Free Study Material from

All Lab Experiments

Mathematical Physics for JAM/NET/Gate Physics

> Differential Equations <

Differential Equations → An equation which contains some derivative.

Order → The order of a diff. eqn. is the highest order of derivative in it.

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 5$$

Degree → The power of highest order of derivative

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + y = 5$$

first order diff eqns. → $\frac{dy}{dx} = f(x, y)$

first degree

$y = f(x)$

Solution Mae → Exact diff eqn.

Separable diff eqn.

Linear diff eqn.

Homogeneous diff eqn.

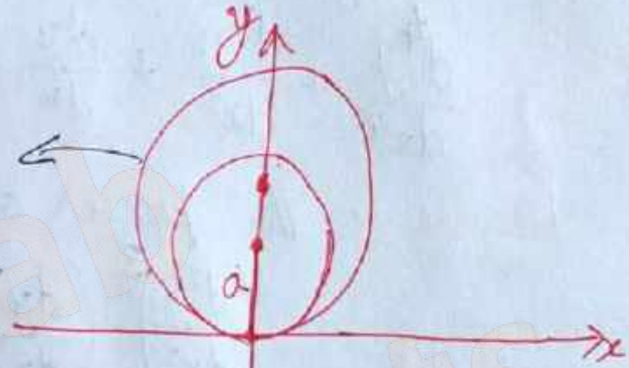
How to create a diff. eqn.

$$x^3 + 2(y^2 + \underline{a}) = \underline{b}$$

no. of parameters = order of diff. eqn.

Q. Create the diff. eqn. for a family of circles passing through origin & whose center lies on y-axis.

$$x^2 + (y-a)^2 = a^2$$



$$2x + 2(y-a)\frac{dy}{dx} = 0$$

$$\frac{\cancel{2}x}{\frac{dy}{dx}} = -\cancel{2}(y-a) \qquad a = -y + \frac{x}{\frac{dy}{dx}}$$

Home work

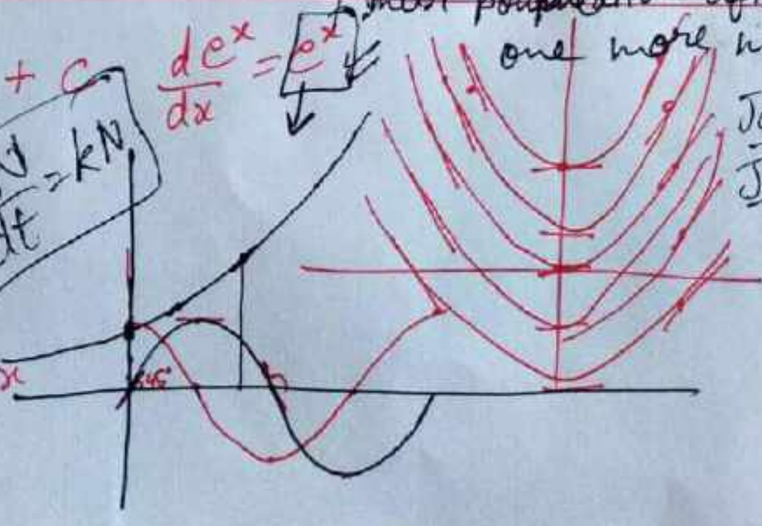
$$(x^2 - y^2)\frac{dy}{dx} = 2xy$$

The increase in population is ~~proportional~~ proportional to the poupulation of that time. if they started at 10^5 & they became 10^7 after one month. what will be their poupulation after one more month.

$$y = x^2 + c \qquad \frac{de^x}{dx} = \boxed{e^x}$$

$$\boxed{\frac{d f(x)}{dx} = f'(x)} \qquad \frac{dy}{dx} = 2x \qquad \boxed{\frac{dN}{dt} = kN}$$

$$\frac{dN}{dt} \propto N \qquad \frac{d \sin x}{dx} = \cos x \qquad \frac{df}{dt} = \frac{d}{dx}$$



