Free Study Material from

All Lab Experiments

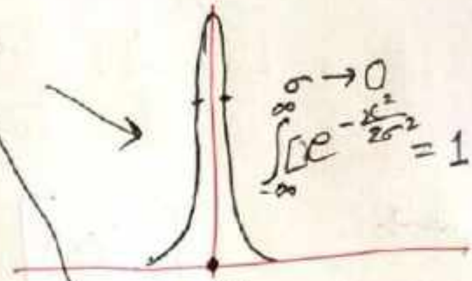
Mathematical Physics for JAM/NET/Gate Physics

> Dirac Delta, Jacobian <

Dirac Delta →

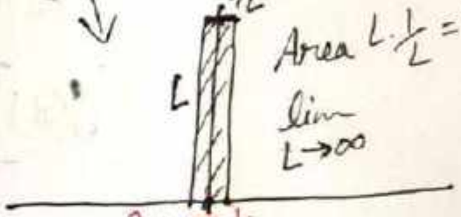
$\delta(x-a)$ when $a=0$

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega(x-a)}\, d\omega$$

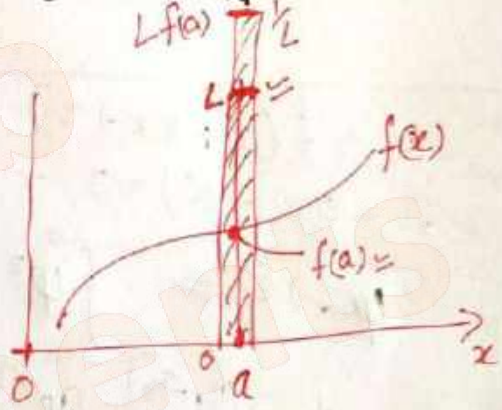


① $$\int_{-\infty}^{+\infty} \delta(x-a)\,dx = \int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a)\,dx = 1$$

$\varepsilon \to$ very very small



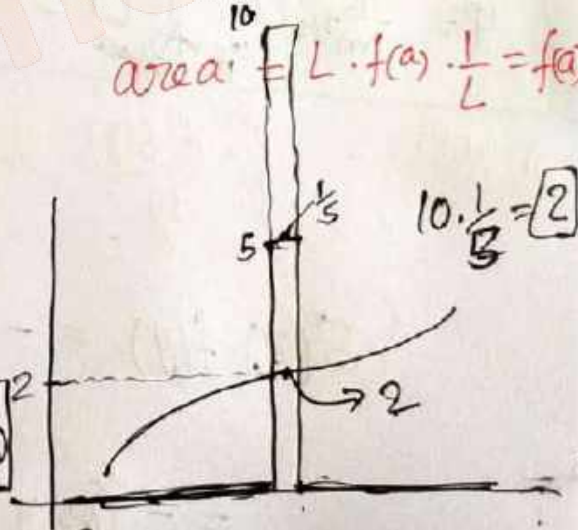
② $$\int_{-\infty}^{+\infty} \delta(x-a)\, \overset{L \cdot f(a)}{f(x)}\, dx = f(a)$$



③ $$\int f(x)\, \delta'(x-a) = -f'(a)$$

④ $$\delta(ax) = \frac{1}{|a|}\delta(x)$$

area $= L \cdot f(a) \cdot \frac{1}{L} = f(a)$



⑤ $$\delta(x^2 - a^2) = \delta\big((x-a)(x+a)\big) = \frac{1}{|2a|}\left[\delta(x-a) + \delta(x+a)\right]$$

⑥ $$\delta\big((x-a)(x-b)\big) = \frac{1}{|a-b|}\left[\delta(x-a) + \delta(x-b)\right]$$

⑦ $$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$$

$\int f(x) \cdot \delta(x-a)$

$f(x)=0$ find roots & let x_i $(x_1\ x_2\ x_3 \ldots x_n)$

using 7 derive 5 & 6
Home work.

Jest 2014 $\int_{-\infty}^{+\infty} f(x)\, \delta(x)\, dx = f(0)$

if x has the dimensions of momentum. Then what are the dimensions of dirac-delta fun.?

