Free Study Material from

All Lab Experiments

All Lab Experiments

Mathematical Physics for JAM/NET/Gate Physics

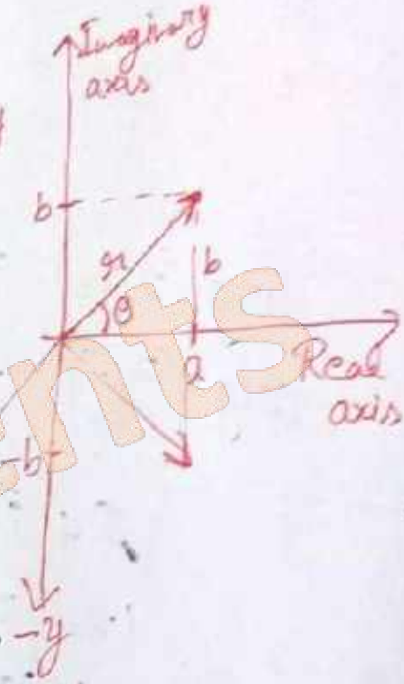
> Complex Numbers <

Complex Numbers

(1) imaginary no. → $\sqrt{-1} = x$ $\quad \boxed{x^2 = -1}$ $\quad \boxed{i \cdot i = -1}$ $\quad -1 \times -1 = 1$

(2) Complex no. → $z = a + ib$

a → Real, ib → imaginary



$\Rightarrow |z| = \sqrt{a^2 + b^2}$ (Magnitude)

$\Rightarrow \bar{z} = a - ib$ (Reflection wrt x-axis)

$\Rightarrow \tan\theta = \dfrac{b}{a} = \dfrac{\cancel{\text{complex}}\ \text{img}}{\text{Real}}$

$\theta = \tan^{-1} b/a$ (arg)

