Free Study Material from

All Lab Experiments

Mathematical Physics

for JAM/NET/Gate Physics

> Fourier Series <

> Fourier and Laplace Transform<

⇒ fourier Series ⇐

⇒ $f(x) \rightarrow$ pariodic → Sinusoidal (Sin, Cos)
→ Non - ״ (sq wave. ract.)

⇒ Any non-sinusoidal fun. can be respresented as the sum of sinusoidal waves.

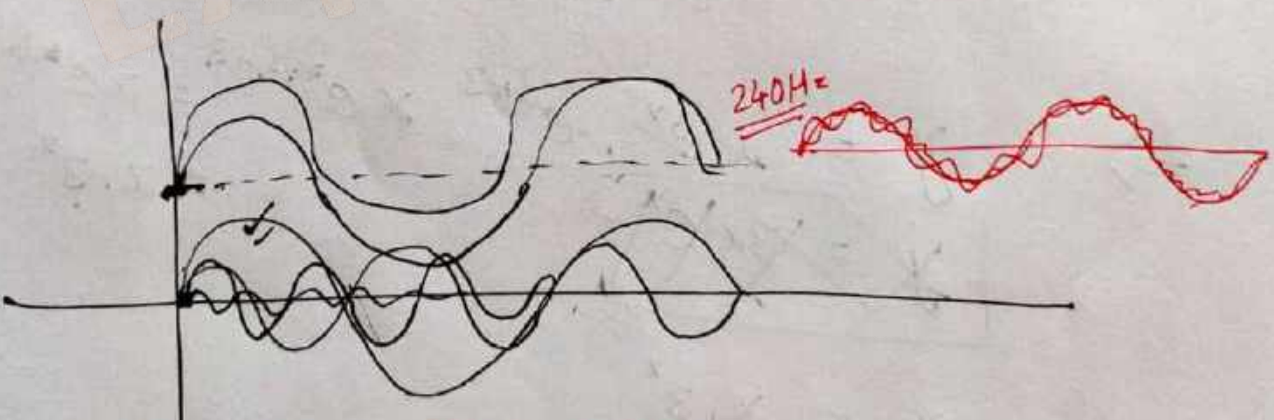
⇒ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$\frac{a_0}{2}$ is your DC part (Baseline)

⇒ $n = 1$ the freq. of wave is the fundamental freq.

⇒ Time Period → $T = \frac{1}{f}$

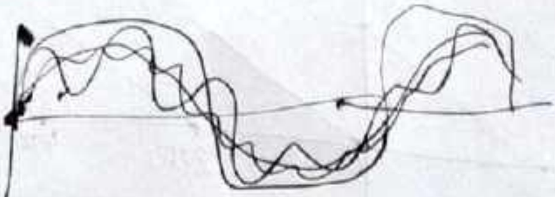
Sa → 240Hz



$n = 2 - \infty$ Harmonics

[Integral multi ~~ps~~ of fun. freq.]

Fourier Series ✓



Taylor ~~Series~~ ✓ Approximation

$\sqrt{26} \approx \sqrt{25}$

$f(x+a) = f(x) + \frac{a}{1} f'(x) + \frac{a^2}{2!} f''(x)$

Determination of Coefficients →

$\Rightarrow$ Period $\rightarrow 2\pi$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{2}{T}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\boxed{}\,dx$$

$$b_n = \frac{2}{T}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\boxed{}\,dx$$

Some imp. Integrals →

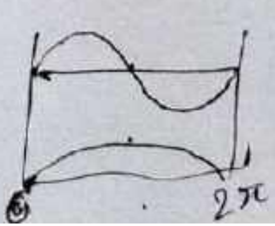
$$\int_0^{2\pi} \sin nx\,dx = 0 \qquad \int_0^{2\pi} \cos nx\,dx = 0$$

$$\int_0^{2\pi} \sin^2 nx\,dx = \pi \qquad \int_0^{2\pi} \cos^2 nx\,dx = \pi$$

$$\int_0^{2\pi} \sin nx \cos mx\,dx = 0 \qquad \int_0^{2\pi} \cos nx \cos mx = \pi\delta_{mn}$$

$\delta = 1 \quad m = n$

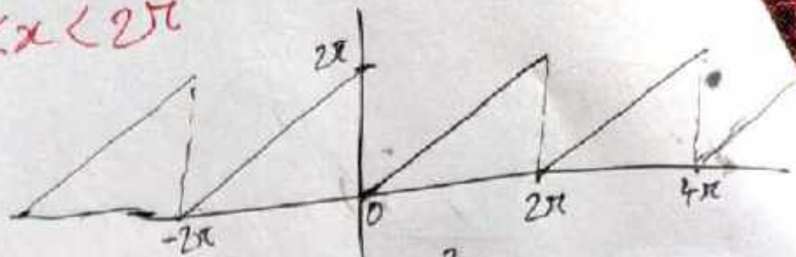
$\delta = 0 \quad m \neq n$



$$\int_0^{2\pi} \sin nx \sin mx = \pi\delta_{mn}$$

⇒ find the fourier Series for 6

$f(x) = x \quad 0 < x < 2\pi$



$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left.\frac{x^2}{2}\right|_0^{2\pi} = \frac{1}{\pi}\frac{4\pi^2}{2} = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx = \frac{1}{\pi}\left[x\frac{\sin nx}{n} - \left[-\frac{\cos nx}{n^2}\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{\cos 0}{n^2}\right] = \frac{1}{1}(1-1) = 0$$

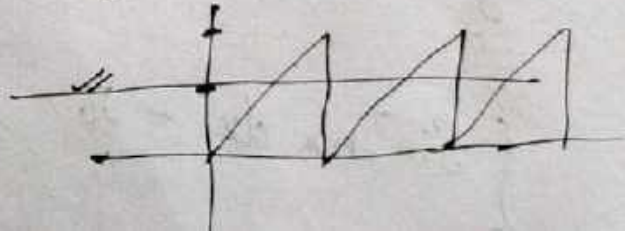
$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left[-\frac{\sin nx}{n^2}\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[2\pi\frac{-\cos 2\pi n}{n}\right] = -\frac{2}{n}\cos 2n\pi = -\frac{2}{n}$$

$$f(x) = \frac{2\pi}{2} - 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$$

1kHz   2kHz   3kHz

n=1



$f(x) \rightarrow$ Even fun.