Test JAM

③

Separable Diff Eqns → You separate the varidbles x & y with thier derivatives & solve.

$$\Rightarrow (e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$$

$$\int\left(\frac{\cos x}{\sin x}\right)dx = \int\left(\frac{-e^y}{(e^y+1)}\right)dy$$

$$\ln(\sin x) = -\ln(e^y+1) + \ln C$$

$$\ln(\sin x)(e^y+1) = \ln C$$

$$\boxed{\sin x\,(e^y+1) = C}$$

Linear Diff Eqn → $\dfrac{dy}{dx} + Py = Q$ ✓

P & Q are the fun. of x.

Integrating factor (I.F.) → $e^{\int P dx}$

Solution → $y\,[I.F.] = \int Q.[I.F.] + C$

$$\Rightarrow (x+1)\frac{dy}{dx} - y = e^x(x+1)^2$$

$$\frac{dy}{dx} \underbrace{- \frac{1}{(x+1)}}_{P} y = \underbrace{e^x(x+1)}_{Q}$$

$$= \quad I.F. = e^{\int -\frac{1}{(x+1)}dx} = e^{-\ln(x+1)} = \frac{1}{(x+1)}$$

Solution $\quad y \cdot \dfrac{1}{(x+1)} = \int e^x \cancel{(x+1)} \cdot \dfrac{1}{\cancel{(x+1)}}\,dx + C$

$$\boxed{y \cdot \frac{1}{x+1} = e^x + C}$$ ✓

Eqn. Reducible to Linear form →

$$\Rightarrow \frac{dy}{dx} + Py = Qy^n \Rightarrow \frac{1}{y^n}\frac{dy}{dx} + P\frac{1}{y^{n-1}} = Q$$

Substitute $\frac{1}{y^{n-1}} = Z$

$$\frac{dz}{dx} + Pz = Q$$

$$\cos y \frac{dy}{dx} + P \sin y = Q$$

Substitute $\sin y = Z$

Then it will change to your linear eqn.

Q. $x^2 dy + y(x+y)\,dx = 0$

$$\frac{dy}{dx} + \frac{y}{x} = -\frac{y^2}{x^2} \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} + \frac{-1}{x}\frac{1}{y} = \frac{1}{x^2}$$

$$-\frac{1}{y} = z \qquad \frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} + \frac{1}{x}z = \frac{1}{x^2}$$

Home work.

I.F.

Homogeneous Eqns → $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)} = \frac{x(x+y)}{y^2}$

To solve such eqns. $y = vx \Rightarrow v = y/x$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Q. $(2xy + x^2)\frac{dy}{dx} = 3y^2 + 2xy$

$$\frac{dy}{dx} = \frac{(3y^2 + 2xy)}{(2xy + x^2)}$$

$$v + x\frac{dv}{dx} = \frac{3(vx)^2 + 2x(vx)}{2x(vx) + (vx)^2} = \frac{3v^2 + 2v}{2v + 1}$$

Home work

$$\frac{dx}{x} = \left(\frac{2v+1}{v^2+v}\right)dv$$

$$y^2 + xy = cx^3$$

Q. for Home-work

$$(1 + e^{x/y})dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$$

$y(0) = 1$

$$\frac{x}{y} = v$$

$$\boxed{e^{x/y} + \frac{x}{y} = \frac{1}{y}}$$

Exact Diff Eqn. $\rightarrow$ $M dx + N dy = 0$

an eqn. is exact when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Solution $\rightarrow$ $\int M dx + \int \left(\begin{matrix}\text{terms of } N \\ \text{not containing } x\end{matrix}\right) dy = C$