In polar co-ordinates →

$x = a = r\cos\theta$

$y = b = r\sin\theta$

$z = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

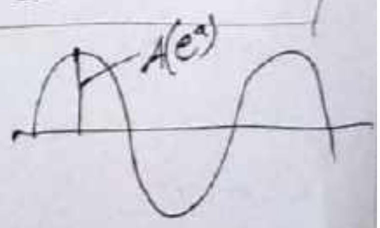
$z = \boxed{r e^{i\theta}}$

$(a+b)(a-b) = a^2 - b^2$

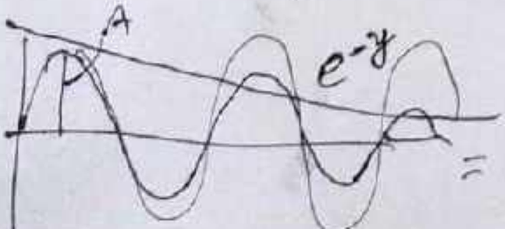
$(a+ib)(a-ib) = a^2 + b^2$

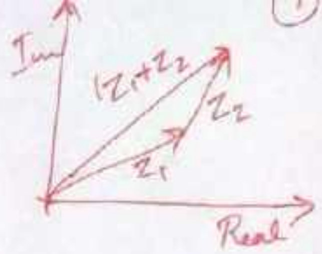
$z\bar{z} = |z|^2$

$e^{a+ib} = e^a \cdot e^{ib} = A\cos\theta$



$= e^a e^{i(x+iy)} = A e^{ix} e^{-y}$





$z \Rightarrow$

$$|z_1+z_2| \le |z_1| + |z_2|$$

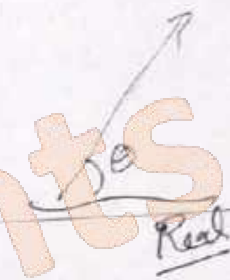
$$|z_1-z_2| \ge |z_1| - |z_2|$$

$$|z_1 z_2| = |z_1||z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$



$$|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$$

$$|z_1+z_2| = |z_1-z_2|$$

Then what is the angle b/w z_1 & z_2

$$|\vec{A}+\vec{B}| = \sqrt{|\vec{A}|^2+|\vec{B}|^2+2|A||B|\cos\theta}$$

$$|z_1+z_2| = \sqrt{|z_1|^2+|z_2|^2+2|z_1||z_2|\cos\theta}$$

Real part $= \frac{1}{2}(z+\bar{z}) = a$

Imaginary part $= \frac{1}{2i}(z-\bar{z}) = b$

$$z = a+ib$$
$$\bar{z} = a-ib$$
$$\overline{2a}$$

$$a+ib$$
$$-a+ib$$
$$\overline{2ib}$$

$$z\bar{z} = |z|^2$$

$$|z| = |\bar{z}|$$

$$\overline{z_1 \pm z_2} = \bar{z}_1 + \bar{z}_2$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

⑤

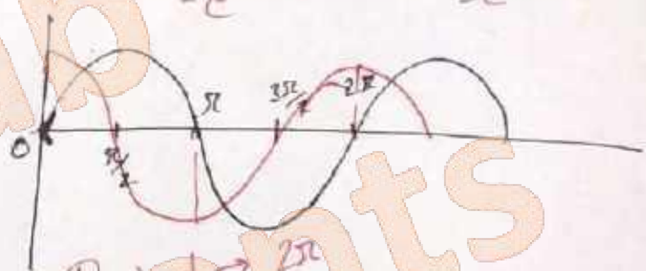
$e^{i\theta} = \cos\theta + i\sin\theta =$

$(e^{i\theta})^n = e^{in\theta} = (\cos\theta + i\sin\theta)^n$

$\longrightarrow (\cos n\theta + i\sin n\theta)$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$= e^{i0} = e^{i(0+2n\pi)}$ $\quad -1 = e^{i\pi} = e^{i(\pi+2n\pi)}$ $\quad -i = e^{i\frac{3\pi}{2}} = e^{i(\frac{3\pi}{2}+2n\pi)}$ $\quad i = e^{i\frac{\pi}{2}} = e^{i(\frac{\pi}{2}+2n\pi)}$



Period → 2π

$\boxed{1^{\frac{1}{2}} = \pm 1}$

Cube roots of Unity →

$y = x^{\frac{1}{3}} = (1)^{\frac{1}{3}}$

$\boxed{y^3 = 1}$ → three roots

$y^3 - 1^3 = 0$

$(y-1)(\qquad) = 0$

$1 = e^{i(0+2n\pi)} = e^{i2n\pi}$

$(1)^{\frac{1}{3}} = e^{\frac{i2n\pi}{3}} = \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$

$n = 0, 1, 2 \rightarrow$

100
101
102

$\underline{1}$, $\underline{-\frac{1}{2} + i\frac{\sqrt{3}}{2}}$, $\underline{-\frac{1}{2} - i\frac{\sqrt{3}}{2}}$

1 $\qquad \omega \qquad \omega^2$

$(1^{\frac{1}{3}})^3 = 1 \qquad \omega^3 = 1 \qquad \omega^6 = 1$

$1 + \omega + \omega^2 = 0$

$(-1)^{\frac{1}{3}} = e^{i\frac{(\pi+2n\pi)}{3}}$

$= \cos \qquad + \quad i\sin$

$n = 0, 1, 2.$

$= \boxed{\qquad}$

$e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow \underline{cis\,\theta}$

$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$(e^{i\theta})^n = e^{in\theta}$

$cis\,\theta_1 \cdot cis\,\theta_2 = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)} = cis(\theta_1+\theta_2)$

$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta \qquad \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$\boxed{z + \frac{1}{z} = 2r\cos\theta} \qquad z - \frac{1}{z} = 2ir\sin\theta$

