B.Sc. Physics (Hons.)
Mechanics (2017)

S. No. of Question Paper : **6672**

Unique Paper Code : **32221102** **HC**

Name of the Paper : **Mechanics**

Name of the Course : **B.Sc. (Hons.) Physics**

Semester : **I**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt *five* questions in all.

Question No. **1** is compulsory.

All questions carry equal marks.

Use of non-programmable scientific calculator is allowed.

1. Attempt any *five* of the following questions :

   (*i*) Locate the centre of mass of a system of three particles masses 1 kg, 2 kg and 3 kg placed at the corners of an equilateral triangle of side 1 m.

All Lab Experiments

(*ii*) When a particle moves under a central force, prove that the particle moves in a fixed plane.

(*iii*) Show that damping has little or no effect on the frequency of a harmonic oscillator if its quality factor is large.

(*iv*) State Kepler's laws of planetary motion.

(*v*) Show that a conservative force can be expressed as $\vec{F} = -\nabla V$, where V is the potential energy.

(*vi*) What is potential energy curve ? Identify stable, unstable and neutral equilibrium from the curve.

(*vii*) Calculate the recessional velocity of a galaxy at a distance of 3×10^9 light years. Is this velocity relativistic ?

(*viii*) Explain the physical significance of negative results obtained from Michelson-Morley experiment. 3×5=15

2. (*a*) Find the centre of mass of a homogeneous semi-circular plate of radius R. 3

All Lab Experiments

(*b*) An object falling in the earth's gravitational field gains mass from surrounding stationary material :

(*i*) Show that :

$$M\frac{dv}{dt} + v\frac{dM}{dt} = Mg,$$

where v is the instantaneous downward velocity of the object when its mass is M. 2

(*ii*) If :

$$\frac{dM}{dt} = kM,$$

where k is a constant, show that the object acquires a terminal velocity and determine this velocity. 4

(*c*) A particle is projected with a velocity of 40 m/s at an elevation of 30°. Calculate (*i*) the greatest height attained (*ii*) the horizontal range and (*iii*) the velocity and direction at a height of 12 m. 2,2,2

3. (*a*) Show that in the case of an elastic and glancing collision between two particles of masses m_1 and m_2 respectively, the maximum value of scattering angle θ_1 in the

All Lab Experiments

laboratory frame corresponds to the scattering angle θ in the centre mass reference frame, where $\theta = \cos^{-1}(-m_2/m_1)$.

Also show that this maximum value of the scattering angle $\theta_1 = \tan^{-1}\left(\frac{m_1^2}{m_2^2} - 1\right)^{1/2}$. 5,2

(*b*) Show that if a heavy particle is incident on a light particle initially at rest, the heavy particle will not bounce backward as a result of collision. 3

(*c*) Prove that in centre of mass system, the magnitude of the velocities of the particles remains unaltered in elastic collision. 5

4. (*a*) Find the moment of inertia of a uniform solid cylinder of radius R, height H and mass M about an axis passing through its centre of mass and perpendicular to its axis of symmetry. 8

(*b*) Show that the ratio of rotational to translational kinetic energy for a solid cylinder rolling down a plane without slipping is 1 : 2. 3

All Lab Experiments

(c) Moment of inertia of a bigger solid sphere about its diameter is I. 64 smaller, equal spheres are made out of bigger sphere. What will be the moment of inertia of such smaller sphere about its diameter ? 4

5. (a) Derive the expressions for gravitational field and potential at a point inside and outside a uniform solid sphere of radius R and mass M. 5,5

(b) Represent the variations of field and potential graphically with respect to distance from the centre of the shell. 5

6. (a) Derive differential equation for a forced harmonic oscillator and find its steady state solution. Obtain the amplitude and phase of the steady state solution. 2,6,2,2

(b) What is the displacement of a particle executing SHM from its mean position when its KE is half of its PE ? 3

7. (a) What is longitudinal and transverse Doppler effect in light. Obtain an expression for the apparent frequency of light pulse in case of longitudinal Doppler effect in a moving frame of reference. 2,9

All Lab Experiments

(*b*) How does mass of a material particle change with velocity ? Show that c is the ultimate speed of the particle in free space. 2,2

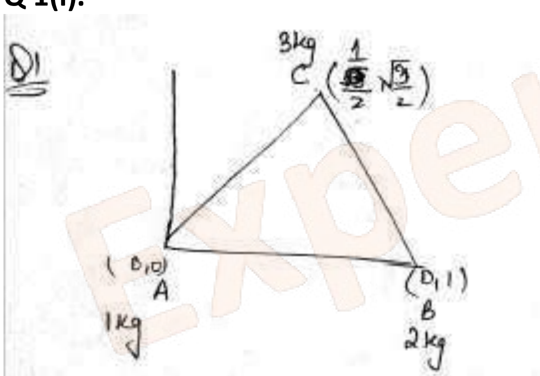
8. (*a*) Derive the relativistic law of variation of mass with velocity. For a relativistic particle, show that : 7,3

$$E^2 = p^2c^2 + m_0^2c^4.$$

(*b*) Find the velocity that an electron must be given so that its momentum is 10 times its rest mass times the speed of light. What is the energy at this speed ?

(Rest mass of electron = 9×10^{-31} kg). 3,2

Q 1(i).



All Lab Experiments

$$x_{cm} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1} m_i} = \frac{1(0) + 2(0) + 3 \times \left(\frac{1}{2}\right)}{1+2+3}$$

$$= \frac{3}{2 \times 6} = \frac{1}{4}$$

$$Y_{cm} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} = \frac{1(0) + 2(1) + \frac{3\sqrt{3}}{2}}{1+2+3}$$

$$= \frac{2 + \frac{3\sqrt{3}}{2}}{6} = \frac{1}{3} + \frac{\sqrt{3}}{4}$$

$\therefore$ centre of mass is $\left(\frac{1}{4}, \frac{1}{3} + \frac{\sqrt{3}}{4}\right)$

Q 1(ii).
Ans.

If the force $\vec{f}$ acting on a body has following characteristics then it is a central force
(i) it depends on the distance between two particles
(ii) it is always directed towards or away from a fixed point.
Gravitational force is an example of central forces. Mathematically if we consider central point as origin

$$\vec{f}(r) = f(r)(\pm\hat{r}) \qquad \text{……………………. (1)}$$

where + stands for repulsion forces and – for attractive forces , $f(r)$ is the magnitude of the central forces and $\hat{r}$ is unit vector in the direction of central forces.

Multiplying equation (1) by $\vec{r}$ on both the sides we get

$$\vec{r} \times \vec{f}(r) = f(r)\{\vec{r} \times \hat{r}\}$$

This gives,

$$f(r) = 0$$

since $\vec{r} \times \hat{r} = 0$

but we know that

$$\vec{f}(r) = m\frac{d^2\vec{r}}{dt^2}$$

Therefore,

$$\vec{r} \times m\frac{d^2\vec{r}}{dt^2} = 0$$

All Lab Experiments

$$\vec{r} \times \frac{d^2\vec{r}}{dt^2} = 0$$

Now,

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \vec{r} \times \frac{d^2\vec{r}}{dt^2} + 0$$

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = 0$$

$$\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = cons\tan t(\vec{h})$$

where $\vec{h}$ is a vector which does not depend on time and is perpandicular to the plane formed by the position vector $\vec{r}$ and velocity $\frac{d\vec{r}}{dt}$. Thus the plane formed by $\vec{r}$ and velocity $\frac{d\vec{r}}{dt}$ will also remains constant. Therefore the particle will always move in the same plane.