$\cos x \rightarrow$ Even fun.

$\sin x \rightarrow$ Odd fun.

$$b_n = 0 = \int_0^{2\pi} f(x) . \sin nx \, dx$$

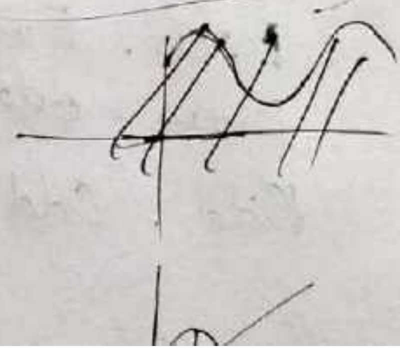
fourier Cosine Series $\rightarrow$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$f(x) \rightarrow$ Odd fun.

$$a_n = 0 \qquad \boxed{a_0 = \int_0^{2\pi} f(x) \, dx = 0}$$

fourier Sine Series $\rightarrow$



$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Even fun. → $f(-x) = f(x)$ Evenfunction

Odd fun. → $f(-x) = -f(x)$ Odd fun.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$x \to -x$

$$= -\int_{a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

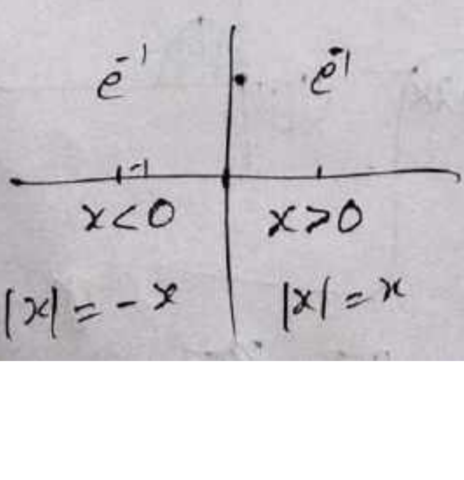
$$\begin{cases} 2\int_{0}^{a} f(x)\,dx & \text{when } f(x) \text{ is even} \\ 0 & \text{when } f(x) \text{ is an odd.} \end{cases}$$

Even . Even = Even

Even . odd = odd

Odd . Odd = even

$f(x) = e^{-|x|}$

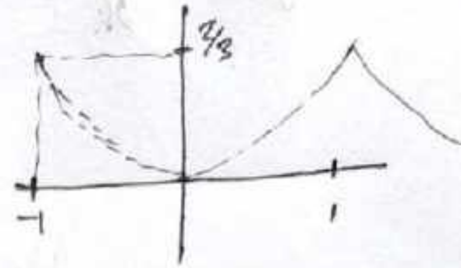


$f(-x) = e^{-x}$

IFR 2013

$$f(x) = a_0 + a_1 \cos\frac{\pi x}{2} + a_2 \sin\frac{\pi x}{2}$$

find a_0 a_1 & a_2



$\Rightarrow$ $a_2 = 0$ [because it is an even fun.]

$\Rightarrow$ $f(x) = a_0 + a_1 \cos\frac{\pi x}{2}$

$\frac{2}{3} = a_0 + a_1(0)$ $\quad\left[\begin{array}{l} x = 1, -1 \\ y = 2/3 \end{array}\right]$