Q. find the value of x & y such that $-3 + ix^2y$ and x^2+y+4i ~~we~~ are the complex conjugate of each other.

Ans. $-3 + ix^2y = x^2 + y - 4i$

$x^2 + y = -3 \qquad x^2y = -4$

$x^2 = -y - 3 \qquad +y(y+3) = +4$

Home work

Q. JAM2013 What is the value of $\sqrt{i}+\sqrt{-i}$

$\sqrt{i}+\sqrt{-i}\,(\quad)$ $\quad (\sqrt{i}+\sqrt{-i})^2$

$(\quad)$

$= i+(-i)+2\sqrt{i}\sqrt{-i}$

$(\quad)^2 = 2$

$(\quad) = \sqrt{2}$

Q. JAM2015 Phase of $(1+i)i$

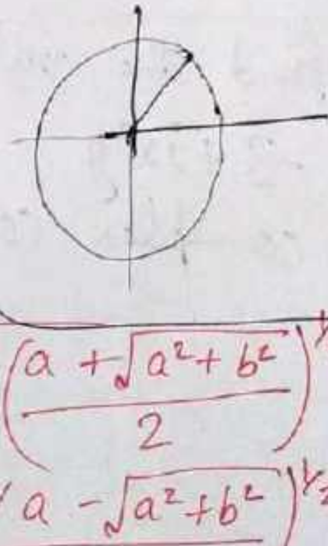
$a+ib \qquad i+i^2 \rightarrow -1+i$

$\tan\theta = \frac{1}{-1} = -1 \qquad 135^\circ \rightarrow \frac{3\pi}{4}$

Q. $|z| = 2$ when $z = x+iy$ then the locus of z represents $\rightarrow$

a) parabola b) hyperbola c) circle d) line

$\sqrt{x^2+y^2} = 2$



Q. Sqr. root of a complex no. $\rightarrow$

$\sqrt{z} \qquad \sqrt{a+ib} = x+iy$

$a+ib = x^2-y^2+2ixy$

$a = x^2-y^2$

$b = 2xy$

$(x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2$

$x^2+y^2 = \sqrt{(a)^2+(b)^2}$

$x = \left(\frac{a+\sqrt{a^2+b^2}}{2}\right)^{1/2}$

$y = \left(\frac{a-\sqrt{a^2+b^2}}{-2}\right)^{1/2}$

find the sqr root of $-15-8i$ =

Q. You have a complex no. z. You multip. it with $(1+i)$. This rotates the vector. what is the angle of rotation.

Ans. $\arg(z_1 \cdot z_2) = (\arg z_1) + (\arg z_2)$

$$\arg[(1+i)z] = \arg(1+i) + \arg(z)$$

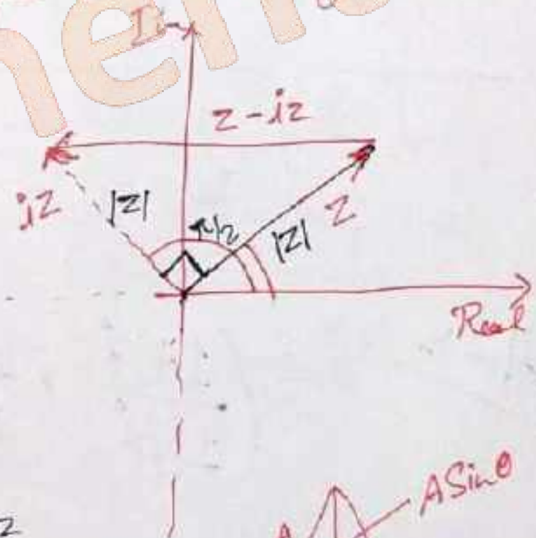
$$= \tan^{-1}\frac{1}{1} + \arg(z)$$

$$\arg((1+i).z) - \arg(z) = 45^\circ \left(\pi/4\right)$$

Q. find the area of the triangle enclosed by z, iz & $z - iz$

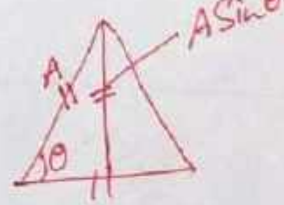
$z = x + iy$ | $z + iz$

$iz = ix - y$



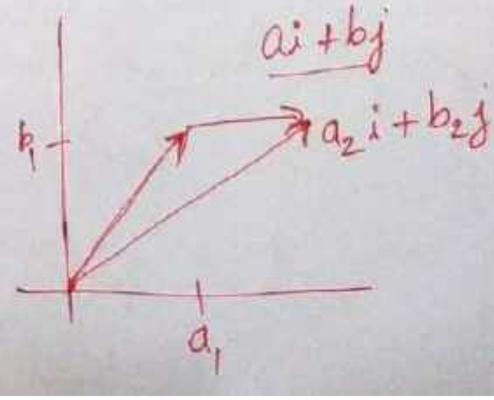
$$\arg(iz) = \arg i + \arg z$$

$$\arg(iz) - \arg(z) = \tan^{-1}\frac{1}{0} = \pi/2$$



$$\text{Area} = \frac{1}{2}|z|\,|z| = \frac{1}{2}|z|^2$$

$$= \frac{1}{2} z\bar{z}$$



Q. $z = (\lambda+3) + i\sqrt{5-\lambda^2}$