Q. $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

1. Check exactness $\rightarrow$ $\frac{\partial M}{\partial y} = 0 + 6x^2y - 6xy^2$

$\frac{\partial N}{\partial x} = 6x^2y - 6xy^2 + 0$

2. $\int (5x^4 + 3x^2y^2 - 2xy^3) dx + \int -5y^4 dy = C$

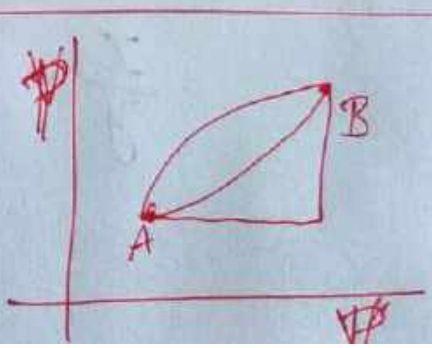
$$\boxed{x^5 + x^3y^2 - x^2y^3 - y^5 = C}$$

In case when this is not exact

1. Check exactness $\longrightarrow$ ✗
2. find I.F.
3. $\boxed{(I.F.)M} dx + \boxed{(I.F.)N} dy = 0 \longrightarrow$ Exact
4. Solution

What are Exact Diff eqns

Independent of your path.



⇒ when eqn. is non-exact.

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Case 1 ⇒ $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ then $I.F. = e^{\int f(x)dx}$

Case 2 ⇒ $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$ then $I.F. = e^{\int f(y)dy}$

Q. $(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0$

$\frac{\partial M}{\partial y} = \qquad \neq \frac{\partial N}{\partial x} =$

$\cancel{I.F.}$ $= \dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = -\frac{3}{y}$ $\qquad I.F. = e^{\int -\frac{3}{y}dy} = \frac{1}{y^3}$

$$\left(\frac{y^4+2y}{y^3}\right)dx + \left(\frac{xy^3+2y^4-4x}{y^3}\right)dy = 0$$

Home work

Case 3 $\Rightarrow$ $M = y f(x,y)$ $N = x f(x,y)$

$$I.F. = \frac{1}{Mx - Ny}$$ when $Mx - Ny \neq 0$

Case 4 $\Rightarrow$ M & N Homogeneous

$x^2y + x^3 + y^3$ $\quad$ $y^3 + x^3$

$$I.F. = \frac{1}{Mx + Ny}$$ when $Mx + Ny \neq 0$

~~Cons~~ Master trick $\rightarrow$ 95%

just consider $x^a y^b$ is your I.F.

$$x^a y^b M\,dx + x^a y^b N\,dy = 0$$

$$\frac{\partial (x^a y^b M)}{\partial y} = \frac{\partial (x^a y^b N)}{\partial x}$$

$(a+2)x^2y + bxy + \text{---} = 2x^2y + 5xy$

$(a+b)x^2y + 3xy = 2x^2y + bxy$

$a+b=2$ $\quad$ $\boxed{x^- y^-}$

$a=0$ $\quad b=-3$

$I.F. = \frac{1}{y^3}$

Q. $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$

find I.F. $\rightarrow$ Let $x^a y^b$ is the I.F.

$$\underbrace{(x^a y^{b+4} + 2x^a y^{b+1})}_{M'}dx + \underbrace{(x^{a+1}y^{b+3} + 2x^a y^{b+4} - 4x^{a+1}y^b)}_{N'}dy = 0$$

Consider this is exact. $\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$

$$(b+4)x^a y^{b+3} + 2(b+1)x^a y^b = (a+1)x^a y^{b+3} + 2a\,x^{a-1}y^{b+4} - 4(a+1)x^a y^b$$