The torque due to central forces can be found out by the above expression. Torque

$$\tau = \vec{r} \times \vec{f}(r) = \vec{r} \times f(r)\hat{r} = 0$$

But, we know,

$$\tau = \frac{d\vec{J}}{dt}$$

This implies thet $\vec{J}$ is a constant quantity and it is known as angular momentum. Thus angular momentum is also conserved. But

$$\vec{J} = \vec{r} \times \vec{p} = \vec{r} \times m\frac{d\vec{r}}{dt} = \vec{h}$$

All Lab Experiments

Q1 (iii).

Ans. The equation of damped harmonic oscillation is—

$$\frac{md^2x}{dt^2} = F = F_{ext} - kx - C\frac{dx}{dt}$$

where F_{ext} is the applied force; k is the spring constant for displacement x and c is the damping coefficient. When $F_{ext} = 0$

Then $$\frac{md^2x}{dt^2} + 2\gamma w_o \frac{dx}{dt} + w_o^2 x = 0$$

where $\omega_o = \sqrt{k/m}$ and $\gamma = \frac{c}{2\sqrt{mk}}$ is the damping ratio

Quality factor $Q = 2\pi \times \frac{\text{Energy stared}}{\text{Energy lost per cycle}} = \frac{1}{2\gamma}$

when Q value is high then the energy stored is much higher than the loss and there is no effect on the frequency of the harmonic oscillator as the damping reduces frequency by $\frac{12.5}{Q^2}$

Q1 (iv).

Kepler's three laws of planetary motion can be stated as follows:

(1) All planets move about the Sun in elliptical orbits, having the Sun as one of the focii. (The Law of Ellipses)

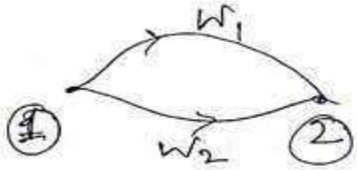
(2) A radius vector joining any planet to the Sun sweeps out equal areas in equal lengths of time. (The Law of Equal Areas)

(3) The squares of the sidereal periods (of revolution) of the planets are directly proportional to the cubes of their mean distances from the Sun. (The Law of Harmonies)

Q 1(v)

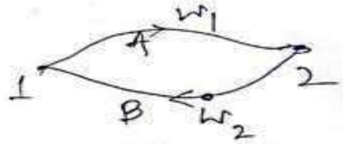
Ans.

A force field (or a region in which a body experiences a force) is said to be conservative if, the work done on the body by the force, in taking it from point 1 to 2 is independent of path that the body follows.



All Lab Experiments

$W_1 = W_2 \Rightarrow W_1(1\to 2) = W_2'(2\to 1)$



$$\Rightarrow \oint \vec{F}\cdot d\vec{s} = 0$$

now, By green's theorem,

$$\oint \vec{F}\cdot d\vec{s} = \int_A (\vec{\nabla}\times\vec{F})\cdot\hat{k}\, dA$$

$$\Rightarrow \int_A (\vec{\nabla}\times\vec{F})\cdot\hat{k}\, dA = 0$$

$$\Rightarrow (\vec{\nabla}\times\vec{F})\cdot d\vec{A} = 0$$ [dA is any arbitrary area bounded by the curve AB]

$$(\vec{\nabla}\times\vec{F})\cdot d\vec{A} = 0$$

Again as $d\vec{A}$ is any arbitrary non zero area, so,

$$(\vec{\nabla}\times\vec{F}) = 0$$

[You can say that what if $\vec{\nabla}\times\vec{F} \neq 0$ but $(\vec{\nabla}\times\vec{F}) \perp d\vec{A}$?

Well, that can't be the case ~~ase~~ as dir^n of curl $\vec{F}$ and $d\vec{A}$ are either same or opposite but not $\perp^r$].

All Lab Experiments

$$\Rightarrow \quad \vec{\nabla} \times \vec{F} = 0$$

$$\Rightarrow \quad \because \vec{\nabla} \times (\vec{\nabla} \phi) = 0$$

$\Rightarrow$ $\vec{F}$ can be represented as

$$\vec{F} = \vec{\nabla} \phi \; ; \; \phi = \text{Scalar quantity}$$

now $\because$ $d\phi = \vec{\nabla}\phi \cdot d\vec{s}$

$\downarrow$ differential of ϕ along a dirⁿ $d\vec{s}$

Also, $\vec{F} \cdot d\vec{s} = \delta W = -dU$

[from conservation of energy or defⁿ of conservative field, whatever]

$$\Rightarrow \quad \vec{\nabla}\phi \cdot d\vec{s} = -dU$$

$$\Rightarrow \quad d\phi = -dU \Rightarrow \phi = -U$$

$$\Rightarrow \quad \vec{F} = \vec{\nabla}\phi = -\vec{\nabla} U \Rightarrow \boxed{F = -\vec{\nabla} U} \; !!!$$

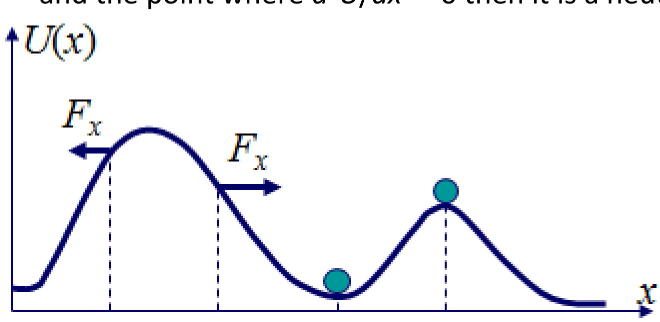
Q 1(vi)

Ans.

In a conservative force field, the potential energy of a particle is in general a function of space. So, changing from point to point a function shows the potential energy of the particle at every point of the space. this curve is called as a potential energy curve.

The condition for a position to be stable or unstable is found from the curvature d^2U/dx^2 at the point.

- If the curvature is upward ($d^2U/dx^2 > 0$) and $dU/dx = 0$ (a minimum in $U(x)$), then the position is stable.
- If the curvature is downward ($d^2U/dx^2 < 0$) and $dU/dx = 0$ (a maximum in $U(x)$), then it is unstable.
- and the point where $d^2U/dx^2 = 0$ then it is a neutral point.



All Lab Experiments

Q 1(vii)

Recessional velocity is most pertinent to distant galaxies, which (due to Hubble's law) redshift proportionally to their distance from the Earth. The redshift is usually interpreted as due to recessional velocity, which can be calculated according to the formula

$$v = H_0 D$$

Where H_o is the Hubble constant, D is the proper distance, and v is the recessional velocity. The recessional velocity of a galaxy (or any cosmological object) at a particular distance is also termed as *Hubble velocity*

$$\begin{aligned} V &= H_0 D \\ &= 3\times10^{-18} \times 3\times10^{9} \times 24\times365\times3600 \\ &= 8.514\times10^{7} \text{ m/s} \end{aligned}$$

Hence relativistic.

Q 1(viii)

Ans. Following important conclusions can be drawn from the negative results of Michelson-Morley experiment.

1. The velocity of light is constant in all directions.
2. The effects of ether in entire space of the universe are undetectable.
3. A new theory with different concepts of space, time and mass is needed. Thus, we must think of different set of transformation in contract to Galilean transformation which failed to give correct results.

