}x^3$$

$Ex) \quad x = p + p^3 \quad , \; p = \frac{dy}{dx}$

by differentiation with respect toy

$$\frac{dx}{dy} = \frac{dp}{dy} + 3p^2 \frac{dp}{dy} \quad , \text{but } \frac{1}{p} = \frac{dx}{dy}$$

$$\frac{1}{p} = (1 - 3p^2)\frac{dp}{dy}$$

$$\frac{dp}{dy} = \frac{1}{p(1+3p^2)}$$

$$\int dy = \int (p + 3p^3)dp$$

$$y = \frac{1}{2}p^2 + \frac{3}{4}p^4$$

$$x = p + p^3 \quad \text{(the origin equation)}$$

we can not delete p from the last tow equations so this the parametric solution.

Ex) $y = 2xp + p$

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

$$p = 2p(2x + 2p)\frac{dp}{dx} \Rightarrow -p = (2x + 2p)\frac{dp}{dx}$$

$$1 = (-\frac{2x}{p} - 2)\frac{dp}{dx} \Rightarrow \frac{dx}{dp} = -\frac{2x}{p} - 2$$

$$\frac{dx}{dp} + \frac{2x}{p} = -2 \quad \text{linear differential equation}$$

$$\mu = \exp\int 2\frac{dp}{p} \quad \text{integral factor}$$

$$e^{2\ln p} = p^2 \Rightarrow p^2 x = \int -2p^2 dp$$

$$p^2 x = -\frac{2p^3}{3} + c$$

$$y'' + y' - 6y = 8e^{3x}$$

$$(D^2 + D - 6)y = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda - 2)(\lambda + 3) = 0$$

$$\lambda_1 = 2 \quad , \quad \lambda_2 = -3$$

$$y_c = C_1 e^{2x} + C_2 e^{\ 3x}$$

$$y_p = \frac{1}{\underset{D \Leftrightarrow a}{D^2 + D - 6}} 8e^{3x} \Rightarrow y_p = \frac{1}{9+3-6} 8e^{3x}$$

$$y_p = \frac{8}{6} e^{3x} = \frac{4}{3} e^{3x}$$

general solution $\quad y = y_c + y_p$

$$y = C_1 e^{2x} + C_2 e^{-3x} + \frac{4}{3} e^{3x}$$