Ans. $\cancel{[f(x)]}\,[\delta(x)]\,[\text{momentum}] = \cancel{[f(0)]}$

$[\delta(x)] = \dfrac{1}{\text{momentum}}$

$\boxed{\frac{1}{2}\hbar} \rightarrow$

$[0] \rightarrow [\hbar]$

Something about Dimensions

$v = f(x^2 + 2x + 3) = m s^{-1}$

$x = 0 \quad f(0) = 3 = m s^{-1}$

$\bigcirc^{T^{3/2}} \quad \bigcirc^{T^{1}}$

$\boxed{\bigcirc^{T^{5/2}}} \quad \bigcirc^{T^{1/2}}$

$e^{\square} \quad \log[\ \ ]$

in 3-D $\psi \to$ Dim

$\int_{-\infty}^{+\infty} \psi^2\, dV = 1$

$\boxed{\psi = \frac{1}{m^{3/2}}}\ \boxed{L^{-3/2}}$ in 3D space $\quad [\psi^2] = \dfrac{1}{[\text{Vol}]}$

DU 2019 $\int_{-\infty}^{+\infty} \dfrac{d\,\delta(y)}{dy} \cdot \sin y\, dy$

$\int \delta'(y)\, f(y) = -f'(0)$

$= -\cos y \Big|_{y=0} = -\cos(0) = -1$ //

Jest 2019
TIFR 2013

(3)

$$\int_{-\infty}^{+\infty} \delta(x^2-\pi^2)\cos x\, dx$$

$$\delta(x^2-a^2) = \frac{1}{|2a|}\left[\delta(x-a)+\delta(x+a)\right]$$

$$\int_{-\infty}^{+\infty} \frac{1}{|2\pi|}\left(\delta(x-\pi)+\delta(x+\pi)\right)\cos x\, dx$$

$$\frac{1}{2\pi}\left[\underset{a=\pi}{\int \delta(x-\pi)\cos x\, dx} + \underset{a=-\pi}{\int \delta(x+\pi)\cos x\, dx}\right]$$

$$= \frac{1}{2\pi}\left[-1-1\right] = -\frac{1}{\pi}$$

Jacobian →

- 1 Change in variable
- 2 Change in volume

$$x_i(x_1\ x_2\ x_3 \ldots x_n) \xrightarrow{f(q_i)} q_i(q_1\ q_2 \ldots q_n)$$

n-variables — n-variables

$$\boxed{x_1 = \overset{f_1(\,)}{3q_1 + 2q_2 + \ldots}}$$

$$\delta x_1 = \frac{\partial f_1}{\partial q_1}\delta q_1 + \frac{\partial f_1}{\partial q_2}\delta q_2 + \ldots \frac{\partial f_1}{\partial q_n}\delta q_n$$

$$\delta x_2 = \frac{\partial f_2}{\partial q_1}\delta q_1 + \ldots$$

$$\vdots$$

$$\delta x_n = \frac{\partial f_n}{\partial q_1}\delta q_1 + \ldots$$

$$\begin{pmatrix}\delta x_1\\ \delta x_2\\ \vdots\\ \delta x_n\end{pmatrix} = \begin{pmatrix}\frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdots & \frac{\partial f_1}{\partial q_n}\\ \vdots & & & \\ \frac{\partial f_n}{\partial q_1} & \cdots & \cdots & \frac{\partial f_n}{\partial q_n}\end{pmatrix}\begin{pmatrix}\delta q_1\\ \delta q_2\\ \vdots\\ \delta q_n\end{pmatrix}$$

≡ Jacobian Matrix

Change in variable

$$\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix} = \left(J\left(\frac{f_1 f_2 \ldots f_n}{q_1 q_2 \ldots q_n}\right)\right)\begin{pmatrix}q_1\\ q_2\\ \vdots\\ q_n\end{pmatrix}$$

2 Change in volume/area/---

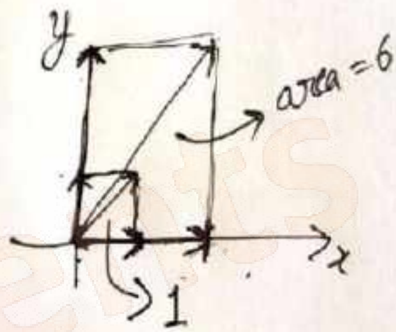
$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} & & \\ & J & \\ & & \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

$$dx_1\, dx_2 \,---\, dx_n = |J|\, dq_1\, dq_2 \,---\, dq_n$$

$\hookrightarrow$ Determinant of Jacobian.