Q 2(a)

the center of mass is given as

$$Y_{CM} = \frac{\int y\, dm}{\int dm}$$

Now, substituting the values $y = r\sin\theta$ & $dm = \sigma r dr d\theta$, we get

$$Y_{CM} = \frac{\int_0^R \int_0^\pi \sigma r^2 \sin\theta d\theta\, dr}{\int_0^R \int_0^\pi \sigma r d\theta\, dr}$$

$$= \frac{\int_0^R \left(\int_0^\pi \sin\theta d\theta\right) r^2\, dr}{\int_0^R \left(\int_0^\pi d\theta\right) r\, dr}$$

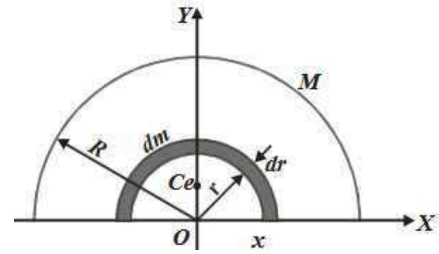
$$= \frac{\int_0^R (2)\, r^2\, dr}{\int_0^R (\pi)\, r\, dr} = \frac{2\int_0^R r^2\, dr}{\pi \int_0^R r\, dr}$$

$$= \frac{2\left[\frac{r^3}{3}\right]_0^R}{\pi\left[\frac{r^2}{2}\right]_0^R}$$

All Lab Experiments

$$= \frac{4R^3}{3\pi R^2} = \frac{4R}{3\pi}$$

$$\boxed{Y_{CM} = \frac{4R}{3\pi}}$$



Q 2(b)

Sol: (i) we know, $f = \frac{dp}{dt} = mg$

$P = mv$.

$f = \frac{d(mv)}{dt} = mg$

$\Rightarrow m\frac{dv}{dt} + v\frac{dm}{dt} = mg$ —①

(ii) Given, $\frac{dM}{dt} = KM$.

equation ① can be written as,

$m\frac{dv}{dt} + vKM = mg$

$\frac{dv}{dt} + vK = g$ (1st order diff. eqⁿ)

$If = e^{\int k dt} = e^{kt}$

$v e^{kt} = \int g e^{kt} dt = \frac{g}{k} e^{kt} + C$

$V = \frac{g}{k} + Ce^{-kt}$

(C): Velocity $= 40$ m/s. $\theta = 30°$.

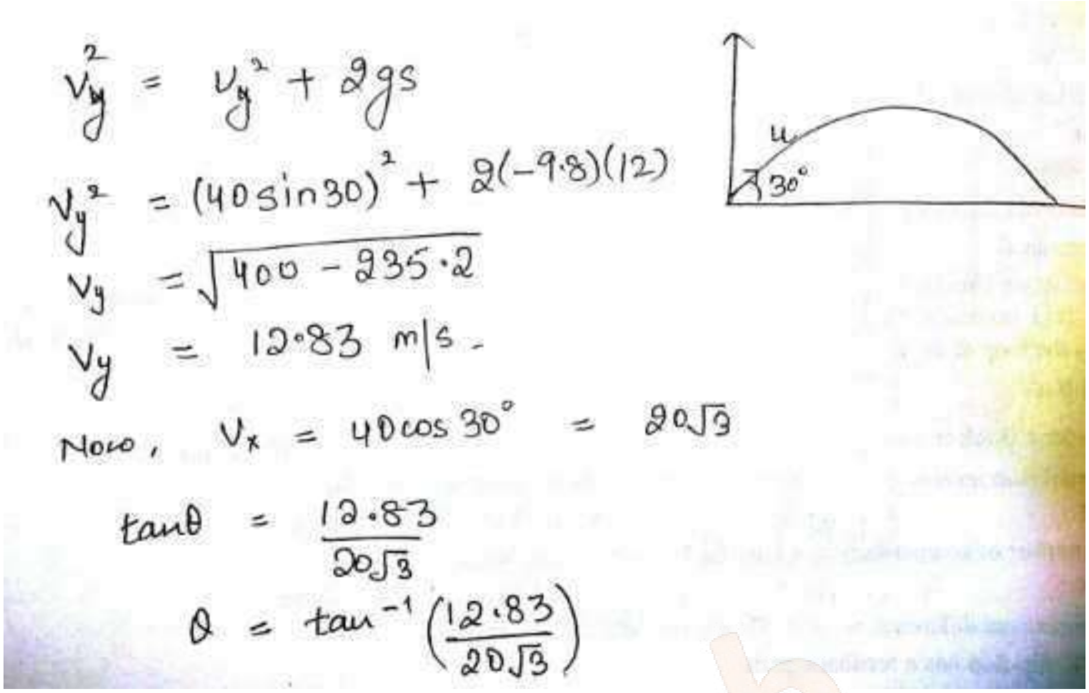
Height, $H = \frac{u^2 \sin^2\theta}{2g} = \frac{(40)^2 \sin^2 30}{2 \times 10}$

$= \frac{(40)^2 \left(\frac{1}{2}\right)^2}{20} = 20m$.

Range, $R = \frac{u^2 \sin 2\theta}{g} = \frac{(40)^2 \sin 60}{10}$

$= \frac{(40)^2 \sqrt{3}}{2 \times 10} = 80\sqrt{3}$ m

All Lab Experiments



$$v_y^2 = u_y^2 + 2gs$$

$$v_y^2 = (40\sin 30)^2 + 2(-9.8)(12)$$

$$v_y = \sqrt{400 - 235.2}$$

$$v_y = 12.83 \text{ m/s}$$

Now, $V_x = 40\cos 30^\circ = 20\sqrt{3}$

$$\tan\theta = \frac{12.83}{20\sqrt{3}}$$

$$\theta = \tan^{-1}\left(\frac{12.83}{20\sqrt{3}}\right)$$

Q3:

Ans. Suppose that, in the center of mass frame, the first particle has velocity $\mathbf{v}_1$ before the collision, and velocity $\mathbf{v}_1'$ after the collision. Likewise, the second particle has velocity $\mathbf{v}_2$ before the collision, and $\mathbf{v}_2'$ after the collision. We know that

$$m_1\,\mathbf{v}_1 + m_2\,\mathbf{v}_2 = m_1\,\mathbf{v}_1' + m_2\,\mathbf{v}_2' = \mathbf{0} \qquad (368)$$

in the center of mass frame. Moreover, since the collision is assumed to be elastic (*i.e.*, energy conserving),

$$v_1' = v_1, \qquad (369)$$

$$v_2' = v_2. \qquad (370)$$

Let us transform to a new inertial frame of reference--which we shall call the *laboratory frame*--which is moving with the uniform velocity $-\mathbf{v}_2$ with respect to the center of mass frame. In the new reference frame, the first particle has initial velocity $\mathbf{V}_1 = \mathbf{v}_1 - \mathbf{v}_2$, and final velocity $\mathbf{V}_1' = \mathbf{v}_1' - \mathbf{v}_2$. Furthermore, the second particle is initially at *rest*, and has the final velocity $\mathbf{V}_2' = \mathbf{v}_2' - \mathbf{v}_2$. The relationship between scattering in the center of mass frame and scattering in the laboratory frame is illustrated in Figure 23.

All Lab Experiments

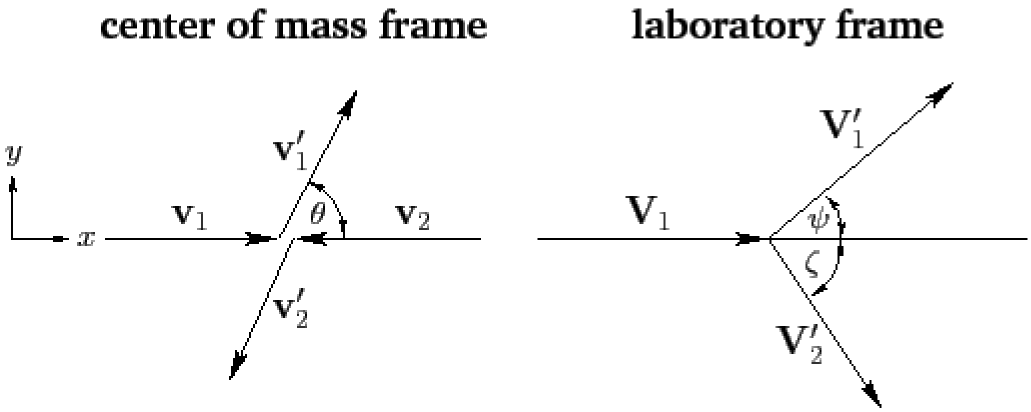


Figure 23: *Scattering in the center of mass and laboratory frames.*

In the center of mass frame, both particles are scattered through the same angle θ. However, in the laboratory frame, the first and second particles are scattered by the (generally different) angles ψ and ζ, respectively.