$b+4 = a+1$

$b = -3$

$2(b+1) = -4(a+1)$

$b+1 = -2(a+1)$

$b = -2-1 = -3$

$2a = 0$

$a = 0$

$$\boxed{\text{I.F.} = \frac{1}{y^3}}$$

JAM 2014

Q. $dz(x,y) + x\,z(x,y)\,dx + y\,z(x,y)\,dy = 0$

$dz(x,y) = -x\,z(x,y)\,dx - y\,z(x,y)\,dy$

$\frac{\partial z}{\partial x} = -xz$

$\frac{\partial z}{z} = -x\partial x$

$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

$z = e^{-\frac{x^2}{2}} + c_1 \quad \cdot z = e^{-y^2/2} + c_2$

$\neq$

$z = \boxed{}$

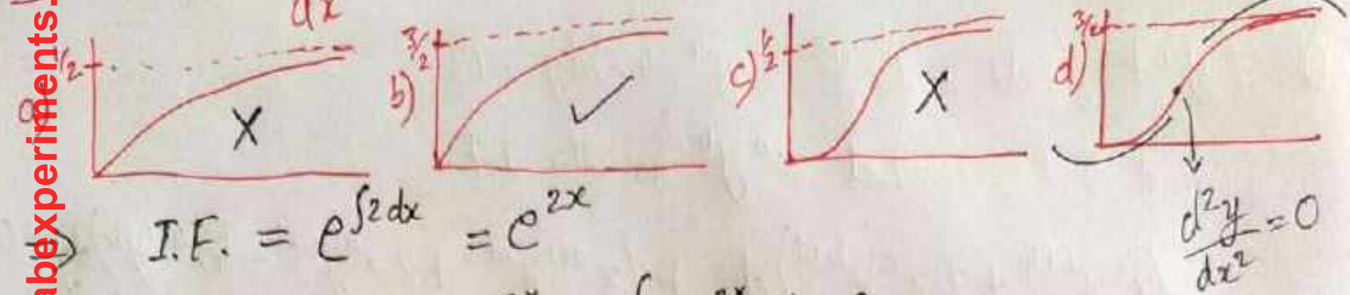
$\frac{dz}{z} = -x\,dx - y\,dy$

$\ln z = -\frac{x^2}{2} - \frac{y^2}{2} + c$

$z = e^{-x^2/2 - y^2/2} \cdot e^c$

$= k\,e^{-\left(\frac{x^2+y^2}{2}\right)}$

JAM 2018 Q. $\frac{df}{dx} + 2f = 3$; $f(0) = 0$ ②



$\Rightarrow$ $I.F. = e^{\int 2\,dx} = e^{2x}$

Solution $y \cdot e^{2x} = \int 3e^{2x} + c$

$y e^{2x} = \frac{3e^{2x}}{2} + c$

$$\boxed{y = \frac{3}{2} + c e^{-2x}}$$

$\Rightarrow$ $f(0) = 0$ $\quad 0 = \frac{3}{2} + c \Rightarrow c = -\frac{3}{2}$

$\Rightarrow$ Solution $\rightarrow$ $y = \frac{3}{2}\left(1 - e^{-2x}\right)$

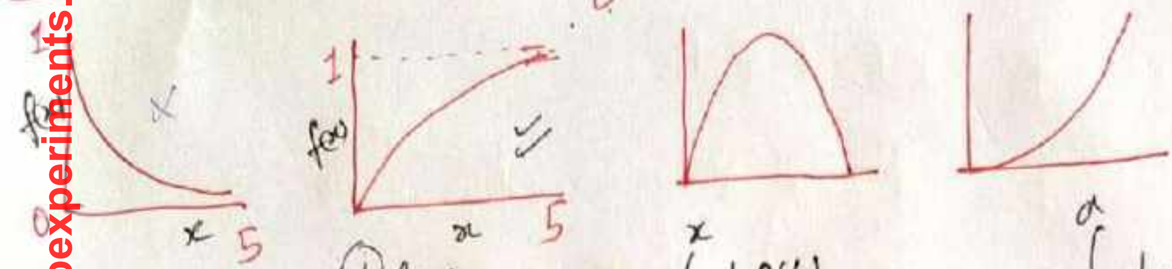
$\Rightarrow$ Graph $\rightarrow$ ① when $x \rightarrow \infty$ $\quad y \rightarrow \frac{3}{2}$

② $y' = -\frac{3}{2} e^{-2x}(-2) = 3e^{-2x}$

$y'' = 3(-2)e^{-2x} = -6e^{-2x} \neq 0$

There is no point of inflexion in this curve so option is valid.

