B.Sc. Physics (Hons)
Mechanics

1. Attempt any *five* of the following questions :

   (*i*) What is the speed of a particle whose relativistic mass is twice its rest mass ?

   (*ii*) Two bodies of different masses are moving with the same kinetic energy of translation. Which has greater momentum ?

   (*iii*) What is potential energy curve of a particle ? What significant information does it give about the motion of the particle ?

   (*iv*) Calculate the period of revolution of an artificial satellite at a height *h* from the surface of the earth, assuming that the satellite takes a circular orbit around earth.

P.T.O.

(v) Find the centre of mass of a thin uniform wire bent in the form of a semi-circle of radius R.

(vi) Find the impulse developed by a force $\mathbf{F} = 4t\hat{i} + (6t^2 - 2)\hat{j} + 12\hat{k}$ from time $t = 0$ to $t = 4$ s.

(vii) A person normally weighing 60 kg stands on a platform which is oscillating up and down harmonically with a time period of 1.0 s and amplitude of 10 cm. If a weighing machine on the platform gives the person's weight against time, what will be the maximum and minimum readings shown by it ?

(viii) A solid sphere of mass 0.1 kg and radius 2.5 cm rolls without slipping with uniform velocity of 0.1 ms^{-1} along a straight line on a horizontal table. Calculate its total energy. 5×3

2. (i) Establish the equation of motion of a damped harmonic oscillator subjected to a resistive force that is proportional to the first power of its velocity. If the damping is less than critical, show that the motion of the system is oscillatory with its amplitude decaying exponentially with time.

(ii) What do you understand by 'logarithmic decrement', 'relaxation time' and 'equality factor' of a weakly damped harmonic oscillator ? 12,3

3. A particle of mass m_1 moves with velocity v_1 in the Laboratory frame of reference and collides elastically with a particle of mass m_2 at rest in the Laboratory frame. If, after the collision, the direction of the first particle makes angles θ and φ with respect to its initial direction in the Laboratory and Centre of mass frames respectively :

(i) Verify that :

$$\tan\theta = \sin\varphi/\{\cos\varphi + (1/A)\}, \text{ where } = m_2/m_1.$$

(ii) If $m_1 = m_2$, then verify that $\theta = \varphi/2$. What is then the maximum value of θ ?

(*iii*) Show that the maximum value of θ for arbitrary A when $A < 1$ is given by $\tan\theta = A/(1 - A^2)^{1/2}$. 7,4,4

4. (*i*) A reference frame S' rotates with respect to another reference frame S with uniform angular velocity ω. Obtain a relation between the acceleration experienced by a particle in the two frames of reference. Highlight the physical significance of each term in the above relation.

(*ii*) Calculate the values of the Centrifugal and the Coriolis forces on a mass of 20 g placed at a distance of 10 cm from the axis of rotating frame of reference, if the angular speed of rotation of the frame be 10 $rads^{-1}$. 9,6

5. (*i*) On the basis of Lorentz transformations derive an expression for length contraction and time dilation.

(*ii*) With what velocity should a rocket move so that every year spent on it corresponds to 4 years on earth.

(*iii*) Two electrons move towards each other, the speed of each being 0.8 c in a Galilean frame of reference. What is their speed relative to each other ? 8,3,4

6. (*i*) Show that the total angular momentum of a system of particles about a fixed point is given by the relation $\mathbf{J} = \mathbf{R} \times \mathbf{P} + \mathbf{J}_{c.m.}$, where $(\mathbf{R} \times \mathbf{P})$ is the angular momentum of the centre of mass about that point and $\mathbf{J}_{c.m.}$, the angular momentum of the system about the centre of mass. What do the two terms in the above relation represent physically ?

(*ii*) Show that the angular momentum of a particle under the influence of a central force always remains constant.

(*iii*) Prove that a projectile launched on level ground reaches its maximum height midway along its trajectory. 6,4,5

7. (*i*) A rocket ascends from rest in a uniform gravitational field by ejecting exhaust gases with a constant speed u relative to the rocket. Assuming that the rate at which mass is expelled is given by $\frac{dM}{dt} = -\gamma M$, where M is the instantaneous mass of the rocket and γ is a constant, find the velocity of the rocket as a function of time. 8

(*ii*) A particle moves along the x-axis under the influence of a force $F = ax^2 - b$, where a and b are positive constants.

(*a*) Find its potential energy as a function of position, taking $U = 0$ at $x = 0$.

(*b*) Determine the positions of equilibrium of the particle and predict the nature of equilibrium at these points.

(*c*) Find the value/s of equilibrium potential energy of the particle. 2,3,2

8. (*i*) Derive expressions for gravitational potential at a point inside and outside a uniform solid sphere of radius R and mass M. Also, represent your results graphically.

(*ii*) (*a*) Find the moment of inertia of a solid sphere of radius R and mass M about its diameter and tangent.

(*b*) Calculate the radius of gyration of a solid sphere of radius 5 cm rotating about its diameter. 7,6,2

Q 1(i)

Ans 1 (i): Given, $m = 2m_0$.

we know, $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$

$$2m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Rightarrow 1 - \frac{v^2}{c^2} = \frac{1}{4}$$

$$\Rightarrow \frac{v^2}{c^2} = \frac{3}{4}$$

$$\Rightarrow v = \frac{\sqrt{3}\,c}{2}$$

Q 1(ii)

Ans. Let m_1, m_2 and v_1, v_2 be the mass and velocity of particle 1 and 2. As both of these objects have same kinetic energy then -

$$\frac{1}{2}m_1v_1^2 = \frac{1}{2}m_2v_2^2$$

or $$m_1 v_1^2 = m_2 v_2^2$$

If $m_1 < m_2$, then

$$v_1 > v_2$$

The momentum is mv.

$$m_1v_1 \cdot v_1 = m_2v_2 \cdot v_2$$

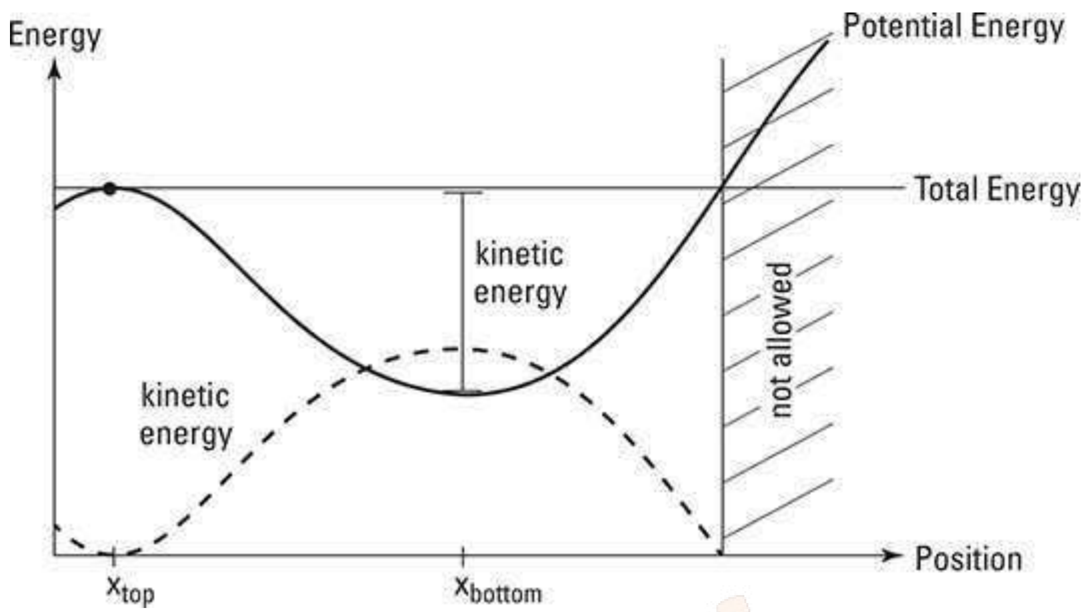
$\therefore$ $$\frac{p_1}{p_2} = \frac{v_1}{v_2} < 1$$

$\therefore$ $$p_1 < p_2 \text{ or } p_2 > p_1$$

$\therefore$ The heavier body has more momentum.