$a_0 = \frac{2}{3}$

$\Rightarrow$ $[x = 0 \quad y = 0]$

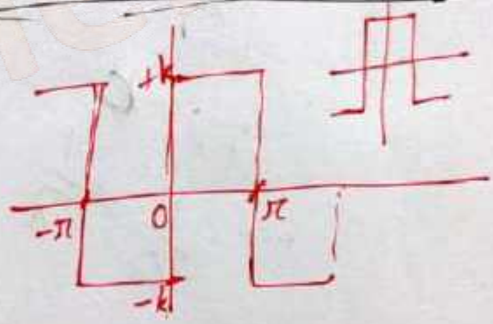
$0 = \frac{2}{3} + a_1(1)$

$a_1 = -\frac{2}{3}$

$$\boxed{a_0 = \frac{2}{3} \quad a_1 = -\frac{2}{3} \quad a_2 = 0}$$

Square wave $\rightarrow$

$$f(x) \begin{cases} +k & 0 \le x \le \pi \\ -k & -\pi < x < 0 \end{cases}$$



$\Rightarrow$ $a_0 = 0 \quad a_n = 0$

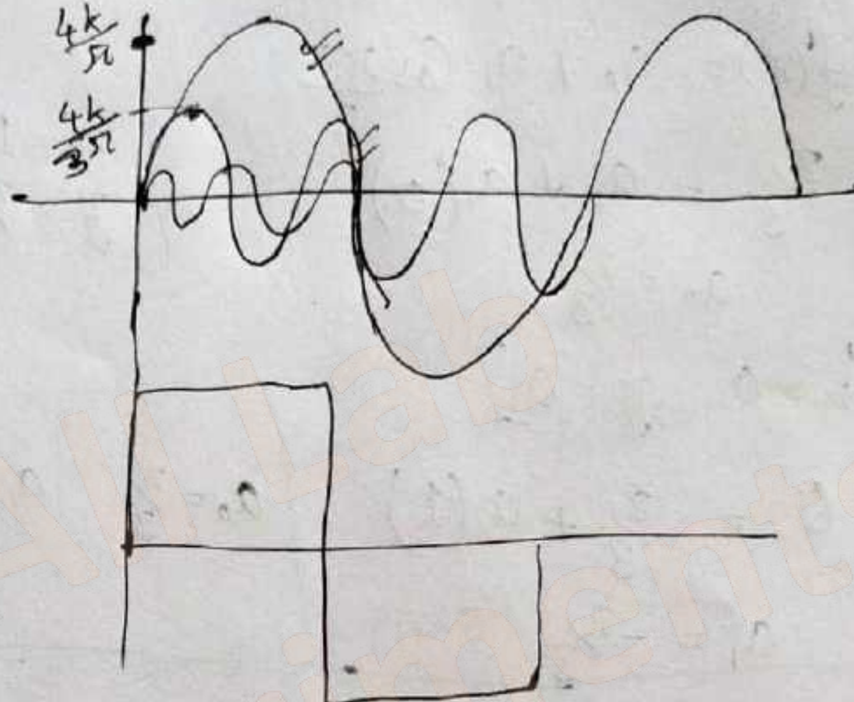
$\Rightarrow$ $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$

$$= \frac{2}{\pi}\int_0^{\pi} k \sin nx\, dx = \frac{2k}{n\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{-2k}{n\pi}\left[\cos n\pi - 1\right]$$

$$= \frac{2k}{\pi n}\begin{cases} 2 & n = \text{odd} \quad \left(b_n = \frac{4k}{n\pi}\right) \\ 0 & n = \text{even} \quad (b_n = 0) \end{cases}$$

$$f(x) = \frac{4k}{\pi}\left(\frac{\sin x}{1} + \cancel{\sin 2x}\,\frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

1 kHz



$\sin kx$

$\downarrow$

$\boxed{\frac{2\pi}{\lambda}}$

$\lambda\nu = c$

$\frac{\nu}{c} = \frac{1}{\lambda}$

$\boxed{\frac{1}{\nu} = T}$

$\frac{2\pi}{\lambda} \Rightarrow \frac{2\pi\nu}{c}$

Half period. $\rightarrow$ $(0 - \pi)$ periodic

$$a_n = \frac{2}{T}\int_0^{\pi} \square \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \square \cos nx\,dx$$

$$b_n = \frac{2}{T}\int_0^{\pi} \square \sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \square \sin nx\,dx$$

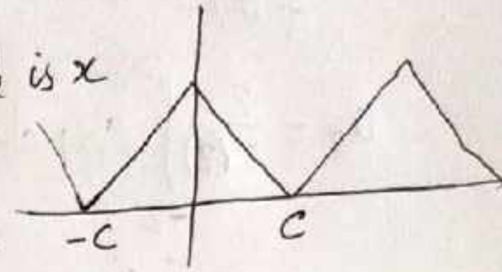
14 June, 2020

Change in Interval →

⇒ If you have a func. periodic in $(-c, c)$

Then $f(x) \longrightarrow 2c$

$\because$ 2c is the period when variable is x



1 ———— $= \dfrac{x}{2c}$

2π ———— $= \dfrac{x}{2c} \cdot 2\pi$

$= \dfrac{x\pi}{c} = z$

$f(x) \longrightarrow 2c \qquad f(z) \longrightarrow 2\pi$

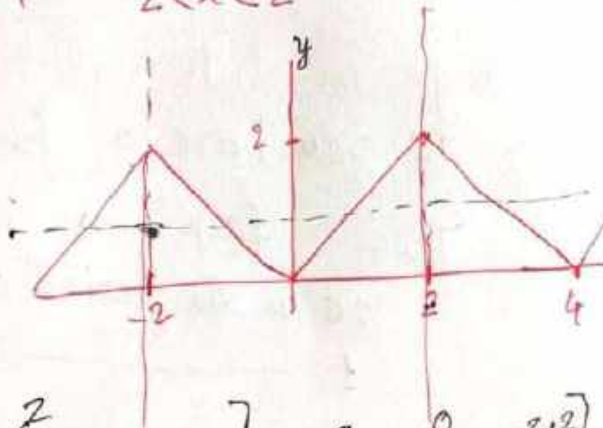
$$a_0 = \frac{2}{2c}\int_{-c}^{c} f(x)\,dx$$

$$a_n = \frac{2}{2c}\int_{-c}^{c} f(x)\cos\left(\frac{\pi}{c}nx\right)dx$$

$$b_n = \frac{2}{2c}\int_{-c}^{+c} f(x)\sin\left(\frac{\pi}{c}nx\right)dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{c}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{c}\right)$$

Q Periodic function $f(x) = |x| \quad -2 < x < 2$



$f(x) = x \quad 0 < x < 2$

$f(x) = -x \quad -2 < x < 0$

$$a_0 = \frac{2}{2(2)} \int_{-2}^{2} f(x)\, dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} (-x)\, dx + \int_{0}^{2} (x)\, dx\right] = \frac{1}{2}\left[-\frac{x^2}{2}\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_{0}^{2}\right]$$

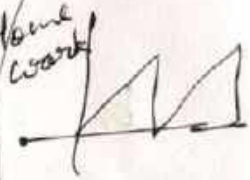
$$= 2$$

b_n = all zero

$$a_n = \frac{2}{2c} \int_{-c}^{c} f(x) \cos\left(\frac{\pi}{c} n x\right) dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} (-x) \cos\left(\frac{\pi}{c} n x\right) dx + \int_{0}^{2} x \cos\left(\frac{\pi}{c} n x\right) dx\right]$$

Home work



$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] - \begin{cases} \dfrac{-8}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

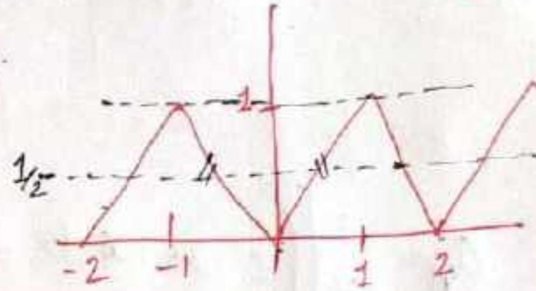
$$f(x) = 1 + \left(-\frac{8}{\pi^2}\right)\left[\frac{\cos\frac{\pi(1)x}{c}}{(1)^2} + \frac{\cos\frac{3\pi x}{c}}{(3)^2} + \frac{\cos\frac{5\pi x}{c}}{(5)^2} + \cdots\right]$$

JAM2015 Period of a fun. is $(0, 2L)$

find a_0 for this fun.