TIFR 2014 $\qquad f(x) = x - \int_0^x dt\, f(t)$ ③



Ans.

$$\frac{d f(x)}{dx} = 1 - \int_0^x d f(t) \qquad \int_0^a dx = x\Big|_0^a$$

$$\frac{d f(x)}{dx} = 1 - \Big[f(t)\Big]_0^x$$

$$\frac{d f(x)}{dx} = 1 - \big[f(x) - f(0)\big]$$

$$\frac{d}{dx} f(x) + \underset{P}{1} f(x) = \underbrace{1 + \underset{0}{f(0)}}_{Q}$$

$$I.F. = e^{\int 1\, dx} = e^{x}$$

$$f(x) = 1 - e^{-x}$$

$y[I F] = \int Q$, $Q \to (1+\square)$

$f(t) = t^3 + 2t^2 - - -$

$f\left(\frac{t^2}{2}\right) X$

$$\frac{d}{dx}\int_0^x \frac{d}{\cancel{dt}}\, \cancel{dt}\, f(t)$$

$$\int_0^x d f(t)$$

$$\int_0^x d f(t) = \frac{d}{dx}\int_0^x dt\, f(t)$$

$$f(t)\Big|_0^x = \big[f(x) - f(0)\big]$$

$$\int \big[f(x) - f(0)\big] dx = \int_0^x f(t)\, dt$$

$f(x) \neq x^3 + x^2 + - - - C$

$$\frac{d}{dx}\int_0^x f(t)\, dt = \int_0^x d f(t)$$

$$\Downarrow$$

$$f(t)\Big|_0^x = f(x) - \underset{0}{f(0)}$$

⇒ Linear Diff eq. of higher order (1 degree) →

$$E = K.E. + P.E.$$

$$E = \frac{p^2}{2m} + V \qquad p \equiv -i\hbar\frac{\partial}{\partial x}$$

$$E = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right)\psi$$

$$\left[\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + (E-V)\psi\right] = 0$$

$$\left[\quad\right]\psi = 0$$

$$\square(\psi_1 + \psi_2) = \square\psi_1 + \square\psi_2$$

Definition of linearity.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

let $y = e^{mx}$

⟶ $(m^2 + Pm + Q)e^{mx} = 0$ [find the roots]

⇒ Complimetary fun.

⟶ $D^2 + PD + Q = f(D) \qquad \frac{d}{dx} \equiv D$

$$P.I. = \frac{1}{f(D)} R$$

⇒ Particular Integral

⇒ Linear diff Eq. of higher order

Step 1 → Make auxiliary Equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q = 0 \Rightarrow m^2 + Pm + Q = 0$$

Step 2 → find roots of this eqn.

m_1 & m_2

⇒ If roots are real & diff.

e^{mx} $\quad C_1e^{m_1x} + C_2e^{m_2x}$

⇒ If roots are real but eql.

$m_1 = m_2$

$$C_1e^{mx} + C_2xe^{mx} + C_3x^2e^{mx}$$

⇒ If your roots are complex no.

$a \pm ib$

$$C_1e^{(a+ib)x} + C_2e^{(a-ib)x}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$e^{ax}e^{ibx} \qquad e^{ax}e^{-ibx}$$

$$\Rightarrow e^{ax}(A\cos bx + B\sin bx)$$

Jest 2014 $\quad f''(x) - 2f'(x) + f(x) = 0$

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$\Rightarrow \underline{C_1e^x + C_2xe^x}$$

P.I. $\rightarrow$ $f(D)y = R$ $\qquad y = \frac{1}{f(D)} R$ ⑤

① $\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)} \quad f(a) \neq 0$

in case when $f(a)=0$ then $\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax} \quad f'(a) \neq 0$

② $\frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$

$D \equiv \frac{d}{dx}$

③ $\frac{1}{f(D^2)} \mathrm{Sin}/\mathrm{Cos}\, ax = \frac{1}{f(-a^2)} \mathrm{Sin}/\mathrm{Cos}\, ax$