3 Determinant → (Physical Meaning)

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \underline{\underline{6}}$$



$$\Downarrow$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Q 2D Cartesian - polar →

$$x = r\cos\theta$$
$$y = r\sin\theta \qquad J(r,$$

$$dx\,dy = J\left(\frac{x,y}{r,\theta}\right) dr\,d\theta$$

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} =$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

$$dx\,dy = \underline{\underline{r}}\,dr\,d\theta$$

JAM 2015 $\quad (x, y) \to (x', y')$

$$x' = \frac{x+y}{\sqrt{2}} \qquad y' = \frac{x-y}{\sqrt{2}}$$

$dx'\,dy' = J\,dx\,dy \Rightarrow$ find the value of J.

$$J\left(\frac{x', y'}{x, y}\right)$$

$$\begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{vmatrix} = \begin{vmatrix} 1/\sqrt{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -1$$

$$dx'\,dy' = |J|\,dx\,dy$$

JAM2018 $(x, y) \to (\xi, \eta)$ 7

$\xi = 2x + 3y \qquad \eta = 3x - 2y$

what is $dx\,dy$ in terms of $\xi\ \eta$

$$(x, y) = J\left(\frac{x, y}{\xi, \eta}\right)(\xi, \eta)$$

$$J^{-1}\left(\frac{x, y}{\xi, \eta}\right)(x, y) = (\xi, \eta) \quad \to J\left(\frac{\xi, \eta}{x, y}\right)$$

Two Properties of Jacobian

① $J\left(\frac{\xi, \eta}{x, y}\right) = J^{-1}\left(\frac{x, y}{\xi, \eta}\right)$

$J J^{-1} = I \qquad \mathrm{Det}(J)\,\mathrm{Det}(J^{-1}) = 1$

② $\mathrm{Det}(J) = \dfrac{1}{\mathrm{Det}(J^{-1})}$

$$dx\,dy = \left|J\left(\frac{x, y}{\xi, \eta}\right)\right| d\xi\, d\eta$$

$$\left|J\left(\frac{\xi, \eta}{x, y}\right)\right| = \begin{vmatrix} \frac{\partial \xi}{\partial x} = 2 & 3 \\ 3 & -2 \end{vmatrix} = -4 - 9 = -13$$

$\left|J\left(\frac{x, y}{\xi, \eta}\right)\right| = -\frac{1}{13}$

$dx\,dy = \left|-\frac{1}{13}\right| d\xi\, d\eta$

$= \frac{1}{13}\, d\xi\, d\eta$

D) 2017 $ds^2 = g_{ij}\, dx^i dx^j$ ⑧

$g_{12}\, dx^1 dx^2$

$\underline{0\, dr\, d\theta}$

$$ds^2 = \underset{g_{11}}{(dr)^2} + \underset{g_{22}}{r^2 (d\theta)^2} + \underset{g_{33}}{r^2 \sin^2 (d\phi)^2}$$

$$dx^i \rightarrow \underset{i=1}{dr} \quad \underset{i=2}{d\theta} \quad \underset{i=3}{d\phi}$$

$$\Gamma_{ij,k} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right]$$

$$\Gamma_{22,1} = \frac{1}{2}\left[\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right]$$

$$= \frac{1}{2}\left[0 + 0 - \frac{\partial r^2}{\partial r}\right]$$

$$= \frac{1}{\cancel{2}}\left[-\cancel{2}r\right] = \boxed{-r}$$

Electrostatics

$$\frac{\partial}{\partial r}\boxed{\frac{\hat{r}}{r^2}}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r^2}r^2 = 0$$

$$\int \frac{\hat{r}}{r^2}\, dV = 4\pi$$

$$\int \underbrace{\nabla \cdot \frac{\hat{r}}{r^2}}_{= 0} ds = 0 \Rightarrow \delta(r)\, 4\pi$$