Q 1(iii)

Ans.: Potential energy curve of a particle is the graphical representation of potential energy with respect to position of the particle.



It tells us about stable and unstable equlibrium points.
Stable equilibrium exists if the net force is zero, and small changes in the system would cause an increase in potential energy.
Unstable equilibrium exists if the net force is zero, and small changes in the system would cause a decrease in potential energy.

Q 1(iv)
Ans.

Ans 1 (d) : Suppose a satellite of mass, m is moving in an orbit of radius, r around earth with velocity v. then,

centripetal force = gravitational force.

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$\Rightarrow v = \sqrt{\frac{GM}{r}}$$

Also, $v = r\omega$

$$\Rightarrow \omega = \sqrt{\frac{GM}{r^3}}$$

Time period, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r^3}{GM}}$

At height, h $r = R+h$

where $R \rightarrow$ radius of earth

$$T = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$$

In terms, of acc. due to gravity:

Also, $mg = \frac{GMm}{r^2}$

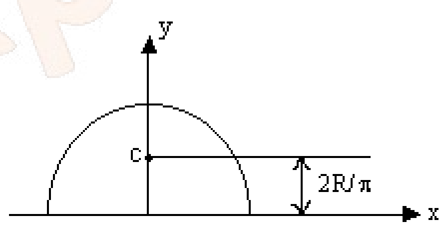
$$GM = gr^2$$

At $r = R+h$, $GM = g(R+h)^2$

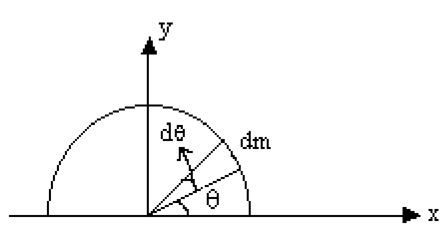
$$T = 2\pi\sqrt{\frac{(R+h)^3}{g(R+h)^2}} = 2\pi\sqrt{\frac{R+h}{g}}$$

Q 1(v)
Ans.

Center of Mass of semi circular wire:



Derivation:



Total length of semicircular wire = R
and elemental length = Rdq

$$\text{So}, y = \int_0^{\pi} \frac{1}{m} y\,dm$$

$$= \frac{1}{m}\int_0^{\pi} R\sin\theta \frac{m}{\pi} d\theta$$

$$= \frac{2R}{\pi}$$

Q 1(vi)

Ans. Impulse,

$$J = \int_{t_1}^{t_2} \left(4ti + (6t^2 - 2)j + 12k\right) dr$$

$$= \int_0^4 4.\frac{t^2}{r} i + \left(\frac{6t^3}{3} - 2t\right) j + 12tk$$

$$= \int_0^4 2t^2 i + \left(2t^3 - 2t\right) j + 12tk$$

$$J = 2(4)^2\, i + [2(4)^3 - 2(4)]\, j + 12(4)\, k$$

$$= 32\, i + (128 - 8)\, j + 48\, k$$

$$= 32\, i + 120\, j + 48\, k$$

Magnitude of

$$J = \sqrt{(32)^2 + (120)^2 + (48)^2}$$

$$= \sqrt{1024 + 14400 + 2304}$$

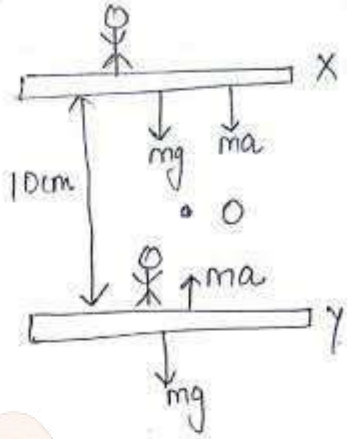
$$= \sqrt{17728}$$

$$= 133.15 \text{ Ns.}$$

Q 1(vii)

Ans.

Q1(vii) Suppose platform oscillates up and down simple harmonically between positions X and Y about its mean position, O as shown



Actual mass of person, m = 60kg

frequency, $\nu = \frac{1}{T} = 1\ sec^{-1}$

amplitude, A = 10 cm = 0.1 m

In SHM, acc., a is max. at extreme points X and Y and is directed towards mean position O.

Then, $a = \omega^2 A = (2\pi\nu)^2 A$

$= (2\pi.1)^2\ 0.1$

$= 3.94\ m/s^2$

(i) when platform is at extreme position X

$W_x = mg + ma = m(g+a)$

$= 60\,(9.8 + 3.94) = 824.4N$

$= 84.12\ Kg$

(ii) when platform is at extreme position, Y.

$W_Y = mg - ma = m(g-a)$

$= 60\,(9.8 - 3.94) = 351.6N$

$= 35.88\ kg$

Hence max. weight = 84.12 Kg

min. weight = 35.88 kg.

1(viii)

Ans.

Ans. Total energy, E, of a rolling body is :

$$E = K_t + K_r$$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}Iw^2$$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}Mk^2 \cdot \frac{v^2}{R^2}$$

$$= \frac{1}{2}Mv^2\left[1 + \frac{k^2}{R^2}\right]$$

$\frac{k^2}{R^2}$ for A solid sphere $= \frac{2}{5}$

$$\therefore \quad E = \frac{1}{2}Mv^2\left[1 + \frac{2}{5}\right]$$

$$= \frac{1}{2} \times \frac{7}{5} Mv^2$$

$$= \frac{7}{10} Mv^2$$

$M = 0.1$ kg

$V = 0.1\ ms^{-1}$

$$\therefore \quad E = \frac{7}{10} \times \frac{1}{10} \times \frac{1}{10} = 7 \times 10^{-3}\ kgms^{-1}$$

Q 2(i) Ans.

A damped harmonic oscillator is displaced by a distance x_0 and released at time $t = 0$. Show that the subsequent motion is described by the differential equation

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = 0,$$

or equivalently

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = 0,$$

with $x = x_0$ and $\dot{x} = 0$ at $t = 0$, explaining the physical meaning of the parameters m, γ and ω_0.