$f(x) = y = x \quad 0 < x < 1$

$f(x) = y = -x \quad -1 < x < 0$

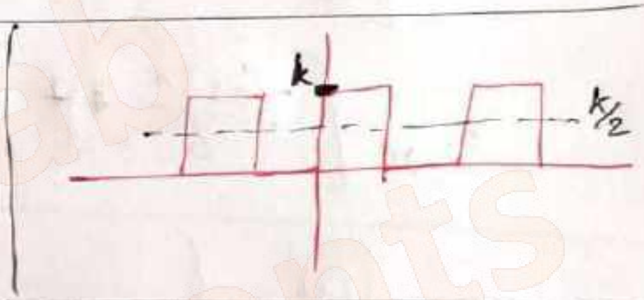


$$a_0 = \frac{2}{2(1)}\left[\int_{-1}^{0} -x\,dx + 1\int_0^1 x\,dx\right] = \boxed{1}$$

baseline (DC part) $= \frac{a_0}{2} = \frac{1}{2}$

JAM2005

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$



$f_1 = \cos^2 x \qquad f_2 = \sin^2 x$

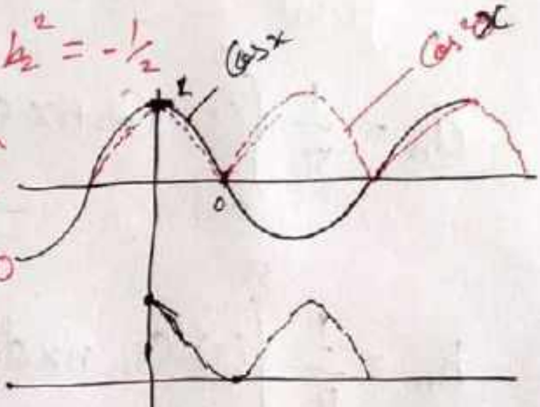
$\begin{bmatrix} a_1^{(1)} & a_2 & a_3 & \cdots \\ b_1 & b_2 & b & \cdots \end{bmatrix} \qquad \begin{bmatrix} a_1^{(2)} & a_2^{(2)} & a_3 & \cdots \\ b_1 & b_2 & b_3 & \cdots \end{bmatrix}$

a) $a_2^1 = \frac{1}{2} \quad b_2^2 = -\frac{1}{2}$ b) $b_2^1 = \frac{1}{2} \quad a_2^2 = -\frac{1}{2}$

d) $b_2^1 = \frac{1}{2} \quad b_2^2 = -\frac{1}{2}$

$\cancel{c)}$ $a_2^1 = \frac{1}{2} \quad a_2^2 = -\frac{1}{2}$

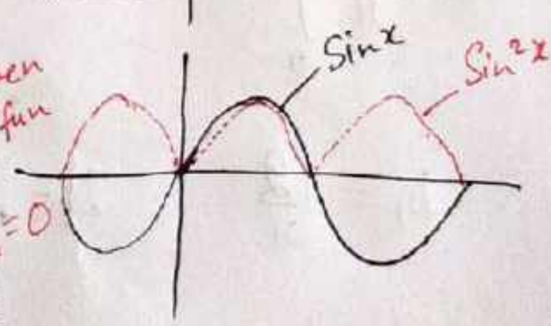
① Even fun So $b_n^1 = 0$



$f_1 = \cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2} + \frac{\cos 2x}{2}$

$f_2 = \sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{\cos 2x}{2}$

② Even fun So $b_n^2 = 0$



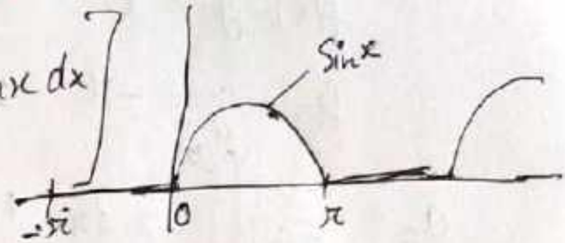
JAM 2007

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$$

④

find the coefficient of $\cos 2x$ (a_2)

$$a_n = \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{0} 0 \cos nx\, dx}_{0} + \int_0^{\pi} \sin x \cos nx\, dx\right]$$



$$a_2 = \frac{1}{\pi}\left[\int_0^{\pi} \sin x \cos 2x\, dx\right] = \frac{1}{2\pi}\left[\int_0^{\pi} \left(\sin 3x + \sin(-x)\right) dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{-\cos 3x}{3} + \cos x\right]_0^{\pi} = \boxed{-\frac{2}{3\pi}}$$

n=1 $a_1 \sin x + b_1 \cos x$
fundamental

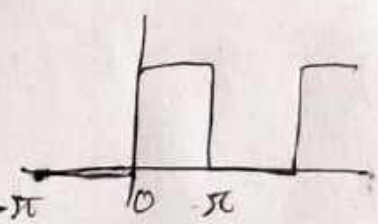
n=2 $a_2 \sin 2x + b_2 \cos 2x$
2nd Harmonic