$\lambda \to$ is a real no parameter

& find the locus of z will be $\to$

parametric
$\downarrow$
eliminate para.

$z = x + iy$

$x = \lambda + 3 \to \lambda = x - 3$

$y = \sqrt{5-\lambda^2}$

$y^2 = 5 - (x-3)^2$

$(x-3)^2 + y^2 = 5 \to$ circle

$r = \sqrt{5}$

center, $+3, 0$

$3 + i0$

Q. find the smallest positive integer for which $\left(\frac{1+i}{1-i}\right)^n = 1$ is a real no.

~~Ans~~ $\left(\frac{1+i}{1-i} \cdot \frac{1+i}{1+i}\right)^n = (i)^n \quad (i)^2 = -1$

$(i)^4 = 1$

Q. solve $x^5 + i = 0$

$x^5 = -i$

$cis(\ ) = -i$

$\cos\theta + i\sin\theta$

Step 1 $\quad x^5 = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} \qquad \theta = \frac{3\pi}{2}$

Step 2 $\quad x^5 = \cos\left(\frac{\pi}{2} + 2n\pi\right) - i\sin\left(\frac{\pi}{2} + 2n\pi\right) \qquad \cos\theta - i\sin\theta$

$\theta = \pi/2 \qquad n = 0, 1, 2, 3, 4$

Step 3 $\quad x = \cancel{\cos(\pi/2}$

$\left[\cos\frac{1}{5}\left(\frac{\pi}{2} + 2n\pi\right) - i\,\frac{1}{5}\sin\frac{1}{5}\left(\frac{\pi}{2} + 2n\pi\right)\right]$

(5)

Solve → $x^5 = 1+i$

$\cos\theta + i\sin\theta$

Step 0 → $x^5 = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

Step 1 $= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Step 2 $= \sqrt{2}\left(\cos\left(\frac{\pi}{4}+2n\pi\right) + i\sin\left(\frac{\pi}{4}+2n\pi\right)\right)$

Step 3 $x = (2)^{1/10}\left(\cos\frac{1}{5}(\quad) + i\sin\frac{1}{5}(\quad)\right)$

Q. $(1-\sqrt{3}\,i)^{100} = x+iy$

$\left(\frac{1+i}{1+\sqrt{3}i}\right)^{100}$

$\frac{e^{i\theta_1}}{e^{i\theta_2}}$

$2^{100}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{100}$

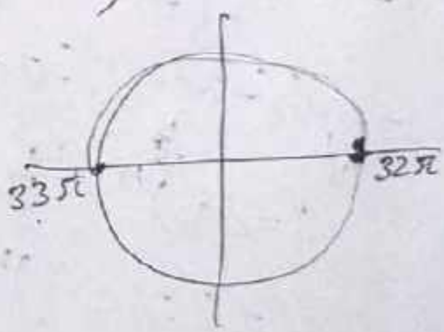
$2^{100}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^{100}$

$2^{100}\left(\cos\frac{100\pi}{3} - i\sin\frac{100\pi}{3}\right)$

$33\frac{1}{3}\pi$

$2^{100}\left(\cos\left(\pi+\frac{\pi}{3}\right) - i\sin\left(\pi+\frac{\pi}{3}\right)\right)$