Defining x- and y-axes, as indicated in Figure 23, it is easily seen that the Cartesian components of the various velocity vectors in the two frames of reference are:

$$\mathbf{v}_1 = v_1\,(1,\,0), \tag{371}$$

$$\mathbf{v}_2 = (m_1/m_2)\,v_1\,(-1,\,0), \tag{372}$$

$$\mathbf{v}_1' = v_1\,(\cos\theta,\,\sin\theta), \tag{373}$$

$$\mathbf{v}_2' = (m_1/m_2)\,v_1(-\cos\theta,\,-\sin\theta), \tag{374}$$

$$\mathbf{V}_1 = (1+m_1/m_2)\,v_1\,(1,\,0), \tag{375}$$

$$\mathbf{V}_1' = v_1\,(\cos\theta + m_1/m_2,\,\sin\theta), \tag{376}$$

$$\mathbf{V}_2' = (m_1/m_2)\,v_1\,(1-\cos\theta,\,-\sin\theta). \tag{377}$$

In the center of mass frame, let E be the total energy, let $E_1 = (1/2)\,m_1\,v_1^2$ and $E_2 = (1/2)\,m_2\,v_2^2$ be the kinetic energies of the first and second particles, respectively, before the collision, and let $E_1' = (1/2)\,m_1\,v_1'^2$ and $E_2' = (1/2)\,m_2\,v_2'^2$ be the kinetic energies of the first and second particles, respectively, after the collision. Of course, $E = E_1 + E_2 = E_1' + E_2'$. In the laboratory frame, let $\mathcal{E}$ be the total energy. This is, of course, equal to the kinetic energy of the first particle before the collision. Likewise, let $\mathcal{E}_1' = (1/2)\,m_1\,V_1'^2$ and $\mathcal{E}_2' = (1/2)\,m_2\,V_2'^2$ be the kinetic energies of the first and second particles, respectively, after the collision. Of course,

$$\mathcal{E} = \mathcal{E}_1' + \mathcal{E}_2'$$

The following results can easily be obtained from the above definitions and Equations (371)-(377). First,

$$\mathcal{E} = \left(\frac{m_1 + m_2}{m_2}\right) E. \tag{378}$$

Hence, the total energy in the laboratory frame is always *greater* than that in the center of mass frame. In fact, it can be demonstrated that the total energy in the center of mass frame is less than the total energy in *any* other inertial frame. Second,

$$E_1 = E_1' = \left(\frac{m_2}{m_1 + m_2}\right) E, \tag{379}$$

$$E_2 = E_2' = \left(\frac{m_1}{m_1 + m_2}\right) E. \tag{380}$$

These equations specify how the total energy in the center of mass frame is distributed between the two particles. Note that this distribution is *unchanged* by the collision. Finally,

$$\mathcal{E}_1' = \left[\frac{m_1^2 + 2\, m_1\, m_2\, \cos\theta + m_2^2}{(m_1 + m_2)^2}\right] \mathcal{E}, \tag{381}$$

$$\mathcal{E}_2' = \left[\frac{2\, m_1\, m_2\, (1 - \cos\theta)}{(m_1 + m_2)^2}\right] \mathcal{E}. \tag{382}$$

These equations specify how the total energy in the laboratory frame is distributed between the two particles after the collision. Note that the energy distribution in the laboratory frame is *different* before and after the collision.

Equations (371)-(377), and some simple trigonometry, yield

$$\tan\psi = \frac{\sin\theta}{\cos\theta + m_1/m_2}, \tag{383}$$

Differentiating Equation (383) with respect to θ, we obtain

$$\frac{d\tan\psi}{d\theta} = \frac{1 + (m_1/m_2)\, \cos\theta}{(\cos\theta + m_1/m_2)^2}. \tag{386}$$

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Thus, $\tan\psi$ attains an extreme value, which can be shown to correspond to a *maximum* possible value of ψ, when the numerator of the above expression is zero: *i.e.*, when

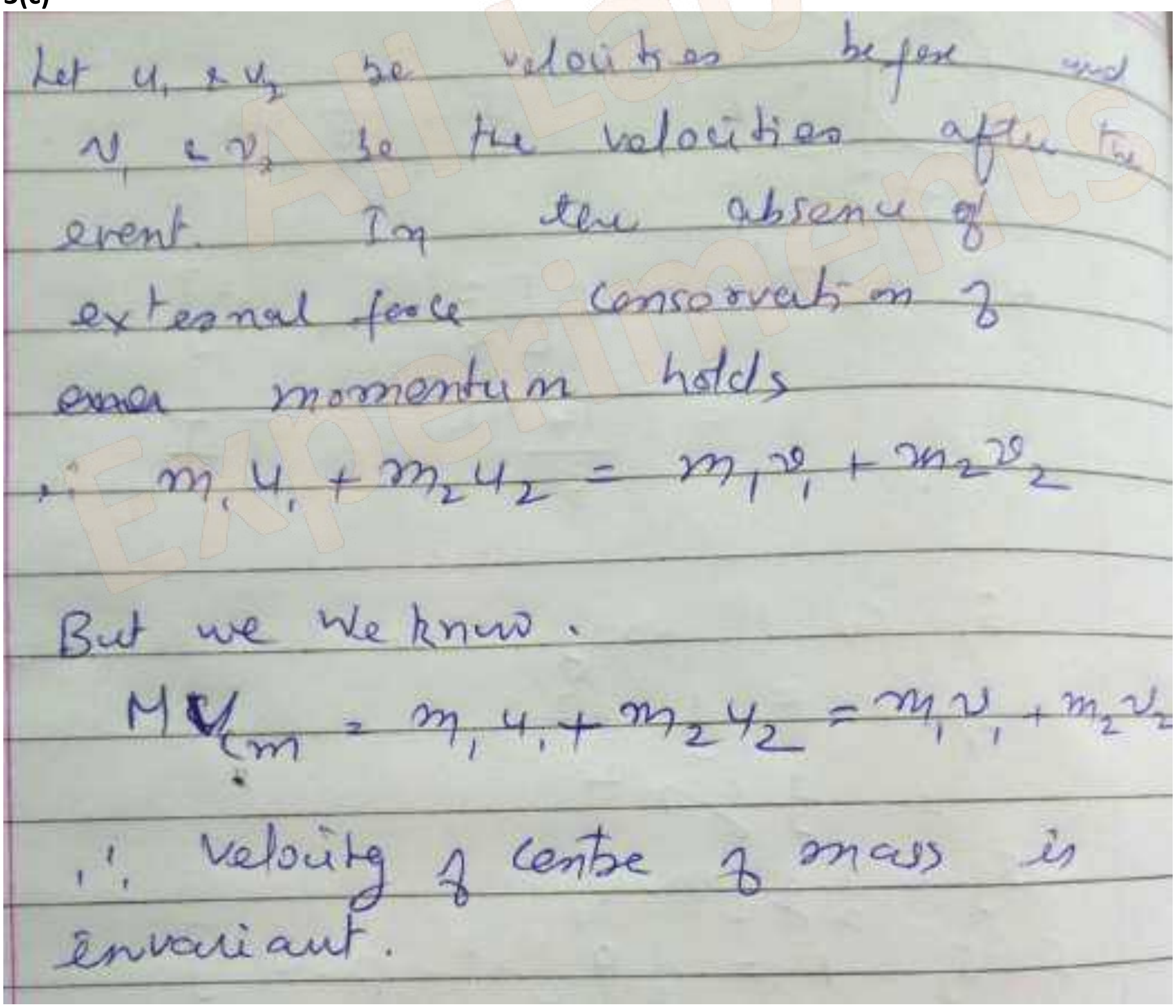
$$\cos\theta = -\frac{m_2}{m_1}.$$

Note that it is only possible to solve the above equation when $m_1 > m_2$. If this is the case then Equation (383) yields:

$$\tan\psi_{max} = \frac{m_2/m_1}{\sqrt{1-(m_2/m_1)^2}}, \tag{388}$$

3(b) Refer your textbook

3(c)



Let u_1 & u_2 be velocities before and v_1 & v_2 be the velocities after the event. In the absence of external force conservation of linear momentum holds

$\therefore m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

But we know.