Q. $f(x) = \begin{cases} 0 & (-\pi, 0) \\ 1 & (0, \pi) \end{cases}$

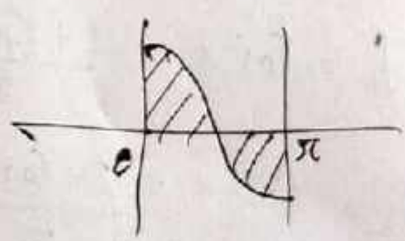
What is the ratio of coefficients of first & third harmonics.

$$a_n = \frac{1}{\pi}\int_0^{\pi} (1) \cos nx\, dx = \frac{1}{\pi} \left.\frac{\sin nx}{n}\right|_0^{\pi}$$

$$= 0.$$



$$b_n = \frac{1}{\pi}\int_0^{\pi} (1) \sin nx\, dx = \frac{-1}{\pi} \left.\frac{\cos nx}{n}\right|_0^{\pi}$$



$$= -\frac{1}{n\pi}\left[(-1)^n - 1\right]$$

$b_1 = +\frac{2}{\pi}$ $\qquad$ $b_3 = +\frac{2}{3\pi}$

$$\boxed{\frac{b_1}{b_3} = 3}$$

Integral Transform

⇒ It changes one fun. to another fun.
↳ when it is done through integration.

⇒ Space → momentum | freq. → Time
$f(x) \longrightarrow F(p)$

$$f(s) = I[f(x)] = \int_a^b \underbrace{K(s,x)}_{\text{Kernel of transformation}} f(x)\, dx$$

⇒ Fourier Transform → $K(s,x) = e^{isx}$

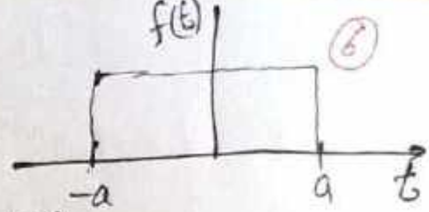
$$f(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Inverse transform →

$$f(x) = F^{-1}[f(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(s)\, e^{-isx}\, dx$$

Q.

$$f(t) = \begin{cases} 1 & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$



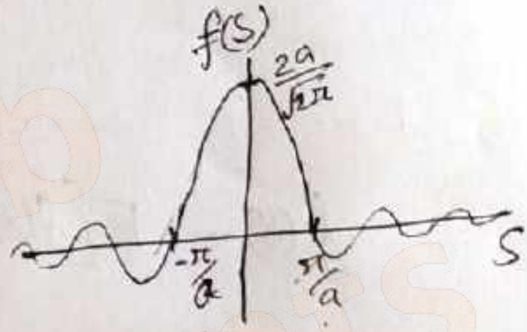
⑥

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1\, e^{ist}\, dt = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ist}}{is} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2i}{is} \left(\frac{e^{ias} - e^{-ias}}{2i} \right)$$

$$= \boxed{\frac{1}{\sqrt{2\pi}} \frac{2}{s} (\sin as)}$$

$$f(s) = \frac{2a}{\sqrt{2\pi}} \left(\frac{\sin as}{as} \right)$$



for zero $\sin as = \sin n\pi$

$$s = \frac{n\pi}{a}$$

Q. $f(t) = \cos at$

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cos at\, e^{ist}\, dt$$

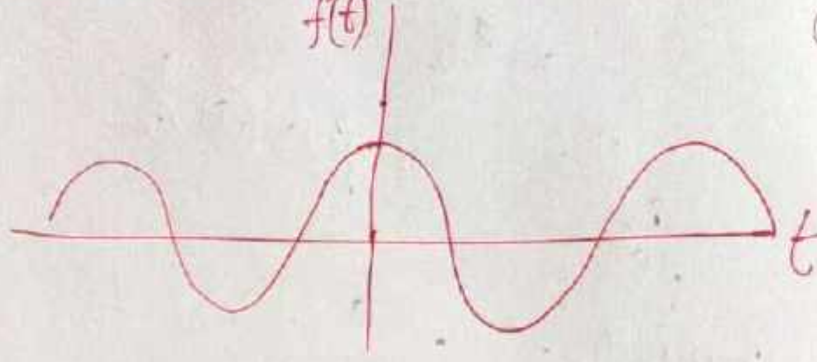
$$\cos at \longrightarrow \frac{e^{iat} + e^{-iat}}{2}$$

$$\delta(s+a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(s+a)t} dt$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(a+s)t} dt + e^{i(s-a)t} dt$$

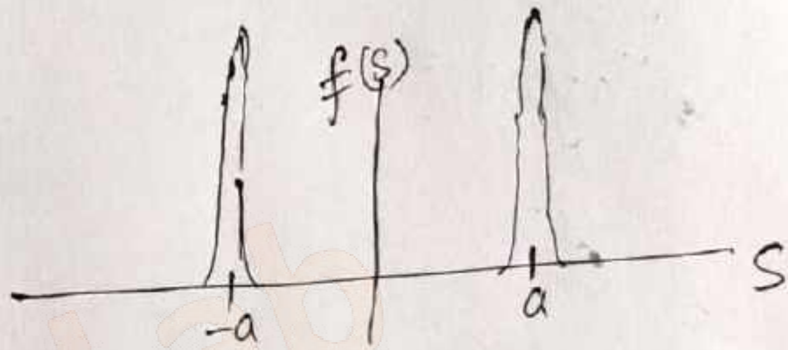
$$= \sqrt{\frac{\pi}{2}} \left[\delta(s+a) + \delta(s-a) \right]$$