$2^{100}\left(-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$



$\boxed{2^{100}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}$

$\boxed{-2^{99} + 2^{99}\sqrt{3}\,i}$

⑥

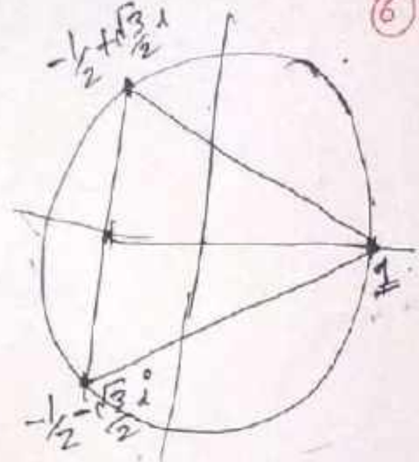
$$2^{100}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{100}$$

$$2^{100}(\omega)^{100} \rightarrow 2^{100}\omega$$

$$2^{100}(\omega)^{99+1}$$

$\omega^3 = 1$

$$\boxed{2^{100}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)}$$



Cube-roots of unity

$$\omega \rightarrow -\frac{1}{2}-\frac{\sqrt{3}}{2}i \leftarrow \omega^2$$

$$\omega^2 \rightarrow -\frac{1}{2}+\frac{\sqrt{3}}{2}i \leftarrow \omega$$

Q. If 1 $\omega \cdot \omega^2$ are cube roots of unity Then find the solution to this series →

$$(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(\ldots\ldots \quad \text{2n terms}$$

$$\boxed{1+\omega+\omega^2=0} \quad \boxed{\omega^3=1}$$

$$\omega^4 = \omega^3 \cdot \omega = \omega$$

$$1-\omega^4+\omega^8$$
$$1-\omega+\omega^2$$

$$= (-\omega-\omega)(-\omega^2-\omega^2)(-\omega-\omega)(-\omega^2-\omega^2)$$

$$= (-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2)\ldots$$

$$= (-2\omega)^n(-2\omega^2)^n$$

$$= (+2\cdot 2\omega^3)^n = 2^{2n}$$

Q. $\left(\frac{1-i}{1+i}\right)^{100} = a+ib$

$$\Rightarrow \left(\frac{e^{-i\pi/4}}{e^{i\pi/4}}\right)^{100} = \left(e^{-i\pi/2}\right)^{100} = e^{-i50\pi} = \cos 50\pi - i\sin 50\pi$$

$$= 1 - i\,\underline{0}$$

$$= \underline{\underline{1}}$$

Q. $|z| = 2|z-1|$ Then the locus of z will make a structure →

$z = x+iy \qquad |z| = \sqrt{x^2+y^2}$

$z-1 = x-1+iy \qquad |z-1| = \sqrt{(x-1)^2+y^2}$

$\sqrt{x^2+y^2} = 2\sqrt{(x-1)^2+y^2} \longrightarrow$ a circle

Home work

$\left(\frac{4}{3}, 0\right) \longrightarrow$ center

$\frac{2}{3} \longrightarrow$ radius

Q. $\arg(z+1) = \frac{\pi}{6}$ & $\arg(z-1) = \frac{2\pi}{3}$

find z

$z+1 = x+1+iy \qquad \arg(z+1) = \tan^{-1}\frac{y}{x+1} = \frac{\pi}{6}$

$z-1 = x-1+iy \qquad \arg(z-1) = \tan^{-1}\frac{y}{x-1} = \frac{2\pi}{3}$

$x \to \frac{1}{2}$

$y \to \frac{\sqrt{3}}{2}$

Home work

$e^x \quad e^{i\theta} = \cos\theta + i\sin\theta$ ——— (1) (2)

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\cdots$$

$x \longrightarrow i\theta$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{\lfloor 2} - \frac{i\theta^3}{\lfloor 3} + \frac{\theta^4}{\lfloor 4} + \cdots$$

$$= \left(1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} \cdots\right) + i\left(\theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} \cdots\right)$$

(1) $\Downarrow$ $\cos\theta$ (2) $\Downarrow$ $\sin\theta$

$\cos\frac{4M}{\pi}$ where $M = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