$MV_{cm} = m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

$\therefore$ velocity of centre of mass is invariant.

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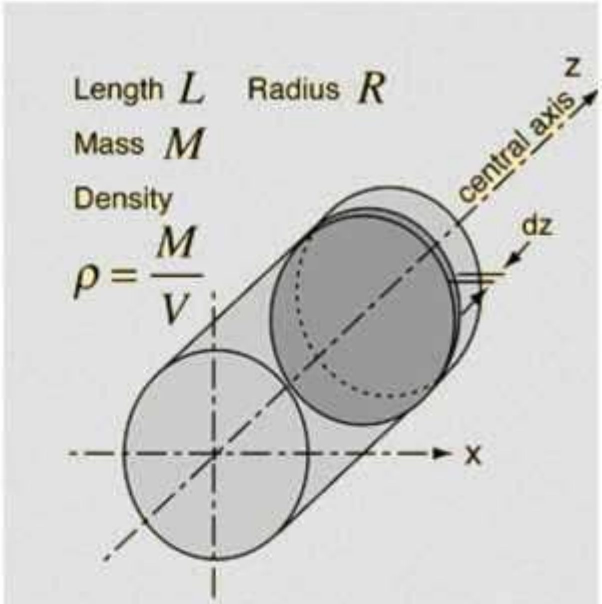
Q 4(a)

Ans. Let us consider a cylinder of length L, Mass M, and Radius R placed so that z axis is along its central axis as in the figure.



We know that its density ρ=Mass/Volume=M/V.
Let us consider that the cylinder is made up of infinitesimally thin disks each of thickness dz. If dm is the mass of one such disk, then
dm = ρ × Volume of disk
or dm=MV×(πR2.dz),
since V=Areal of circular face × length=$\pi R^2 L$, we obtain
dm=$M\pi R^2 L\times(\pi R^2.dz)$
or dm=MLdz ......(1)
We know that moment of inertia of a circular disk of mass m and of radius R about its central axis is is same as for a cylinder of mass M and radius R and is given by the equation
Iz = $12mR^2$. In our case
dIz =$12dmR^2$......(2)
Observe from figure 2, that this moment of inertia has been calculated about z axis. In the problem we are required to find moment of inertia about transverse (perpendicular) axis passing through its center. Knowing that the desired axis of rotation is transverse, therefore we need to apply perpendicular axis theorem which states:
The moment of inertia about an axis which is perpendicular to the plane contained by the remaining two axes is the sum of the moments of inertia about these two perpendicular axes, through the same point in the plane of the object. It follows that
dIz = dIx + dIy .....(3)
Also from symmetry we see that moment of inertia about x axis must be same as moment of inertia about y axis.
∴ dIx = dIy ......(4)
Combining the equations (3) and (4) we obtain
dIx = dIz2, Substituting Iz from (2), we get
dIx = $1/2\times1/2dmR^2$
or dIx = $1/4dmR^2$
Let the infinitesimal disk be located at a distance z from the origin which coincides with the center of mass. Now we make use of the parallel axis theorem about the x axis which states: The moment of inertia about any axis parallel to that axis through the center of mass is given by
IParallel axis = ICenter of Mass + Mass×d^2
where d is distance of parallel axis from Center of mass.
dIx=$1/4dmR^2+dmz^2$ ......(5)

Insert the value of dm calculated in (1) in moment of inertia equation (5) to express it in terms of z then integrate over the length of the cylinder from the value of z = −L/2 to z = +L/2

$Ix=\int dIx=\int 1/4(M/L)dzR^2 + \int z^2 M/Ldz$

$Ix=1/4MLR^2z+MLz33]$,

ignoring constant of integration because of it being definite integral.

$Ix = 1/4.MR^2/L\ [L/2 − (−L/2)] + M/3L.[(L/2)^3 − (−L/2)^3]$

or $Ix =1/4MLR^2/L + M2L^3/3L/2^3$

or $Ix=1/4MR^2+1/12ML^2$

Q 4(b)

Ans. In a pure translatory motion, all the particles in the body, at any instant of time, have equal velocity and acceleration. Kinetic energy is a scalar quantity with no direction associated with it.

$$\begin{aligned} KE_{\text{translation}} &= \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 + \cdots + \frac{1}{2}m_Nv^2 \\ &= \frac{1}{2}(m_1 + m_2 + \cdots + m_N)v^2 \\ &= \frac{1}{2}M_{\text{body}}v^2. \end{aligned}$$

In case of a rigid body in pure rotation, all the particles on the body rotates in circular motion with their centers lying on the same axis

$$KE_{PureRotation} = \frac{1}{2}I_{\text{rot}}\omega^2,$$

$KE = 1/2.\ (1/2.MR^2)(v/R)^2 = 1/4.\ Mv^2$

Hence the ratio of these K.E. is 1:2.

4(c) : Moment of interia of bigger sphere

$$I_1 = \frac{2}{5} mR_1^2$$

Its volume is $V_1 = \frac{4}{3}\pi R^3$

Now, it is divided in 64 spheres.

$$V_1 = 64 V_2$$

$$\frac{4}{3}\pi R_1^3 = 64 \frac{4}{3}\pi R_2^3$$

$$\frac{R_1}{R_2} = 4$$

New, MOI of sphere

$$I_2 = \frac{2}{5} mR_2^2 = \frac{2}{5} m \frac{R_1^2}{4^2}$$

$$I_2 = \frac{1}{16} I_1$$

All Lab Experiments

Q 5(a)

Ans.

1. Gravitational potential

(*a*) **At a point outside the solid sphere.** Let P be a point distant r from the centre O of a solid sphere, of mass M and radius R, outside the sphere, *i.e*, with $r > R$, (Fig. 12.14), where the gravitational potential due to the sphere is to be determined.

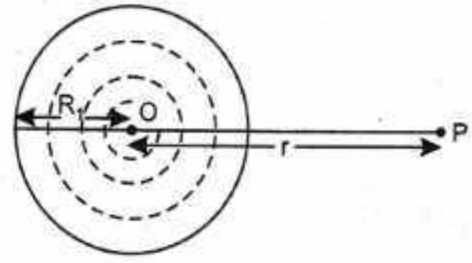


Fig. 12.14

Imagine the sphere to consist of a number of spherical shells (shown dotted), one inside the other, concentric with the sphere, and of masses m_1, m_2, m_3. etc. Then, as we have seen under §12.13, 1 (*a*), above, *gravitational potential* at P due to each spherical shell = – (*mass of spherical shell*) × *G*/*R*. So that potentials at *P* due to different shells are $-m_1\,G/R$, $-m_3\,G/R/m_3\,G/R$ etc. And, therefore, *potential at P due to all the shells constituting the sphere, i.e., due to the whole solid sphere* is given by $V = (m_1 + m_2 + m_3 + \ldots)G/R$, *because potential is a scalar quantity.*

Clearly, $(m_1 + m_2 + m_3 + \ldots) = M$. the mass of the solid sphere. So that,

gravitational potential at P due to the solid sphere, i.e., $V = -\dfrac{M}{r}G.$

Again, therefore, *the sphere behaves as though its whole mass is concentrated at its centre.*

(*b*) **At a point on the surface of the solid sphere.** Clearly, if the point *P* lies on the surface of the solid sphere, we have $r = R$, the radius of the sphere.