$\frac{1}{D} \equiv \int dx$

④ $\frac{1}{f(D)} e^{ax}\phi(x) = e^{ax} \frac{1}{f(D+a)} \phi(x)$

⑤ $\frac{1}{D+a} \phi(x) = e^{-ax} \int e^{ax} \phi(x)\, dx$

$\frac{1}{f(D)} = \frac{1}{1-D} x^3$

$(1-D)^{-1} x^3$

$1 + D + \frac{D^2}{2} + \frac{D^3}{3} + \cdots$

JNU 2017 $\quad \frac{d^2y}{dx^2} + y + 1 = 0$

$\frac{d^2y}{dx^2} + y = -1$

$D^2 + 1 = -1$

A.E. $\rightarrow$ $m^2 + 1 = 0$

$m = \pm i$

C.F. $\rightarrow$ $A \cos x + B \sin x$

P.I. $= \frac{1}{D^2+1} -1$

$= \frac{-1}{D^2+1} e^{0x} = \frac{-1}{(0)^2+1}$

$= \frac{-1}{1} = -1$

$\frac{1}{(D+i)(D-i)} \left(\frac{A}{D+i} \quad \frac{B}{D-i} \right) x^0$

$y = A \cos x + B \sin x - 1$

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Q. $D^3y - 3D^2y + 4Dy - 2y = e^x + \cos x$

Ans. A.E. $\Rightarrow m^3 - 3m^2 + 4m - 2 = 0$

roots $\rightarrow 1,\ 1 \pm i$

C.F. $\rightarrow c_1 e^x + e^x(c_2 \cos x + c_3 \sin x)$

P.I. $\rightarrow \dfrac{1}{D^3 - 3D^2 + 4D - 2} e^x + \cos x$

$\rightarrow x \dfrac{1}{3D^2 - 6D + 4} e^x + \dfrac{1}{D^3 - 3D^2 + 4D - 2} \cos x$

$\rightarrow \dfrac{xe^x}{3(1)^2 - 6(1) + 4} + \dfrac{1}{D(-1^2) - 3(-1) + 4D - 2} \cos x$

$\rightarrow xe^x + \dfrac{1}{-D + 3 + 4D - 2}$

$\boxed{D^2 \rightarrow -a^2 \quad \cancel{(-1)^x} \quad -(1^2)}$

$\rightarrow xe^x + \dfrac{1}{3D+1} \times \dfrac{3D-1}{3D-1} \cos x$

$\rightarrow xe^x + \dfrac{3D-1}{9D^2 - 1} \cos x$

$= \dfrac{3(-\sin x) - \cos x}{9(-1) - 1} = \dfrac{1}{10}(3\sin x + \cos x)$

T.S. = C.F. + P.I.

Q. $(D^2-4D+4)y = x^3 e^{2x}$

A.E. $\rightarrow$ $m^2-4m+4=0$

$m = 2, 2$

C.F. $= (C_1 + C_2 x)e^{2x}$

P.I. $\dfrac{1}{D^2-4D+4} x^3 e^{2x}$

$= e^{2x} \cdot \dfrac{1}{(D+2)^2 - 4(D+2)+4} x^3$

$= e^{2x} \dfrac{1}{D^2 \cancel{+4} \cancel{+4D} \cancel{-4D} \cancel{-8} \cancel{+4}} x^3$

$= e^{2x} \dfrac{1}{D^2} x^3 = e^{2x} \dfrac{1}{D} \dfrac{x^4}{4} = e^{2x} \cdot \dfrac{x^5}{5.4}$