Solution: The forces on the mass m are $F_s = -kx = -m\omega_0^2 x$ due to the spring and $F_f = -m\gamma\dot{x}$ due to friction γ. The equation follows from Newton's law $m\ddot{x} = F_s + F_f$.

The characteristic polynomial for ansatz $x(t) = e^{\lambda t}$ is $\lambda^2 + \gamma\lambda + \omega_0^2 = 0$ leading to eigenfrequencies

$$\lambda_{1,2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}.$$

We get (i) overdamping when $\gamma > 2\omega_0$ and hence solutions do not oscillate, (ii) critical damping for $\gamma = 2\omega_0$ and (iii) underdamping for $\gamma < 2\omega_0$. Different solutions are shown in Fig. 1. The general solution is given by

$$x(t) = \Re\left\{A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}\right\} .$$

and can be simplified for the different situations (writing $\alpha = \sqrt{|\omega_0^2 - \gamma^2/4|}$) for the three cases

(i) $x(t) = e^{-\gamma t/2}[A\cosh(\alpha t) + B\sinh(\alpha t)]$ or equivalently $x(t) = e^{-\gamma t/2}(Ce^{\alpha t} + De^{-\alpha t})$

(ii) $x(t) = e^{-\gamma t/2}(A + Bt)$

(iii) $x(t) = e^{-\gamma t/2}[A\cos(\alpha t) + B\sin(\alpha t)]$

using the standard procedure for degenerate roots of the characteristic polynomial in (ii).

By matching the initial conditions we find for the different cases

(i) $A = x_0$ and $B = x_0\gamma/(2\alpha)$ or equivalently $C = x_0(\alpha + \gamma/2)/(2\alpha)$ and $D = x_0(\alpha - \gamma/2)/(2\alpha)$

(ii) $A = x_0$ and $B = x_0\gamma/2$

(iii) $A = x_0$ and $B = x_0\gamma/(2\alpha)$

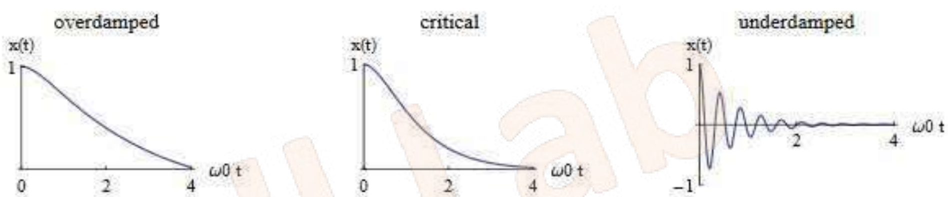


Figure 1: Oscillator displacement for different dampings.

The energy stored in the harmonic oscillator is the sum of kinetic and elastic energy

$$E(t) = \frac{m\dot{x}(t)^2}{2} + \frac{m\omega_0^2 x(t)^2}{2} .$$

In order to proceed for the lightly damped case it is easiest to write $x(t) = A\cos(\alpha t - \phi)e^{-\gamma t/2}$ and thus $\dot{x}(t) = -A\alpha\sin(\alpha t - \phi)e^{-\gamma t/2} - \gamma x(t)/2$. Since lightly damped means $\gamma \ll \omega_0$ we may neglect the second term in $\dot{x}(t)$ and approximate $\alpha \approx \omega_0$. Then the expression for the energy simplifies to

$$E(t) = \frac{m\omega_0^2}{2} A^2 e^{-\gamma t} .$$

A radian corresponds to the time difference $\tau = 1/\omega_0$ and so we find the energy lost per radian

$$E_L = E(0) - E(1/\omega_0) = \frac{m\omega_0^2}{2} A^2(1 - e^{-\gamma/\omega_0}) \approx \frac{m\omega_0\gamma}{2} A^2 .$$

by expanding $e^{-\gamma/\omega_0} \approx 1 - \gamma/\omega_0$ for $\gamma \ll \omega_0$. Hence the result $Q = E(0)/E_L = \omega_0/\gamma$ follows as required.

We now turn to the forced damped harmonic oscillator. The solutions to the homogeneous equation will damp out on a time scale $1/\gamma$. At times $t \gg 1/\gamma$ only terms arising from the particular solution will remain. These terms describe the stationary state[2]. We work out a particular solution using the ansatz $x(t) = \Re\left\{\mathcal{A}(\omega)e^{i\omega t}\right\}$ and find

$$\mathcal{A}(\omega) = \frac{F}{m(\omega_0^2 - \omega^2 + i\gamma\omega)} = |\mathcal{A}(\omega)|e^{-i\varphi} ,$$

where

$$|\mathcal{A}(\omega)| = \frac{F}{m\sqrt{((\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2)}} \quad \text{and} \quad \varphi = \begin{cases} \arctan\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) & \text{for} \quad \omega \le \omega_0 \\ \pi + \arctan\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) & \text{for} \quad \omega > \omega \end{cases}$$

Magnitude $|\mathcal{A}(\omega)|$ and phase φ are shown in Fig. 2 as a function of ω. The velocity is given by $\dot{x}(t) = \Re\{i|\mathcal{A}(\omega)|\omega e^{i\omega t}\} = -|\mathcal{A}(\omega)|\omega\cos(\omega t - (\varphi + \pi/2))$, i.e. there is an additional shift of $\pi/2$ compared to the displacement. The additional factor of ω shifts the maximum amplitude of $\dot{x}(t)$ compared to that of $x(t)$. Amplitude and phase of $\dot{x}(t)$ are shown in Fig. 2.

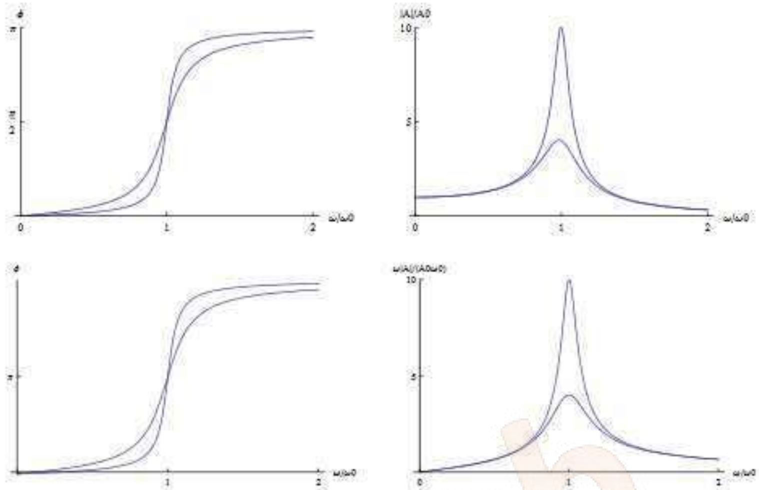


Figure 2: Displacement and velocity response to periodic driving for $\gamma = \omega_0/10$ and $\gamma = \omega_0/4$.

The maximum of the displacement amplitude is found by solving $\mathrm{d}|\mathcal{A}(\omega)|/\mathrm{d}\omega = 0$ giving a resonance frequency $\omega_x^2 = \omega_0^2 - \gamma^2/2$. For the maximum velocity amplitude we solve $\mathrm{d}|\omega\mathcal{A}(\omega)|/\mathrm{d}\omega = 0$ and find the resonance frequency $\omega_{\dot{x}} = \omega_0$.