So that, *gravitational potential at a point on the surface of a solid sphere* $= -\dfrac{M}{R}G.$

(c) **At a point inside the solid sphere.** Let the point *P* now lie inside the solid sphere at a distance *r* from the centre *O* of the sphere, (Fig. (12.15), *i.e.*, now $r < R$.

The solid sphere may be imagined to be made up of *an inner solid sphere of radius r* surrounded by a number of spherical shells, concentric with it and with their radii ranging from *r* to *R*. The potential at *P* due to the whole solid sphere is then clearly equal to the sum of the potentials at *P* due to the inner solid sphere and all the spherical shells outside it.

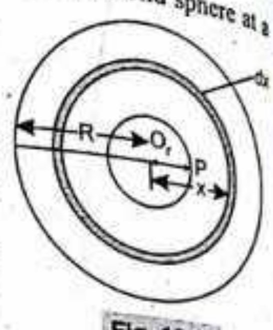


Fig. 12.15

Clearly, point *P* lies on the surface of the inner solid sphere of radius *r* and inside all the spherical shells of radii greater that *r*. So that, *potential at P due to the inner solid sphere of radius r.*

$$= -\frac{\textit{mass of the sphere}}{r}G = -\frac{4}{3}\pi r^3 \rho G/r = -\frac{4}{3}\pi r^2 \rho G,$$

because mass of the inner solid sphere $= \frac{4}{3}\pi r^2 \rho$, where ρ is the *volume density* of the sphere.

To determine the potential at *P* due to all the outer shells, let us consider one such shell of radius *x* and thickness *dx*, *i.e.*, of *volume* = *area* × *thickness* = $4\pi x^2 dx$ and hence of *mass* = $4\pi x^2 dx\rho$. Since potential at a point inside a shell is the same as that at a point on its surface [§ 12.13, 1 (*c*)], we have

$$\textit{potential at P due to this shell} = -\frac{4\pi x^2 dx\rho}{x}G = -4\pi x dx \rho G.$$

∴ *Potential at P due to all the shells*

$$= \int_r^R -4\pi\rho G x\,dx$$

All Lab Experiments

$$= -4\pi\rho G\int_r^R x\,dx = -4\pi\rho G\left[\frac{x^2}{2}\right]_r^R$$

$$= -4\pi\rho G\left(\frac{R^2-r^2}{2}\right)* = -\frac{4}{3}\pi\rho G.\frac{3(R^2-r^2)}{2}$$

$$= -\frac{4}{3}\pi\rho G\frac{(3R^2-3r^2)}{2}$$

$\therefore$ *Potential at P due to the whole solid sphere = potential at P due to inner solid sphere + potential at P due to all the spherical shells*

$$= -\frac{4}{3}\pi r^2\rho G - \frac{4}{3}\pi\rho G\left(\frac{3R^2-3r^2}{2}\right)$$

$$= -\frac{4}{3}\pi\rho G\left(r^2+\frac{3R^2}{2}-\frac{3r^2}{2}\right) = -\frac{4}{3}\pi\rho G\left(\frac{3R^2-r^2}{2}\right)$$

$$= -\frac{4}{3}\pi R^2\rho G\left(\frac{3R^2-r^2}{2R^3}\right).$$ [Multiplying and dividing by R^3]

Clearly, $\frac{4}{3}\pi R^3\rho$ is the mass of the whole solid sphere *i.e., M.*

$\therefore$ *Gravitational potential at P due to the solid sphere, i.e.,*

$$V = -\frac{M(3R^2-r^2)}{2R^3}G.$$

It follows at once, therefore, that if the point *P lies at the centre of the sphere*, we have $r = 0$. So that,

gravitational potential at the centre of the solid sphere

$$= -M\left(\frac{3R^2}{2R^2}\right)G = -\frac{3}{2}\cdot\frac{M}{R}G.$$

But $-\frac{M}{R}G$, as we know, is the gravitational potential on the *surface* of the sphere.

We thus have *gravitational potential at the centre of solid sphere* = $\frac{3}{2}$ *time the gravitational potential on its surface.*

Or, *gravitational potential at the centre of the solid sphere: gravitational potential on the surface of the sphere : : 3 : 2.*

This means, in other words, that *the gravitational potential due to a solid sphere has its maximum (negative) value at its centre.*

2. Gravitational field

(*a*) **At a point outside the solid sphere.** We know that the gravitational potential at a point *P* outside a solid sphere distant *r* from its centre (*i.e.*, with $r > R$) is given by $V = -MG/r$, (§12.15. 1 (*a*).

And, since intensity of the gravitational field at a point is equal to the potential gradient there, we have

Gravitational field due to a solid sphere at a point P distant r from its centre ($r > R$), *i.e.*, $E = -dV/dr = -\frac{d}{dr}\left[-\frac{MG}{r}\right] = -\frac{MG}{r^2}$, *the same as though the whole mass (M) of the sphere were concentrated at its centre.*

All Lab Experiments

(b) **At a point on the surface of the solid sphere.** For a point on the surface of the solid sphere, obviously, $r = R$, the radius of the sphere. We, therefore, have gravitational *field at a point on the surface of the solid sphere,* i.e.,

$$E = -MG/R^2.$$

(c) **At a point inside the solid sphere.** As we know, the gravitational potential at a point inside a solid sphere distant r from its centre (*i.e.*, with $r < R$) is given by the potential gradient there, we have

gravitational field due to a solid sphere at a point P inside it, distant r from its centre, i.e.,

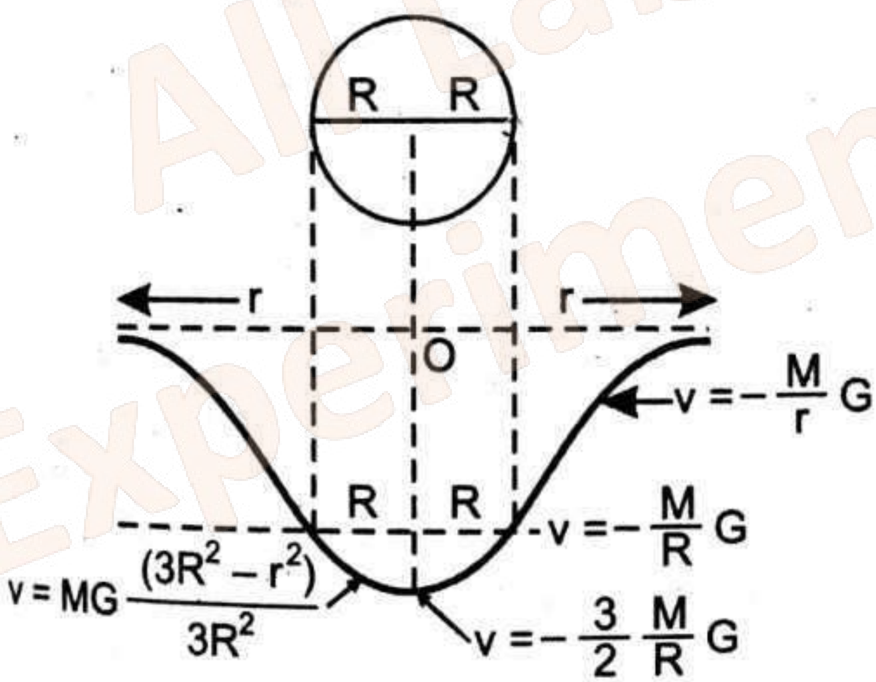
$$E = -\frac{dV}{dt} = -\frac{d}{dr}\left[-MG\frac{(3R^2 - r^2)}{2R^3}\right]$$

$$= -MG\frac{2r}{2R^3} = -\frac{MG}{R^3}r^*, \text{ showing that } E \propto r.$$

Thus, *the intensity of the gravitational field at a point inside a solid sphere is directly proportional to the distance of the point from the centre of the sphere.*

Q 5(b)

Ans.