$M \rightarrow$ E-values

$\rightarrow$ E-vectors

$\rightarrow$ Construct P Matrix

$D \rightarrow P^{-1}MP$

$M \rightarrow PDP^{-1}$

$\cos\frac{4M}{\pi} = P\cos D\,P^{-1}$

If D is a diagonal Matrix then Sin D or Cos D are also diagonal Matrix

$\sin\begin{pmatrix} \pi/4 & 0 \\ \end{pmatrix} = \begin{pmatrix} \sin\pi/4 & 0 \\ \end{pmatrix}$

$D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad D^2 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$

$$\sin D = D - \frac{D^3}{\lfloor 3} + \frac{D^5}{\lfloor 5} - \frac{D^7}{\lfloor 7}$$

$$= \begin{bmatrix} a_{11} - \frac{a_{11}^3}{\lfloor 3} + \frac{a_{11}^5}{\lfloor 5} & 0 \\ 0 & a_{22} - \frac{a_{22}^3}{\lfloor 3} + \frac{a_{22}^5}{\lfloor 5} \end{bmatrix} \cdots$$

$$= \begin{bmatrix} \sin a_{11} & 0 \\ 0 & \sin a_{22} \end{bmatrix}$$

Q. $\log z = \log(x+iy) = a+ib$

Log of complex no

$x \to r\cos\theta \;,\; y \to r\sin\theta$

$z = re^{i\theta}$

$\log z = \log re^{i\theta} = \log r + \log e^{i\theta}$

$= \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$

Q. $i^i = e^{\log i^i} = e^{\log i^i} = e^{i\log i}$

$i\log i$

Home work

Q. $i^{i^{\cdot^{\cdot^{\infty}}}} = a+ib \longrightarrow i^{a+ib} = a+ib$

Home work

Hyperbolic functions → Hyperbolic

$2\,\mathrm{Real} = z + \frac{1}{z}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$$

$$\tanh x = \frac{e^{+x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\sin ix = i\sinh x$$
$$\cos ix = \cosh x$$
$$\tan ix = i\tanh x$$

$$\sinh ix = i\sin x$$
$$\cosh ix = \cos x$$
$$\tanh ix = i\tan x$$

$$\sin^2 x + \cos^2 x = 1$$
$$x \to ix$$
$$(i)^2\sinh^2 x + \cosh^2 x = 1$$
$$\cosh^2 x - \sinh^2 x$$

$$\frac{d\sinh x}{dx} = \cosh x$$

$$\frac{d\cosh x}{dx} = \sinh x$$

$$\sin(x+iy)$$
$$\sin x\cos iy + \cos x\sin iy$$
$$\sin x\cosh y + i\cos x\sinh y$$

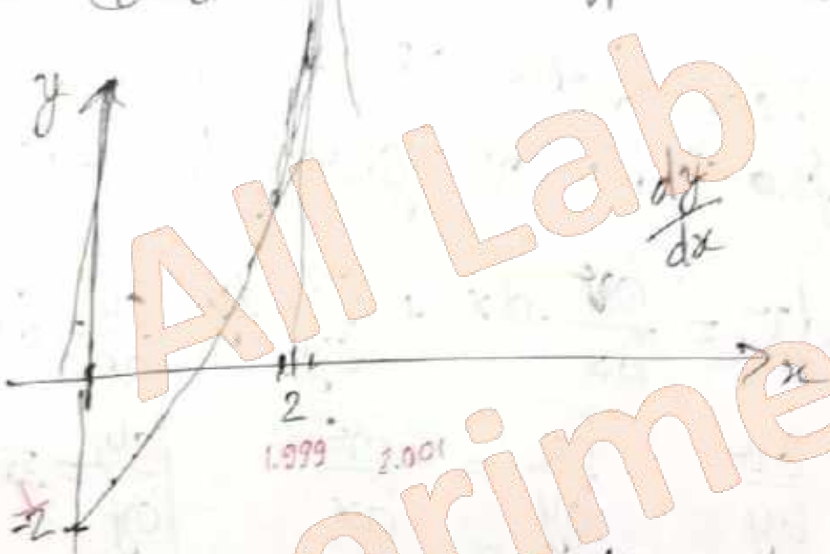
$$e^{\theta} = \cosh\theta + \sinh\theta$$
$$e^{-\theta} = \cosh\theta - \sinh\theta$$
$$e^{\pm n\theta} = \cosh n\theta \pm \sinh n\theta$$