We write the full width half maximum as $\Delta\omega = \omega_2 - \omega_1$ with $\mathcal{A}(\omega_{i=1,2}) = \mathcal{A}(\omega_x)/2$. We take the square of this expression and find

$$\frac{1}{(\omega_0^2 - \omega_i^2)^2 + \gamma^2\omega_i^2} = \frac{1}{4}\frac{1}{(\omega_0^2 - \omega_x^2)^2 + \gamma^2\omega_x^2}$$

Q 2 (ii)

This can be re-written as $\gamma^4 + \gamma^2(\omega_i^2 - 4\omega_0^2) + (\omega_0^2 - \omega_i^2)^2 = 0$. This can in principle be solved for ω_i but since we have assumed the oscillator to be lightly damped and have worked out quantities like Q only to lowest order in γ/ω_0 we instead only look for a solution valid to this order. We thus substitute $\omega_i \approx \omega_0(1 + \beta\gamma)$ obtaining $\gamma^4 + \gamma^2(\beta\gamma\omega_0 - 3\omega_0^2) + \beta^2\gamma^2\omega_0^2 = 0$. We now ignore any terms of $\mathcal{O}(\gamma^3)$ and $\mathcal{O}(\gamma^4)$ and thus get the approximate solution $\beta = \pm\sqrt{3}$ and thus $\omega_0^2 - \omega_i^2 \approx \pm\sqrt{3}\gamma\omega_0$ to lowest order. A Taylor series expansion in γ/ω_0 yields

$$\Delta\omega = \omega_2 - \omega_1 = \omega_0\left(\sqrt{1 + \sqrt{3}\gamma/\omega_0} - \sqrt{1 - \sqrt{3}\gamma/\omega_0}\right) \approx \sqrt{3}\gamma\,.$$

Hence $\Delta\omega/\omega_0 = \sqrt{3}\gamma/\omega_0$.

Near resonance $\omega \approx \omega_0$ and we thus find for the energy of the oscillator

$$E = \frac{m}{2}\dot{x}(t)^2 + \frac{m\omega_0^2}{2}x(t)^2 \approx \frac{F^2}{2m\gamma^2}\,.$$

The average supplied power is given by

$$P = \overline{F\cos(\omega t)\dot{x}(t)} = -F|\mathcal{A}(\omega)|\omega\overline{\cos(\omega t)\sin(\omega t - \varphi)} = -F|\mathcal{A}(\omega)|\omega\overline{\cos(\omega t)(\sin(\omega t)\cos(\varphi) - \cos(\omega t)\sin(\varphi))}$$

Near resonance we have $\varphi \approx \pi/2$ and $\omega \approx \omega_0$ so that

$$P = \frac{F|\mathcal{A}(\omega_0)|\omega_0}{2} = \frac{F^2}{2m\gamma}\,.$$

In the steady state the energy dissipated per radian must be equal to the energy supplied by the external force per radian $E_L = P\tau = P/\omega_0$. Thus

$$Q = \frac{E}{E_L} = \frac{\omega_0}{\gamma} \quad \text{and} \quad \frac{\Delta\omega}{\omega_0} = \frac{\sqrt{3}}{Q}\,.$$

Q 3

Ans. Suppose that, in the center of mass frame, the first particle has velocity $\mathbf{v}_1$ before the collision, and velocity $\mathbf{v}_1'$ after the collision. Likewise, the second particle has velocity $\mathbf{v}_2$ before the collision, and $\mathbf{v}_2'$ after the collision. We know that

$$m_1\,\mathbf{v}_1 + m_2\,\mathbf{v}_2 = m_1\,\mathbf{v}_1' + m_2\,\mathbf{v}_2' = \mathbf{0} \tag{368}$$

in the center of mass frame. Moreover, since the collision is assumed to be elastic (*i.e.*, energy conserving),

$$v_1' = v_1, \tag{369}$$

$$v_2' = v_2. \tag{370}$$

Let us transform to a new inertial frame of reference--which we shall call the *laboratory frame*--which is moving with the uniform velocity $-\mathbf{v}_2$ with respect to the center of mass frame. In the new reference frame, the first particle has initial velocity $\mathbf{V}_1 = \mathbf{v}_1 - \mathbf{v}_2$, and final velocity $\mathbf{V}_1' = \mathbf{v}_1' - \mathbf{v}_2$. Furthermore, the second particle is initially at *rest*, and has the final velocity $\mathbf{V}_2' = \mathbf{v}_2' - \mathbf{v}_2$. The relationship between scattering in the center of mass frame and scattering in the laboratory frame is illustrated in Figure 23.

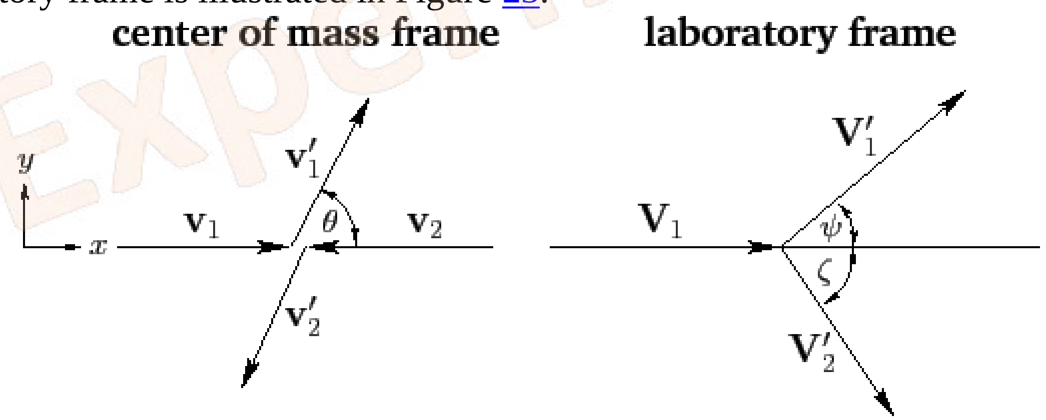


Figure 23: *Scattering in the center of mass and laboratory frames.*

In the center of mass frame, both particles are scattered through the same angle θ. However, in the laboratory frame, the first and second particles are scattered by the (generally different) angles ψ and ζ, respectively.