(a) Gravitational potential due to a uniform solid sphere.

Q 6(a)

Let's drive our damped spring-object system by a sinusoidal force. Suppose that the x-component of the driving force is given by

$$F_x(t) = F_0 \cos(\omega t) \ , \tag{23.6.1}$$

where F_0 is called the *amplitude* (maximum value) and ω is the *driving angular frequency*. The force varies between F_0 and $-F_0$ because the cosine function varies between $+1$ and -1. Define $x(t)$ to be the position of the object with respect to the equilibrium position. The x-component of the force acting on the object is now the sum

$$F_x = F_0 \cos(\omega t) - kx - b\frac{dx}{dt} \ . \tag{23.6.2}$$

Newton's Second law in the x-direction becomes

$$F_0 \cos(\omega t) - kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \ . \tag{23.6.3}$$

We can rewrite Eq. (23.6.3) as

$$F_0 \cos(\omega t) = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx \ . \tag{23.6.4}$$

We shall now use complex numbers to solve the differential equation

$$F_0 \cos(\omega t) = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx \ . \tag{23.D.1}$$

We begin by assuming a solution of the form

$$x(t) = x_0 \cos(\omega t + \phi) \ . \tag{23.D.2}$$

where the amplitude x_0 and the phase constant ϕ need to be determined. We begin by defining the complex function

$$z(t) = x_0 e^{i(\omega t + \phi)} \ . \tag{23.D.3}$$

Our desired solution can be found by taking the real projection

$$x(t) = \text{Re}(z(t)) = x_0 \cos(\omega t + \phi) \ .$$

Our differential equation can now be written as

$$F_0 e^{i\omega t} = m\frac{d^2z}{dt^2} + b\frac{dz}{dt} + kz \ .$$

All Lab Experiments

We take the first and second derivatives of Eq. (23.D.3),

$$\frac{dz}{dt}(t) = i\omega x_0 e^{i(\omega t+\phi)} = i\omega z \ . \tag{23.D.6}$$

$$\frac{d^2 z}{dt^2}(t) = -\omega^2 x_0 e^{i(\omega t+\phi)} = -\omega^2 z \ . \tag{23.D.7}$$

We substitute Eqs. (23.D.3), (23.D.6), and (23.D.7) into Eq. (23.D.5) yielding

$$F_0 e^{i\omega t} = (-\omega^2 m + bi\omega + k)z = (-\omega^2 m + bi\omega + k)x_0 e^{i(\omega t+\phi)} \ . \tag{23.D.8}$$

We divide Eq. (23.D.8) through by $e^{i\omega t}$ and collect terms using yielding

$$x_0 e^{i\phi} = \frac{F_0 / m}{((\omega_0{}^2 - \omega^2) + i(b/m)\omega)} \ . \tag{23.D.9}$$

where we have used $\omega_0{}^2 = k/m$. Introduce the complex number

$$z_1 = (\omega_0{}^2 - \omega^2) + i(b/m)\omega \ . \tag{23.D.10}$$

Then Eq. (23.D.9) can be written as

$$x_0 e^{i\phi} = \frac{F_0}{my} \ . \tag{23.D.11}$$

Multiply the numerator and denominator of Eq. (23.D.11) by the complex conjugate $\overline{z}_1 = (\omega_0{}^2 - \omega^2) - i(b/m)\omega$ yielding

$$x_0 e^{i\phi} = \frac{F_0 \overline{z}_1}{m z_1 \overline{z}_1} = \frac{F_0}{m} \frac{((\omega_0{}^2 - \omega^2) - i(b/m)\omega)}{((\omega_0{}^2 - \omega^2)^2 + (b/m)^2 \omega^2)} \equiv u + iv \ . \tag{23.D.12}$$

where

$$u = \frac{F_0}{m} \frac{(\omega_0{}^2 - \omega^2)}{((\omega_0{}^2 - \omega^2)^2 + (b/m)^2 \omega^2)} \ , \tag{23.D.13}$$

All Lab Experiments

$$v = -\frac{F_0}{m}\frac{(b/m)\omega}{((\omega_0{}^2-\omega^2)^2+(b/m)^2\omega^2)} \ . \qquad (23.D.14)$$

Therefore the modulus x_0 is given by

$$x_0 = (u^2+v^2)^{1/2} = \frac{F_0/m}{((\omega_0{}^2-\omega^2)^2+(b/m)^2\omega^2)} \ , \qquad (23.D.15)$$

and the phase is given by

$$\phi = \tan^{-1}(v/u) = \frac{-(b/m)\omega}{(\omega_0{}^2-\omega^2)} \ . \qquad (23.D.16)$$

Oscillator Equation. The solution to is given by the function

$$x(t) = x_0\cos(\omega t+\phi) \ , \qquad (23.6.5)$$

where the amplitude x_0 is a function of the driving angular frequency ω and is given by

$$x_0(\omega) = \frac{F_0/m}{\left((b/m)^2\omega^2+(\omega_0{}^2-\omega^2)^2\right)^{1/2}} \ . \qquad (23.6.6)$$

The phase constant ϕ is also a function of the driving angular frequency ω and is given by

$$\phi(\omega) = \tan^{-1}\left(\frac{(b/m)\omega}{\omega^2-\omega_0{}^2}\right) \ . \qquad (23.6.7)$$

In Eqs. (23.6.6) and (23.6.7)

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad (23.6.8)$$

is the natural angular frequency associated with the undriven undamped oscillator. The x -component of the velocity can be found by differentiating Eq. (23.6.5),

$$v_x(t) = \frac{dx}{dt}(t) = -\omega x_0\sin(\omega t+\phi) \ , \qquad (23.6.9)$$

where the amplitude $x_0(\omega)$ is given by Eq. (23.6.6) and the phase constant $\phi(\omega)$ is given by Eq. (23.6.7).

Q 6(b)

Ans.

$$K.E = \frac{1}{2} P.E$$

$$\frac{1}{2} m\omega^2(A^2 - x^2) = \frac{1}{2}\left(\frac{1}{2} m\omega^2 A^2\right)$$

$$A^2 - x^2 = \frac{1}{2} A^2$$

$$\frac{A^2}{2} = x^2$$

$$x = \frac{A}{\sqrt{2}}$$ where A is max. amplitude

Q 7(a)

Ans. The longitudinal Doppler Effect considers the simpler case of a source moving directly towards you or away from you along a straight line. The transverse Doppler effect, on the other hand, considers what is observed when the observer is displaced in a direction perpendicular to the direction of the motion.