$\cos at$



Optics

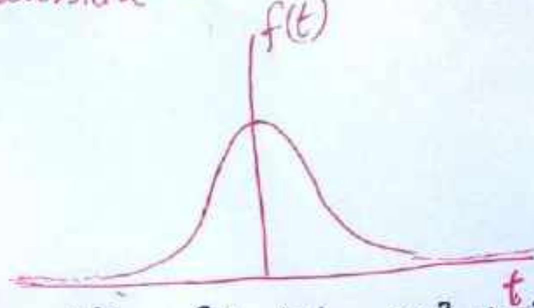
$[\delta(s+a)+\delta(s-a)]$



Q. $\sin at$ Home work

⇒ fourier transform of a Gaussian →

$$f(t) = e^{-\alpha t^2}$$



$$f(S) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\alpha t^2} e^{ist}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\alpha\left(t^2 - \frac{ist}{\alpha}\right)} dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\alpha\left[t^2 - \frac{ist}{\alpha} + \left(\frac{is}{2\alpha}\right)^2 - \left(\frac{is}{2\alpha}\right)^2\right]}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\alpha\left[\left(t - \frac{is}{2\alpha}\right)^2\right]} e^{\alpha\left(\frac{is}{2\alpha}\right)^2} dt$$

$$= \frac{e^{-\frac{s^2}{4\alpha}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha\underbrace{\left(t - \frac{is}{2\alpha}\right)}_{k}{}^2} dt$$

$$\boxed{\begin{aligned} k &= t - \frac{is}{2\alpha} \\ dk &= dt \end{aligned}}$$

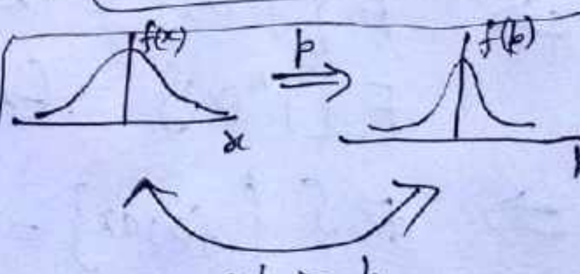
$$= \frac{e^{-s^2/4\alpha}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\underbrace{\alpha k^2}_{z}} dk$$

$$= \frac{2e^{-s^2/4\alpha}}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\alpha k^2} dk$$

$\alpha k^2 = z$
gamma fun. $\Gamma\frac{1}{2}$
$2\sqrt{\frac{\pi}{\alpha}}$

$$= \frac{\cancel{2}\, e^{-s^2/4\alpha}}{\sqrt{2\cancel{\pi}}} \cdot \cancel{2}\sqrt{\frac{\cancel{\pi}}{\alpha}}$$

$$\boxed{= \frac{1}{\sqrt{2\alpha}}\, e^{-s^2/4\alpha}}$$



$$\Delta x \cdot \Delta p \geq \frac{h}{4\pi}$$

Q fourier transfom of dirac delta →

$$\cancel{F} F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x)\, e^{+ist}\, dt$$

$$= e^{i(0)s} = 1$$

$$\boxed{e^{ias}}$$

$$\int f(x)\, \delta(x-a) = f(a)$$

Properties of fourier transform →

$\Rightarrow$ $F\{f(x)\} = f(S)$

$\Rightarrow$ Change in scale → $F\{f(ax)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

$\Rightarrow$ Shifting property → $F\{f(x-a)\} = e^{isa} f(S)$

$\Rightarrow$?? $F\{e^{iax} f(x)\} = f(s+a)$

$\Rightarrow$ Modulating property → $F\{f(x) \cos ax\} = \frac{1}{2}\left[F(s+a) + F(s-a)\right]$

$\Rightarrow$ $F\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n} f(S)$

$\Rightarrow$ $F\{f'(x)\} = -is\, F(S)$

$F\{f^n(x)\} = (-is)^n F(S)$

$\Rightarrow$ $F\left\{\int_a^x f(x)\,dx\right\} = \frac{F(S)}{-is}$

DU2015 Let $F(\omega) \xrightarrow{\text{f. trans.}} f(t)$

$G(\omega) \xrightarrow{\text{f. trans.}} g(t)$

The relation b/w $g(t) = f(t+a)$

find the relation b/w $F(\omega)$ & $G(\omega)$

$$F\{g(t)\} = F\{f(t+a)\}$$

$$G(\omega) = e^{-i\omega a} F\{f(t)\}$$

$$= e^{i\omega a} F(\omega)$$

DU2018 $F(k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$

If we have $g(x)$ then find $G(k)$ when

$g(x) = 1$ from -1 to 1.

a) $\sqrt{\dfrac{2}{\pi}} \dfrac{\sin k}{k}$ b) $\dfrac{1}{\sqrt{\pi}} \dfrac{\exp k}{k}$ c) $\sqrt{\dfrac{2}{\pi}} \cos k$

d) $\dfrac{1}{\sqrt{\pi}} \exp \dfrac{-k}{k}$

DU2019 $F\{e^{-\alpha x^2}\} = \dfrac{1}{\sqrt{\alpha}} e^{-k^2/4\alpha}$

Then find $F\{x^2 e^{-\alpha x^2}\} = (-i)^2 \dfrac{d^2}{dk^2}\left(\dfrac{1}{\sqrt{\alpha}} e^{-k^2/4\alpha}\right)$

$$= \frac{-1}{\sqrt{\alpha}} \frac{d}{dk}\left(e^{-k^2/4\alpha} \cdot \left(-\frac{2k}{4\alpha}\right)\right) = +\frac{1}{2\alpha\sqrt{\alpha}} \frac{d}{dk}\left(k e^{-k^2/4\alpha}\right)$$

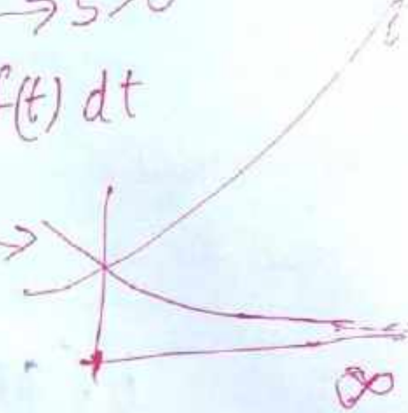
$$= \frac{1}{2\alpha^{3/2}}\left(k\left(e^{-k^2/4\alpha}\left(-\frac{k}{2\alpha}\right)\right) + e^{-k^2/4\alpha}\right)$$

$$= \frac{e^{-k^2/4\alpha}}{2\alpha^{3/2}}\left(-\frac{k^2}{2\alpha} + 1\right) = \left(\frac{-k^2 + 2\alpha}{4\alpha^{5/2}}\right) e^{-k^2/4\alpha}$$