P.I. $= \boxed{e^{2x} \dfrac{x^5}{20}}$

$y = (C_1 + C_2 x)e^{2x} + e^{2x} \dfrac{x^5}{20}$

Q $\frac{d^2f}{dx^2} - (3-4i)f = 0$

A.E. $\Rightarrow \quad m^2 - (3-4i) = 0$

$$m = \sqrt{3-4i}$$

$$= \sqrt{4+(i)^2-4i} = \pm\sqrt{(2-i)^2}$$

$$m = \pm(2-i)$$

Solution $\rightarrow \quad y = C_1 e^{(2-i)x} + C_2 e^{-(2-i)x}$

HCU2010 $\quad \frac{d^2x}{dt^2} + \omega^2 x = ce^{-\alpha t}$

A.E. $\rightarrow \quad m^2 + \omega^2 = 0 \qquad m = \pm i\omega$

C.F. $\rightarrow \quad A\cos\omega t + B\sin\omega t$

P.I. $\rightarrow \quad \frac{1}{D^2+\omega^2} ce^{-\alpha t} = \frac{ce^{-\alpha t}}{\alpha^2+\omega^2}$

Solution $\quad y = \text{C.F.} + \text{P.I.}$

Legendre's Homo. Diff. Eqn. $\rightarrow$

$$(a+bx)^n \frac{d^ny}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_ny = X$$

$$a+bx = e^z \qquad z = \log(a+bx)$$

$$D(D-1)(D-2)\cdots(D-(n-1))$$

$\Rightarrow$ Solution is in terms of z

$\Rightarrow$ Then substi. z to get final ans.

DU 2014 $\frac{d^3y}{dx^3} + \frac{1}{x}\frac{d^2y}{dx^2} + \frac{1}{x^2}\frac{dy}{dx} = 0$

multiply by x^3

$$x^3\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

$\Rightarrow z = \log x$

$x = e^z$

C.F. →

$$D(D-1)(D-2) + D(D-1) + D = 0$$

$$D^3 - 2D^2 - \cancel{D^2} + 2D + \cancel{D^2} - \cancel{D} + \cancel{D} = 0$$

$$D^3 - 2D^2 + 2D = 0$$

$$D(D^2 - 2D + 2) = 0$$

$$D = 0 \quad D = 1 \pm i$$

C.F. → $C_1 e^{0z} + C_2 e^{(1+i)z} + C_3 e^{(1-i)z}$

Solution $y = C_1 + C_2 x^{(1+i)} + C_3 x^{(1-i)}$

JNU 2012 $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

$\Rightarrow z = \log x$

$x = e^z$

$$(D(D-1) - 2D + 2)y = 0$$

$$D^2 - 3D + 2 = 0$$

$$D = 1, 2$$

$$C.F. = C_1 e^{1.z} + C_2 e^{2.z}$$

$$= C_1 x + C_2 x^2$$

$y = e^{mx}$ $\qquad y = C_1 y_1 + C_2 y_2 + \_$

m $\qquad P.I. = \frac{1}{\boxed{}}\boxed{}$

FR2016

③

$$x\frac{dy}{dx} = y(\ln y - \ln x + 1)$$

$$y = vx \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\cancel{x}\left(v + x\frac{dv}{dx}\right) = v\cancel{x}\left(\ln v + \cancel{\ln x} - \cancel{\ln x} + 1\right)$$

$$\cancel{v} + x\frac{dv}{dx} = v\ln v + \cancel{v}$$

$$x\frac{dv}{dx} = v\ln v$$

$$\frac{dv}{v\ln v} = \frac{dx}{x}$$

$$\ln(\ln v) = \ln x + \ln c$$

$$\ln v = xc$$

$$v = e^{xc}$$

$v = y/x \rightarrow$ $$y = xe^{xc}$$

Boundary Cond. when $x = 1$ $y = 3$

$$3 = 1e^{c}$$

$$c = \ln 3 =$$

$$\Rightarrow \quad y = \cancel{xe^{x}.e^{\ln 3}} \quad xe^{x.\ln 3} = xe^{\ln 3^{x}}$$

$$= x.3^{x}$$

$$\boxed{\cancel{y = 3xe^{x}}}$$

$$\Rightarrow \quad y = x.3^{x}$$

$$\Rightarrow \quad y(3) = 3.3^{(3)}$$

$$= \underline{\underline{81}}$$