Defining x- and y-axes, as indicated in Figure 23, it is easily seen that the Cartesian components of the various velocity vectors in the two frames of reference are:

$$\mathbf{v}_1 = v_1\,(1,\,0), \tag{371}$$

$$\mathbf{v}_2 = (m_1/m_2)\,v_1\,(-1,\,0), \tag{372}$$

$$\mathbf{v}_1' = v_1\,(\cos\theta,\,\sin\theta), \tag{373}$$

$$\mathbf{v}_2' = (m_1/m_2)\,v_1(-\cos\theta,\,-\sin\theta), \tag{374}$$

$$\mathbf{V}_1 = (1+m_1/m_2)\,v_1\,(1,\,0), \tag{375}$$

$$\mathbf{V}_1' = v_1\,(\cos\theta + m_1/m_2,\,\sin\theta), \tag{376}$$

$$\mathbf{V}_2' = (m_1/m_2)\,v_1\,(1-\cos\theta,\,-\sin\theta). \tag{377}$$

In the center of mass frame, let E be the total energy, let $E_1 = (1/2)\,m_1\,v_1^2$ and $E_2 = (1/2)\,m_2\,v_2^2$ be the kinetic energies of the first and second particles, respectively, before the collision, and let $E_1' = (1/2)\,m_1\,v_1'^2$ and $E_2' = (1/2)\,m_2\,v_2'^2$ be the kinetic energies of the first and second particles, respectively, after the collision. Of course, $E = E_1 + E_2 = E_1' + E_2'$. In the laboratory frame, let $\mathcal{E}$ be the total energy. This is, of course, equal to the kinetic energy of the first particle before the collision. Likewise, let $\mathcal{E}_1' = (1/2)\,m_1\,V_1'^2$ and $\mathcal{E}_2' = (1/2)\,m_2\,V_2'^2$ be the kinetic energies of the first and second particles, respectively, after the collision. Of course,

$$\mathcal{E} = \mathcal{E}_1' + \mathcal{E}_2'$$

The following results can easily be obtained from the above definitions and Equations (371)-(377). First,

$$\mathcal{E} = \left(\frac{m_1 + m_2}{m_2}\right) E. \tag{378}$$

Hence, the total energy in the laboratory frame is always *greater* than that in the center of mass frame. In fact, it can be demonstrated that the total energy in the center of mass frame is less than the total energy in *any* other inertial frame. Second,

$$E_1 = E_1' = \left(\frac{m_2}{m_1 + m_2}\right) E, \tag{379}$$

$$E_2 = E_2' = \left(\frac{m_1}{m_1 + m_2}\right) E. \tag{380}$$

These equations specify how the total energy in the center of mass frame is distributed between the two particles. Note that this distribution is *unchanged* by the collision. Finally,

$$\mathcal{E}_1' = \left[\frac{m_1^2 + 2\, m_1\, m_2 \cos\theta + m_2^2}{(m_1 + m_2)^2}\right] \mathcal{E}, \tag{381}$$

$$\mathcal{E}_2' = \left[\frac{2\, m_1\, m_2\, (1 - \cos\theta)}{(m_1 + m_2)^2}\right] \mathcal{E}. \tag{382}$$

These equations specify how the total energy in the laboratory frame is distributed between the two particles after the collision. Note that the energy distribution in the laboratory frame is *different* before and after the collision.

Equations (371)-(377), and some simple trigonometry, yield

$$\tan\psi = \frac{\sin\theta}{\cos\theta + m_1/m_2}, \tag{383}$$

when $m_1 = m_2$

$$\tan\theta = \frac{\sin\phi}{\cos\phi + 1}$$

$$\tan\theta = \frac{2\sin(\phi/2)\cos(\phi/2)}{2\cos^2\phi/2}$$

$$= \tan\left(\frac{\phi}{2}\right)$$

$$\Rightarrow \theta = \frac{\phi}{2}$$

The denominator can be 0 for $\cos\phi = -1$.

$\therefore \tan\theta = \infty$

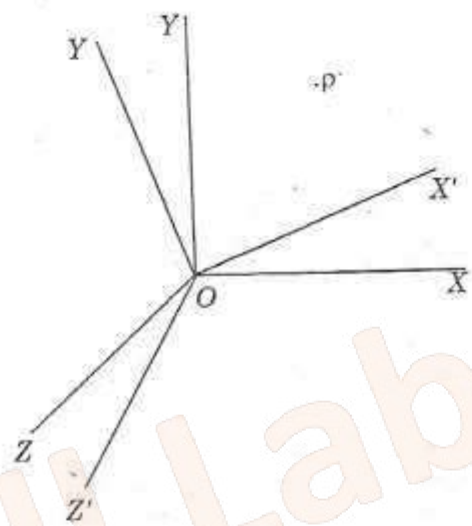
Hence, in this case, θ can have any value up to limiting value of 90°.

Q 4(i)

Ans.

relation.

Ans. Consider two frames of reference S (X, Y, Z) and S' (X', Y', Z') [Fig.] They have common origin and S' is rotating with an angular velocity w about the axis Y' relative to S. Here S is inertial frame of reference and S' is non-inertial frame of reference.



The observer O in frame S observes that the observer O' tn frame S' is rotating with an angular velocity w. The observer & observes that O is rotating with an angular velocity - w. Consider a particle P in space.

Position vector r of P in reference frame S is given by

$$r = ix + jy + kz \qquad \text{...(i)}$$

The position vector r' *of* P in reference frame S' is given by

$$r' = i'x' + j'y' + k'z' \qquad \text{...(ii)}$$

As both the systems have the same origin O,

$$r = r'$$

$$\therefore \quad r = i'x' + j'y' + k'z' \qquad ...(iii)$$

According to observer O', the frame of references S' is not rotating and its unit vectors remains constant, therefore differentiating equation (ii)

$$\frac{dr'}{dt} = i'\frac{dx'}{dt} + j'\frac{dy'}{dt} + k'\frac{dz'}{dt}$$

$$\therefore \quad V' = i'\frac{dx'}{dt} + j'\frac{dy'}{dt} + k'\frac{dz'}{dt} \qquad ...(iv)$$

V' is the velocity of P measured by O' relative to its own frame of reference $S'(x', y', z')$

According to observer O, the frame S' is rotating and its unit vectors are also changing in direction. Therefore differentiating equation (iii) with respect to time

$$\frac{dr}{dt} = i'\frac{dx'}{dt} + j'\frac{dy'}{dt} + k'\frac{dz'}{dt} + x'\frac{di'}{dt} + y'\frac{dj'}{dt} + z'\frac{dk'}{dt} \qquad ...(v)$$

The end points of unit vectors i', j', k' are in uniform circular motion with angular velocity w relative to observer O,

$$\therefore \quad \frac{di'}{dt} = w \times i'$$

$$\frac{dj'}{dt} = w \times j'$$

$$\frac{dk'}{dt} = w \times k'$$

Substituting these values in equation (v)

$$\frac{dr}{dt} = \left[i'\frac{dx'}{dt} + j'\frac{dy'}{dt} + k'\frac{dz'}{dt}\right] + [(w \times i')x' + (w \times j')y' + (w \times k')z']$$

or

$$V = V' + w \times r \qquad ...(vi)$$

Here V is the velocity of P as observed by O and V is the velocity of P as observed by O'.