Laplace Transformation →

$s > 0$

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$



$$\mathcal{L}[1] = \int_0^\infty e^{-st}\,dt$$

$$= \left.\frac{e^{-st}}{-s}\right|_0^\infty = 0 - \frac{1}{-s} = \frac{1}{s}$$

$$\mathcal{L}[1] = \frac{1}{s}$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

$$\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$$

$$\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$$

$$\mathcal{L}[\sinh at] = \frac{a}{s^2-a^2}$$

$$\mathcal{L}[\cosh at] = \frac{s}{s^2-a^2}$$

$$\mathcal{L}[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[f^n(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots$$

$$\mathcal{L}(t^{-1/2}) = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma\frac{1}{2}}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

Properties of Laplace Transformation →

1. Scaling $\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

2. Shifting $\mathcal{L}[e^{at} f(t)] = F(s-a)$

3. Laplace trans. of a derivative

$$L[f'(t)] = s F(s) - f(0)$$

$$L[f''(t)] = s^2 F(s) - s f(0) - f'(0)$$

$$L[f^n(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots$$

4. Laplace trans. of an integral

$$\mathcal{L}\left[\int_0^t f(t)\, dt\right] = \frac{F(s)}{s}$$

5. $\mathcal{L}[t^n \cdot f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$

6. $\mathcal{L}\left[\frac{1}{t} f(t)\right] = \int_s^\infty F(s)\, ds$

Q. $\mathcal{L}(t^2 e^t \sin 4t)$

$$\mathcal{L}(\sin 4t) = \frac{4}{s^2 + 4^2}$$

$$\mathcal{L}(e^t \sin 4t) = \frac{4}{(s-1)^2 + 4^2} = \frac{4}{s^2 + 1 - 2s + 16}$$

$$\mathcal{L}(t^2 e^t \sin 4t) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{4}{s^2 - 2s + 17}\right)$$

$$= \frac{8(3s^2 - 6s - 13)}{(s^2 - 2s + 17)^3}$$

Jest 2014 $\mathcal{L}[e^{-2t}\sin 4t]$

$$\mathcal{L}(\sin 4t) = \frac{4}{s^2+4^2}$$

$$\mathcal{L}[e^{-2t}\sin 4t] = \frac{4}{(s+2)^2+4^2} = \frac{4}{s^2+4+4s+16}$$

$$= \frac{4}{s^2+4s+20}$$

Jest 2018 $\mathcal{L}\left[\frac{1}{2a^3}(\sin at - at\cos at)\right]$

$$= \frac{1}{2a^3}\left[\mathcal{L}(\sin at) - \mathcal{L}(at\cos at)\right]$$

$$= \frac{1}{2a^3}\left[\frac{a}{s^2+a^2} - a\left[(-1)^1\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)\right]\right]$$

$$= \frac{1}{2a^2}\left[\frac{1}{s^2+a^2} + \frac{(s^2+a^2)-s(2s)}{(s^2+a^2)^2}\right]$$

$$= \frac{1}{2a^2}\left[\frac{1}{s^2+a^2} + \frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{1}{2a^2}\left[\frac{s^2+a^2+a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{1}{(s^2+a^2)^2}$$

Taylor Series → (Taylor Approximation)

$$f(a+h) = f(a) + h\,f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots$$

e.g. $\sqrt{26} = \sqrt{25+1}$ $\quad f(x) = \sqrt{x}$ $\quad f(a+h) \Rightarrow$ $a = 25$, $h = 1$

$$f(26) = f(25) + 1\,f'(25) + \frac{1^2}{\lfloor 2} f''(25)$$

$f(x) = \sqrt{x}$ $\qquad f(25) = 5$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\qquad f'(25) = \frac{1}{10}$

$f''(x) = -\frac{1}{4}x^{-3/2}$ $\qquad f''(25) = -\frac{1}{4x^{3/2}} = -\frac{1}{4(125)} = -\frac{1}{500}$

$$f(25+1) = 5 + 1\left(\frac{1}{10}\right) + \frac{1}{2}\left(-\frac{1}{500}\right) \cdots$$

$$= 5 + \frac{1}{10} - \frac{1}{1000}$$

Maclaurin Series →

$$f(0+h) = f(0) + h f'(0) + \frac{h^2}{\lfloor 2} f''(0) + \cdots$$

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \cdots$$

$f(x) = \sin x$ $\qquad f(0) = \sin 0 = 0$

$f'(x) = \cos x$ $\qquad f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$ $\qquad f''(0) = 0$

$f'''(x) = -\cos x$ $\qquad f'''(0) = -1.$

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots$$

Q. HCU 2014 find the taylar Series of $\sin x$ around $\pi/2$.

$$x = a + h = \frac{\pi}{2} + h \qquad f(x) = \sin x, \quad x = \langle \pi/2 \rangle$$

$$h = \left(x - \frac{\pi}{2}\right)$$

(2)

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{\lfloor 2} f''(a) - - - -$$

$$= \sin\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^2}{\lfloor 2}\left(-\sin\frac{\pi}{2}\right) - - -$$

$$= 1 - \frac{1}{\lfloor 2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{\lfloor 4}\left(x - \frac{\pi}{2}\right)^4 - - - - -$$

NET 2011 Taylar Series of $\sin x$ around $\pi/4$

Home work

JAM 2018 find the coefficient of x^3 in the expansion of $\sin(\sin x)$.