You can derive the relativistic Doppler shift from the Lorentz transformations. Let's start in the frame of the moving rocket, and let's take two events corresponding to nodes in the emitted wave (i.e. $1/f$). Then in the rocket's frame the two events are (0, 0) and (τ, 0), where τ is the period of the radiated wave. To see what the period of the radiation is in our frame we just have to use the Lorentz transformations to transform these two spacetime points into our frame. For simplicity we'll take our rest frame and the frame of the rocket to coincide at t=0. This is convenient because then the first event is just (0, 0) in both frames. Now the Lorentz transformations tell us:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$x' = \gamma(x - vt)$$

If we're tranforming from the rocket's frame to ours, and the rocket is moving at velocity v wrt us, then we have to put the velocity in as $-v$, and we're transforming the point (τ, 0). Putting these in the Lorentz transformations we find that the point (τ, 0) in the rocket's frame transforms to the point ($\gamma\tau$, $\gamma v\tau$) in our frame.

The last step is to note that if we're sitting at the origin in our frame the light from the event at ($\gamma\tau$, $\gamma v\tau$) takes a time $\gamma v\tau/c$ to reach us. So the time we see the second event is $\gamma\tau+\gamma v\tau/c$ and this is equal to the period of the radiation, τ' in our frame:

$$\tau' = \gamma\tau + \gamma v\tau/c$$

We just need to rearrange this to get the usual formula. Noting that $f' = 1/\tau'$ and $f = 1/\tau$ we take the reciprocal of both sides to get:

$$f' = f\frac{1}{\gamma(1 + v/c)}$$

To simplify this note that:

All Lab Experiments

$$\frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}}$$
$$= \sqrt{(1 - \frac{v}{c})(1 + \frac{v}{c})}$$

and substituting this back in our expression for f' we get:

$$f' = f\frac{\sqrt{(1 - v/c)(1 + v/c)}}{1 + v/c}$$
$$= f\frac{\sqrt{(1 - v/c)}}{\sqrt{1 + v/c}}$$
$$= f\sqrt{\frac{c - v}{c + v}}$$

Q 7(b).

Ans

The relativistic mass formula is articulated as,

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Where,

the rest mass is $\mathbf{m_o}$,

the velocity of the moving body is **v** ,

Velocity of light is **c**

As the velocity of the particle (v) approaches the velocity of light (c) the mass of the object becomes infinity, which is not possible. Beyond c the mass becomes imaginary, which is also not possible. Hence the ultimate speed of the particle in free space is c.

Q 8(a)

Ans.

According to Newtonian mechanics the mass of a body does not change with velocity. However, conservation laws, especially here the law of conservation of momentum, hold for any inertial system. Hence, in order to maintain the momentum conserved in any isolated system, mass of the body must be related to its velocity. So according to Einstein, the mass of the body in motion is different from the mass of the body at rest. We consider two inertial frames S and S' as in Figure 8.5.

All Lab Experiments

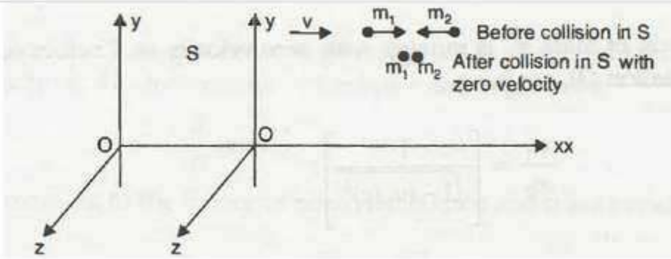


Figure 8.5 Collision between masses viewed from stationary and moving frames of reference

We now consider the collision of two bodies in S' and view it from the S. Let the two particles of masses m_1 and m_2 are travelling velocity u ' and-u ' parallel to x-axis in S'. The two bodies collide and after collision they coalesced into one body.

In System S : Before Collision: Mass of bodies are m1 and m2 . Let the their velocities are u_1 and u_2 respectively.

In System S: After Collision: Mass of the coalesced body is $(m_1 + m_2)$ and the velocity is v .

Using law of addition of velocities

$$u_1 = \frac{u' + v}{1 + \frac{u'v'}{c^2}}$$ and $$u_2 = \frac{-u' + v}{1 - \frac{u'v}{c^2}}$$

Applying the principle of conservation of momentum of the system before and after the

collision, we have,

$m_1 u_1 + m_2 u_2 = (m_1 + m_2)v$

$$m_1\left[\frac{u' + v}{1 + \frac{u'v}{c^2}}\right] + m_2\left[\frac{-u' + v}{1 - \frac{u'v}{c^2}}\right] = (m_1 + m_2)v$$

$$m_1\left[\frac{u' + v}{1 + \frac{u'v}{c^2}} - v\right] = m_2\left[v - \frac{-u' + v}{1 - \frac{u'v}{c^2}}\right]$$

$$\frac{m_1}{m_2} = \left[\frac{1 + \frac{u'v}{c^2}}{1 - \frac{u'v}{c^2}}\right]$$

All Lab Experiments

Now, using equations (1) and (2), we have

$M_1 / m_2 = [\sqrt{1-(u_2/c)^2} / \sqrt{1-(u_1/c)^2}\,]$

Let the body of mass m_2 is moving with zero velocity in S before collision, i.e., $u_2 = 0$,

hence, using equation (3), we have,

$m_1 / m_2 = 1 / \sqrt{1-(u_1/c)^2}$

Using common notation as $m_1 = m$, $m_2 = m_0$, $u_1 = v$, we have by using equation (4).

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Total energy and momentum are conserved in an isolated system, and the rest energy of a particle is invariant. Hence these quantities are in some sense more fundamental than velocity or kinetic energy, which are neither. Let us look into how the total energy, rest energy, and momentum of a particle are related.

We begin with Eq. (1.23) for total energy,

Total energy $$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \qquad (1.23)$$

and square it to give

$$E^2 = \frac{m^2c^4}{1 - v^2/c^2}$$

From Eq. (1.17) for momentum,

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Momentum $$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

we find that

$$p^2c^2 = \frac{m^2v^2c^2}{1 - v^2/c^2}$$

Now we subtract p^2c^2 from E^2:

$$E^2 - p^2c^2 = \frac{m^2c^4 - m^2v^2c^2}{1 - v^2/c^2} = \frac{m^2c^4(1 - v^2/c^2)}{1 - v^2/c^2}$$

$$= (mc^2)^2$$

All Lab Experiments

Q 8(b)
Ans.

Solution : (*i*) Given that p = momentum = $10m_0c$.

We know that p = mass × velocity = $mv = \frac{m_0 v}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}$

$$\therefore \qquad 10\, m_0 c = \frac{m_0 v}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$$

This gives, $1-\frac{v^2}{c^2} = \left(\frac{v}{10c}\right)^2 = \frac{v^2}{100c^2}$

or $\qquad 1 = \left(1+\frac{1}{100}\right)\frac{v^2}{c^2} \quad$ or $\quad \frac{v^2}{c^2} = \frac{100}{101} \qquad$ ....(1)

$$v = \left(\frac{100}{101}\right)^{1/2} .c = (0.990)^{1/2}\, c = 0.995c$$

$$= 0.995 \times 3 \times 10^{10} \text{ cm/sec} = 2.985 \times 10^{10} \text{ cm/sec.}$$

(*ii*) We know that rest mass of electron

$$= m_0 = 9 \times 10^{-28} \text{ gm.}$$

$$m = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} = \frac{9 \times 10^{-28}}{\left(1 - \frac{100}{101}\right)^{1/2}}, \text{ by (1)}$$

$$= 9 \times 10^{-28} \times \sqrt{(101)} \text{ gm}$$

$$= 9 \times 10^{-28} \times 10.0499 = 90.449 \times 10^{-28} \text{ gm}$$

Energy of the electron at the speed $v = 0.995c$ is

$$E = mc^2 = (90.449 \times 10^{-28})(3 \times 10^{10})^2 = 814.04 \times 10^{-8}.$$

All Lab Experiments