Acceleration

The acceleration of P as measured by O relative to reference frame S' is given by

$$a = \frac{dV}{dt} = i\frac{dVx}{dt} + j\frac{dVy}{dt} + k\frac{dVz}{dt} \quad ...(vi$$

The acceleration of P as measured by O' relative to reference frame S' i given by

$$a' = i'\frac{dV'x}{dt} + j'\frac{dV'y}{dt} + k'\frac{dV'z}{dt} \quad ...(viii$$

Differentiating equation (vi) with respect to time, taking w to be constant

$$\frac{dV}{dt} = \frac{dV'}{dt} + w \times \frac{dr}{dt} \quad ...(ix$$

Also from equation (iv)

$$V' = i'V'x + J'V'o + k'V'z \quad ...(x)$$

Differentiating with respect to time

$$\frac{dV'}{dt} = \left[i'\frac{dV'x'}{dt} + j'\frac{dV'y'}{dt} + k'\frac{dV'z'}{dt}\right] + \left[V'x'\frac{di'}{dt} + V'y'\frac{dj'}{dt} + V'z'\frac{dk'}{dt}\right]$$

$$\frac{dV'}{dt} = a' + w \times V' \quad ...(xi)$$

Also $$\frac{dr}{dt} = V = V' + w \times r \quad ...(xii)$$

Substituting these values in equation (ix)

$$\frac{dV}{dt} = a' + w \times V' + w \times V' + w \times (w \times r)$$

or $$a = a' + 2w \times V' + w \times (w \times r) \quad ...(xiii)$$

Here a is the acceleration of P as observed by O and a' acceleration of P as observed by O'. The terms 2w×V is called the *Coriolis acceleration* and w×(w×r) corresponds to centripetal acceleration.

Also $$a' = a - 2w \times V' - w \times (w \times r) \quad ...(xiv)$$

Here $-w \times (w \times r)$ is called centrifugal acceleration because of its negative sign.

Q 4(ii)

Ans.

Given : $m = 20\,g = 20 \times 10^{-3}\,kg$

$r = 10\,cm = 10 \times 10^{-2}\,m = 0.1\,m$

$\omega = 10\,rad^{-1}$

centrifugal force, $f_c = \frac{mv^2}{r} = m\omega^2 r$

$= 20 \times 10^{-3} \times 10 \times 10 \times 0.1$

$= 0.2\,J$

Coriolis force, $f = 2mv\omega$

$= 2m\omega^2 r$

$= 2 \times f_c$

$= 0.4\,J$

Q 5(i)

Ans.

Let there are two inertial frames of references S and S'. S is the stationary frame of reference and S' is the moving frame of reference. At time t=t'=0 that is in the start, they are at the same position that is Observers O and O' coincides. After that S' frame starts moving with a uniform velocity v along x axis. The speed v is relativistic speed that is comparable to the speed of the light.

Let an object is placed in the frame S'. The coordinate of the initial point (A) of the object will be x1 (see the second line till A from S in figure) according to the observer in S and the coordinate of the final point will be will be x2 according to same observer.

The coordinate of the initial point (A) of the object will be x'1 (see the second last line till A from S' in figure) according to the observer in S' and the coordinate of the final point will be will be x'2 according to same observer.

Therefore the length of the object as seen by observer O' in s' will be

$L' = x'2 - x'1$ (1)

The length L' is called the proper length of the object. **Proper length** is defined as the length of the object measured by the observer which is in the same frame in which the object is placed.

The apparent length of the object from frame S at any time t will be

$L = x2 - x1$ (2)

As we have already derived and discussed Lorentz transformation equations for space and time and now use Lorentz transformation equations for space, that is

$x'1 = (x1 - vt)/(\sqrt{1 - v^2/c^2})$ (3)

$x' 2 = (x2 - vt)/(\sqrt{1 - v^2/c^2})$ (4)

By putting equations (3) and (4) in equation (1) and solving, we get

L’ = (x2 – x1)/ (√1 – v²/c²)

Substitute equation (2) in above equation,

L’ = L/(√1 – v²/c²)

Or Apparent length that is the length from frame S will be

L = L’(√1 – v²/c²)   (5)

This is the relation of the length contraction in relativity.

Time dilation

Imagine a gun placed at the position (x', y', z') in S'. Suppose it fires two shots at times t_1' and t_2' measured with respect to S'. In S´´ the clock is at rest relative to the observer. The time interval measured by a clock at rest relative to the observer is called the *proper time interval*. Hence, $t_0 = t_2' - t_1'$ is the time interval between the two shots for the observer in S'. Since the gun is fixed in S', it has a velocity v with respect to S in the direction of the positive X-axis. Let $t = t_2 - t_1$ represent the time interval between the two shots as measured by an observer in S.

From inverse Lorentz transformations we have

$$t_1 = \frac{t_1' + vx'/c^2}{\sqrt{1-(v^2/c^2)}}$$

and

$$t_2 = \frac{t_2' + vx'/c^2}{\sqrt{1-(v^2/c^2)}}$$

$$\therefore \quad t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1-(v^2/c^2)}}$$

or

$$t = \frac{t_0}{\sqrt{1-(v^2/c^2)}}$$

Q 5(ii) Ans.

Ans.

$$\Delta t_{Space} = \Delta t_{Earth}\sqrt{1-\frac{v^2}{c^2}}$$

$$\Delta t_{Space} = 1 \text{ year}$$

$$\Delta t_{Earth} = 4 \text{ year}$$

$$1 = 4\sqrt{1-\frac{v^2}{c^2}}$$

$$1-\frac{v^2}{c^2} = \frac{1}{16}$$

$$\frac{v^2}{c^2} = 1-\frac{1}{16}$$

$$\therefore \quad \frac{v^2}{c^2} = \frac{15}{16}$$

$$\frac{v}{c} = \frac{\sqrt{15}}{16} = 0.968$$

$$V = 0.968\,c$$

Q 5(iii) Ans.

Ans. One of the particles has a velocity 0.8 C and the other −0.8 C. 8, 3, 4

The theorem of velocities has to be used. Let us consider a sys tem S in which the particle having velocity −0.8 C is at res t. The laborator (S') is moving with velocity 0.8 C relative to S.

$$u' = 0.8\text{ C}$$

$$v = 0.8\text{ C}$$

$$u_x = \frac{u'+v}{1+\frac{u'v}{C^2}}$$

$$= \frac{0.8\text{ C} + 0.8\text{ C}}{1+\frac{0.8\text{ C}\times 0.8\text{ C}}{C^2}}$$

$$= \frac{1.6}{1.64}\text{C}$$

$$\therefore \quad = 0.9755\text{ C}.$$

Q 6(i) Ans.

Ans. Thus, if R and V be the *position vector* and *velocity* respectively of the centre of mass of a system of particles relative to a fixed or a reference point, and r_c and v_c the *position vector* and *velocity* of a particle of mass m of the system, *relative to the centre of mass*, the **position vector and velocity of the particle *relative to the fixed or reference point* will clearly be $r = R + r$, and $v = V + v_c$ respectively. Hence, the *angular momentum of the system about the fixed or the reference point* will, as just explained above, be given by**m

$$J = \Sigma m(R + r_c) \times (V + v_c) = \Sigma m(R \times V) + \Sigma m(R \times v_c) + \Sigma m(r_c \times V) + \Sigma m\ (r_c \times v_c)$$

Now, $r_c = (r - R)$ and $\therefore\ mr_c = mr - mR$

or $\Sigma mr_c = \Sigma mr - \Sigma mR = \Sigma mr - MR$

[$\therefore\ \Sigma m = M$, mass of the system.]