Ans. $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7}$

$$\sin(\sin x) = \sin x - \frac{\sin^3 x}{\lfloor 3} + \frac{\sin^5 x}{\lfloor 5} - - - -$$

$$= \left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - - -\right) - \frac{1}{\lfloor 3}\left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - - -\right)^3$$

Coefficients of x^3

$$= -\frac{1}{\lfloor 3} - \frac{1}{\lfloor 3} = -\frac{1}{6} - \frac{1}{6} = -\frac{2}{6} = -\frac{1}{3} = \underline{\underline{-0.33}}$$

Some Sample Series →

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$

$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

$|x| < 1$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots$

$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$

$|x| \leq 1$

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \cdots$

$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$

$\Downarrow$

$\frac{1}{2} \ln\left[\frac{1+x}{1-x}\right]$

TIFR 2011

Binomial Theorem →

$$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + \cdots {}^nC_n x^0 y^n$$

$${}^nC_r = \frac{n!}{(n-r)!\, r!}$$

$(1+x)^{-1} = 1 - x + x^2 - x^3 \cdots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

GP Series →

$a + ar + ar^2 + \cdots$ n-terms

Sum of series $= a\frac{1-r^n}{1-r}$

when $n \to \infty$ in case $r < 1$ $r^n \to 0$

Sum of ∞ GP series is $\frac{a}{1-r}$.

DP $-1 \leq x \leq 1$ find the sum

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} = \tan^{-1} x$$

Jest 2012 as $x \to 1$ find the value of

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \text{---}$$

$\tan^{-1} x$ & $x \to 1$ then $\tan^{-1} 1$

$$\tan^{-1} 1 = \pi/4$$

Jest 2013 $\left(1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 4} + \text{---}\right)^2 - \left(1 + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 5} + \text{---}\right)^2$

$$= (\cosh 1)^2 - (\sinh 1)^2$$

$$= \cosh^2 1 - \sinh^2 1 = 1$$

Convergence / Divergence →

Sequence & Series → Sum of those numbers.
↳ Representation of numbers in a row

Convergence & Div. →

power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n = a_0 + a_1(z-a)^1 + a_2(z-a)^2 + \cdots$$

After addision if you get a finite no. then it is convergent
& if you don't get a finite no. then divergent.

$$1 + 2 + 2^2 + 2^3 + 2^4 + \cdots 2^n \cdots = \infty \text{ [Divergent]}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^n} \cdots = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$$

[Convergent]

u_n is your nth term of a series

$\lim_{n\to\infty} u_n = 0$ then Conv. / div.

$\lim_{n\to\infty} u_n \neq 0$ then for sure it is divergent.

$\lim_{n\to\infty} 2^n = \infty$ divergent

$$(1 + 2 + 3 + \quad n) = \frac{n}{2}(n+1)$$

$u_0 + u_1 + \cdots\cdots\cdots u_n + u_{n+1}$ or

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ Converge

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$ Diverge

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ Then this test fails

G.P. →

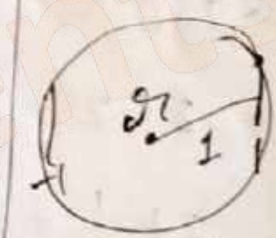
$1 + r + r^2 + r^3 + \cdots\cdots\cdots r^n + r^{n+1}$

$\lim_{n\to\infty}\left|\frac{r^{n+1}}{r^n}\right| = \lim_{n\to\infty}|r|$

$|r| < 1$ then Convergent

$|r| > 1$ then divergent.

$r = 1$ then also divergent.



R = 1

$\sum_{n=0}^{\infty}\frac{r^n}{\lfloor n} = 1 + r + \frac{r^2}{\lfloor 2} + \frac{r^3}{\lfloor 3} + \cdots + \frac{r^n}{\lfloor n} + \frac{r^{n+1}}{\lfloor n+1}$

$\lim_{n\to\infty}\left|\frac{r^{n+1}}{\lfloor n+1}\cdot\frac{\lfloor n}{r^n}\right| \Rightarrow \lim_{n\to\infty}\frac{|r|}{(n+1)} = 0$

This series is convergent.

$R = \infty$

Q $\sum_{n=0}^{\infty} r^n \lfloor\!\underline{n} = r^0 \lfloor\!\underline{0} + r^1 \lfloor\!\underline{1} + r^2 \lfloor\!\underline{2} + \dots r^n \lfloor\!\underline{n} + r^{n+1} \lfloor\!\underline{n+1}$

$$\lim_{n\to\infty} \left| \frac{r^{n+1} \lfloor\!\underline{n+1}}{r^n \lfloor\!\underline{n}} \right| = \lim_{n\to\infty} |r|(n+1) = \infty$$

- but $|r| = 0$ then this expression gives 0 only then this converges otherwise diverges.

$$\boxed{R = 0}$$

Radius of Convergence →

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$f(z) = \sum_{n=0}^{\infty} \frac{\lfloor\!\underline{2n}}{(\lfloor\!\underline{n})^2} (z-a)^n$$

$$R = \lim_{n\to\infty} \left| \frac{\lfloor\!\underline{2n}}{(\lfloor\!\underline{n})^2} \times \frac{(\lfloor\!\underline{n+1})^2}{\lfloor\!\underline{2(n+1)}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{\lfloor\!\underline{2n}}}{\cancel{(\lfloor\!\underline{n})^2}} \times \frac{(n+1)^2 \cancel{(\lfloor\!\underline{n})^2}}{(2n+2)(2n+1)\cancel{\lfloor\!\underline{2n}}} \right| = \frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}\left(1+\frac{1}{n}\right)^2}{2\cancel{n^2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)} = \frac{1}{4}$$

$|z-a| < \frac{1}{4}$ only then it is convergent.