But, as we know, the inherent property of the centre of mass demands that

$$MR = m_1r_1 + m_2r_2 + ... = \Sigma mr$$

We, therefore, have

$$\Sigma mr - MR = 0 \text{ or } \Sigma mr_c = 0.$$

Similarly, $\Sigma mv_c = 0$

So that, the above relation for J simplifies to

$$J = R \times MV + \Sigma(r_c \times mv_c)$$

Here, clearly, $\Sigma(r_c \times mv_c)$ is the angular momentum of the system about the centre of mass, say, $J_{c.m.}$ and $MV = P$, the momentum of the centre of mass or the total linear momentum of the system. We therefore, have

$$J = R \times P + J_{c.m.}$$

i.e., the total angular momentum of the system about the fixed or the reference point is the vector sum of the angular momentum of the centre of mass about that point and the angular momentum of the system about the centre of mass.

Q 6(ii) Ans.

Ans. A central force is a force which always acts towards or away from a fixed point. Let "F" be the central force acting on a particle. Then it is represented as : $F = \hat{r} f(r)$ where r is a unit vector along the direction of r and is equal to r/r and/(r) is a scalar function of the distance r. When we apply the central forces the torque acting on a particle is given by

$$\vec{\tau} = \frac{\overrightarrow{dL}}{dt} \quad \text{where } \overline{L} \text{ is the angular momentum.}$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$= \vec{r} \times \vec{r} f(r)$$

$$= f(r)\vec{r} \times \frac{\vec{r}}{r}$$

But $\quad \vec{r} \times \vec{r} = 0$

$\therefore \quad \vec{\tau} = 0$

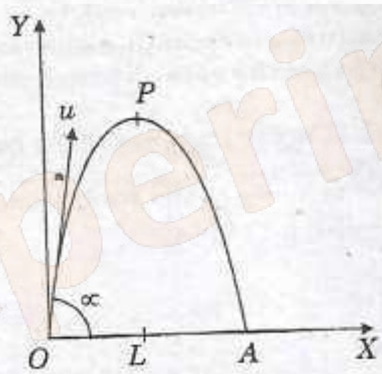
$\therefore$ $\vec{\tau}$ for central force field about origin in zero

showing that $\overline{L}$ = constant

The angular momentum of a particle moving under the influence of a central force always remains constant.

Q 6(iii) Ans.

Ans. Let o be the point of sending the projectile with an initial velocity u at an angle a with the horizontal.



The vertical component of velocity = $u \sin \alpha$

using the equation for L otion upwards,

$$s = ut - \frac{1}{2}gt^2$$

$$y = u \sin \alpha t - \frac{1}{2}gt^2$$

When the projectile comes back at A, $y = 0$

$$\therefore \qquad 0 = u \sin \alpha t - \frac{1}{2}gt^2$$

Solving the above,

$$t = 0 \text{ or } t = \frac{2u\sin\alpha}{g}$$

$t = 0$, corresponds t point O and $t = \frac{2u\sin\alpha}{g}$ corresponds to point A.

The time of going up and down is the same. Therefore, the time for reaching the highest point is $\frac{1}{2} \cdot \frac{2u\sin\alpha}{g}$ or $\frac{u\sin\alpha}{g}$.

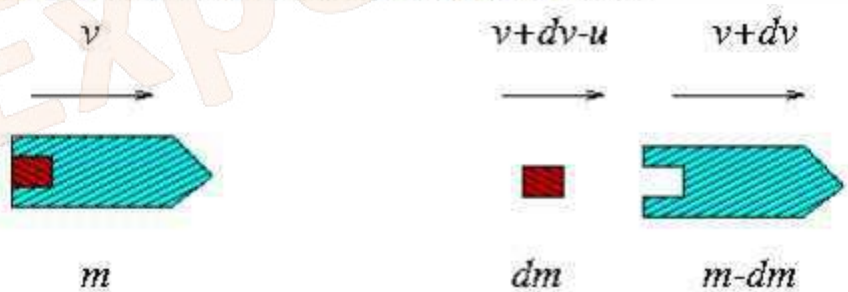
The horizontal component of velocity is $u \cos \alpha$ and it is constant. Horizontal range is u cos$(2t)$ or $\frac{u\cos\alpha(2u\sin\alpha)}{g}$ or $\frac{u^2\sin 2\alpha}{g}$.

The time for mid-point of the horizontal range is $\frac{1}{2}t\left(\text{or} \frac{u\sin\alpha}{g}\right)$.

This clearly shows that the projectile reaches the highest point in the same time in which it covers half the horizontal range.

Q 7(i) Ans. To find the rocket's velocity as a function of time, we first need to find how the velocity changes with respect to time, the differential equation mentioned in the problem statement.

The relation $dm/dt = \gamma m$, $\gamma > 0$ may seem inconsistent with other usages, but the problem statement is correct; the fuel is being exhausted at a rate $\gamma m > 0$. A more mathematically precise statement would be $\left|\frac{dm}{dt}\right| = \gamma m$. In any event, the expulsion of the exhausted fuel must provide an upward thrust to the rocket.



As suggested by the figure, take the system to be the rocket and fuel combination. The small square (red, if viewed in color) represents the differential mass dm of fuel ejected in the differential time dt . The initial momentum in the forward (upward, but to the right in the figure) direction is

$$p_{\text{initial}} = mv \tag{4.11}$$

where m is the combined mass of the rocket and fuel. The final momentum is

$$\begin{aligned} p_{\text{final}} &= (m - dm)(v + dv) + dm(v + dv - u) \\ &= (mv + m\,dv - v\,dm) + (v\,dm - u\,dm) \\ &= mv + m\,dv - u\,dm \end{aligned} \tag{4.12}$$

and so the change in momentum is

$$dp = p_{\text{final}} - p_{\text{initial}} = m\,dv - u\,dm\,. \tag{4.13}$$

Note that in going from the second line in (4.12) to the third, the second-order differential terms dmdv cancelled nicely; if they had not cancelled, we would have discarded them anyway.

the change in momentum divided by the differential time dt is the net external force,

$$\begin{aligned} \frac{m\,dv - u\,dm}{dt} &= -mg - bmv \\ \frac{dv}{dt} &= u\frac{1}{m}\frac{dm}{dt} - g - bv \\ &= \gamma u - g - bv. \end{aligned} \tag{4.15}$$

Q 7(ii) Ans.

Ans. (a)

$$F = ax^2 - b$$

$$F dx = -dU$$

$$\therefore \quad -dU = F dx = (ax^2 - b)dx$$

$$dU = (b - ax^2)dx$$

$$U = \int (b - ax^2)\,dx$$

$$U = bx - a\frac{x^3}{3} + C$$

Boundary condition,

$$U = 0 \text{ at } x = 0$$

$$\therefore \quad 0 = C$$

$$\therefore \quad U = bx - a\frac{x^3}{3}$$

(ii) For equilibrium, $F = 0$

$$\therefore \quad ax^2 - b = 0$$

$$\therefore \quad x = \pm\sqrt{\frac{b}{a}}$$

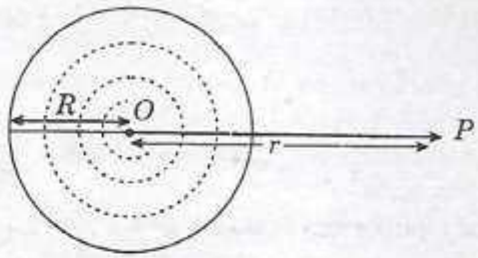
Now $\quad U = bx - a\dfrac{x^3}{3}$

Q 8(i) Ans.

Ans. Gravitational potential due to a solid sphere.

1. Gravitational potential

(a) At a point outside the solid sphere. Let P be a point distant r from the centre O of a solid sphere, of mass M and radius R, outside the sphere, *i.e.*, with $r>R$, (Fig.), where the gravitational potential due to the sphere is to be determined.



Imagine the sphere to consist of a number of spherical shells (shown dotted), one inside the other, concentric with the sphere, and of masses m_1 m_2, m_3 etc. Then, *gravitational potential* at P due to each spherical shells = - (*mass of spherical shell*)× G/R. So that potentials at P due to the different shells are $-m_1 G/R$, $-m_2$ G/R, $-m_3 G/R$ etc. And, therefore, *potential at P due to all the shells constituting the sphere, i.e., due to the whole solia sphere* is given by $V = -(m_1 + m_2 + m_3 + ...) G/R$, *because potential is a scalar quantity.*

Clearly, $(m_1 + m_2 + m_3 + \ ...) = M$, the mass of the solid sphere. So that, gravitational potential at P due to the solid sphere, *i.e.*,

$$V = -\frac{M}{R}G.$$

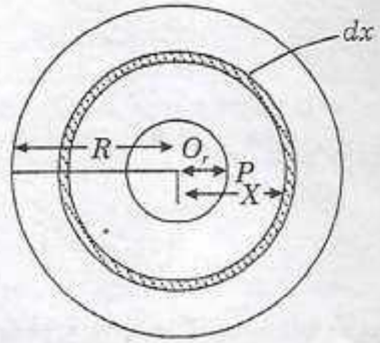
Again, therefore, *the sphere behaves as though its whole mass is concentrated at its centre.*

(b) *At* a point on the surface of the solid sphere. Clearly, if the point P lies on the surface of the solid sphere, we have $r = R$, the radius of the sphere.

So that, *gravitational potential at a point on the surface of a solid sphere*

$$= -\frac{M}{R}G.$$

(c) At a point inside the solid sphere. Let the point P now lie in-side the solid sphere at a distance r from the centre O of the sphere, (Fig.), *i.e.*, now $r < R$.



The solid sphere may be imagined to be made up of *an inner solid sphere of radius r* surrounded by a number of spherical shells, concentric with it and with their radii ranging from r to R. The potential at P due to the whole solid sphere is then clearly equal to the sum of the potentials at P due to the inner solid sphere and all the spherical shells outside it.

Clearly, point P lies on the surface of the *inner* solid sphere of radius r and inside all the spherical shells of radii greater than r. So that, *potential at P due to the inner solid sphere of radius r.*

$$= -\frac{\text{mass of sphere}}{r}G$$

$$= -\frac{4}{3}\pi r^3 \rho G/r = -\frac{4}{3}\pi r^2 \rho G,$$

because *mass of the inner solid sphere* $= \frac{4}{3}\pi r^3 \rho$, where ρ is the *volume density* of the sphere.

To determine the potential at P due to all the outer shells, let us consider one such shell of radius x and thickness dx, *i.e.*, of *volume = area × thickness* = $4\pi x^2 dx$ and hence of *mass* $= 4\pi x^2 dx\rho$.

Since potential at a point inside a shell is the same as that at a point on its surface, we have

potential at P due to this shell $= -\frac{4\pi x^2 dx\rho}{x}G = -\,4\pi x dx \rho G$

∴ *potential at P due to all the shells*

$$= \int_r^R -4\pi\rho G x dx = -4\pi\rho G\int_r^R x dx$$

$$= -4\pi\rho G\left[\frac{x^2}{2}\right]_r^R$$

$$= -4\pi\rho G\left(\frac{R^2-r^2}{2}\right)$$

$$= -\frac{4}{3}\pi\rho G\cdot\frac{3\left(R^2-r^2\right)}{2}$$

$$= -\frac{4}{3}\pi\rho G\cdot\frac{\left(3R^2-3r^2\right)}{2}$$

∴ potential at P due to the whole solid sphere = potential at P due to inner solid sphere + potential at P due all the outer spherical shells

$$= -\frac{4}{3}\pi r^2\rho G - -\frac{4}{3}\pi\rho G\left(\frac{3R^2-3r^2}{2}\right)$$

$$= -\frac{4}{3}\pi\rho G\left(r^2 + \frac{3R^2}{2} - \frac{3r^2}{2}\right)$$

$$= -\frac{4}{3}\pi\rho G\left(\frac{3R^2 - r^2}{2}\right)$$

$$= -\frac{4}{3}\pi R^3\rho G\left(\frac{3R^2 - r^2}{2R^3}\right)$$

[Multiplying and dividing by R^3]

Clearly, $\frac{4}{3}\pi r^3\rho$ is the mass of the whole solid sphere, *i.e.* M.

$\therefore$ gravitational potential at P due to the solid sphere, *i.e.*,

$$V = -\frac{M\left(3R^2 - r^2\right)}{2R^3}G$$

It follows at once, therefore, that if the point P lies *at the centre of the sphere*, we have $r = 0$. So that, *gravitational potential at the centre of the solid sphere*

$$= -M\left(\frac{3R^2}{2R^3}\right)G$$

$$= -\frac{3}{2}\cdot\frac{M}{R}G$$

But $-\frac{M}{R}G$, as we know, is the gravitational potential on the surface of the sphere.

Q 8(ii) a Ans.

The radius of the sphere is R. Let us consider a disk of radius ,r located at a distance, x from the center of the sphere. For purposes of calculation here in, let us state its thickness to be dx

$$r = \sqrt{R^2 - x^2}$$

The volume of this disk is thus:

$$dV = \pi r^2 dx = \pi(R^2 - x^2)dx$$

The mass of this disk is:

$$dm = \rho dV = \pi\rho(R^2 - x^2)dx$$

We know that, over a uniform body:

$$I = \int r^2 dm$$

Therefore,

$$I = \int r^2 (\pi\rho(R^2 - x^2)dx$$

$$= \pi\rho \int_0^R (R^2 - x^2)^2 dx$$

Carrying out the integration, we obtain:

$$I = \frac{8\pi\rho}{15} R^5$$

And since $M = \rho V = \frac{4\pi\rho R^3}{3}$

Substituting the given value for ρ

$$I = 2/5MR^2$$

Using parallel axis theorem,
Moment of inertia across the tangent is,

$I_t = I + MR^2$

$I_t = (7/5)\ MR^2$

Q 8(ii) b Ans.

(b) $K^2 = \frac{I}{M} = \frac{2}{5}\frac{MR^2}{M}$

$\therefore$ $K^2 = \frac{2}{5}R^2$

$= \frac{2}{5} \times 5 \times 5$

$= 10\ \text{cm}^2$

$K = \sqrt{10}\ \text{cm}$

$= 3.16\ \text{cm}$