→ functions of Complex Variable →

$z = x + iy \qquad f(z) = z^2 + 2z + 5$

$\qquad\qquad\qquad\qquad = u(x,y) + i\,v(x,y)$

→ Continuous & Differentiable



$\dfrac{1}{x-2}$

$\dfrac{dy}{dx}$

An Analytical function → A function $f(z)$ is said to be analytical at a point z_0, if it is differentiable in the neighbourhood of z_0.

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h,) - f(x)}{h}$$

C-R equations (Cauchy-Reim. Eqn.)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Q. Let a function $f(z)$ is analytical - its real part is $u = 3x^2y - y^3$ Then find out its imaginary part (v).

Method 1 → Use CR equns.

Method 2 → Check both CR equns. from the v given in answers.

Method 3 → $dv = \frac{\partial v}{\partial x}.dx + \frac{\partial v}{\partial y}dy$

$\frac{\partial u}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = 3x^2 - 3y^2$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\quad$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$dv = \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy$$

$$v = \int \boxed{(3y^2 - 3x^2)dx + 6xy\,dy}$$

Home work

$$v = -x^3 + 3xy^2 + C$$

CR eqns in polar form →

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial v}{r\partial \theta} \quad \Rightarrow \frac{\partial u}{r\partial \theta} = -\frac{\partial v}{\partial r}$$

$$\Rightarrow dv = \left(\frac{\partial v}{\partial r}\right)dr + \frac{\partial v}{\partial \theta}.d\theta$$

Q ⇒ If $f(z)$ is an analytical fun.
& $u = -r^3 \sin 3\theta$ then find its imaginary.

Home work

$i\, r^3(\cos 3\theta + i\sin 3\theta) + iC$
$i r^3 (e^{3\theta}) + iC$

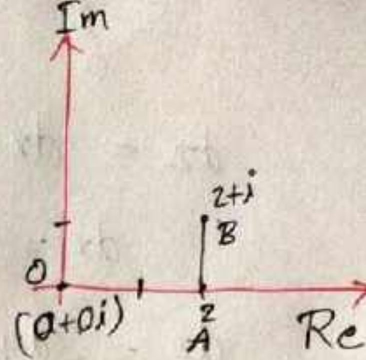
A Complex function is Harmonic

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Big| \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Complex Integration →

(i) Line integration →

$$\int_0^{2+i} (\bar{z})^2 \, dz \qquad \text{Path } O-A-B.$$



$z = x + iy \qquad \bar{z} = x - iy$

$dz = dx + i\,dy$

$(\bar{z})^2 = (x - iy)^2 = x^2 + (iy)^2 - 2xiy$

$$= \int_0^{2+i} (x^2 - y^2 - 2ixy)(dx + i\,dy)$$

$OA \rightarrow \; y = 0, \; dy = 0 \qquad AB \rightarrow \; x = 2, \; dx = 0$

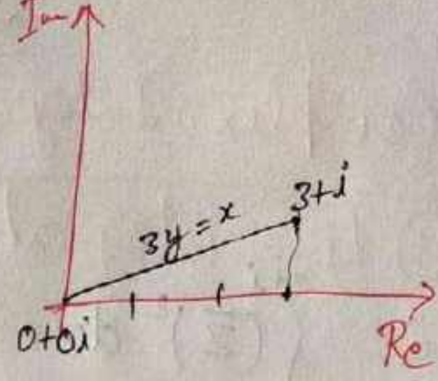
$$= \int_{0}^{2} x^2 \, dx + i \int_{0}^{1} (4 - y^2 - 4iy)\, dy$$

OA   AB

Home work

$$= \frac{1}{3}(14 + 11i)$$

IIT JAM 2019 $\int_0^{3+i} (\bar{z})^2 dz$ along $3y = x$



$dz = dx + i dy$

$dx = 3dy$

$dz = 3dy + i dy = (3+i) dy$

$$\int_0^1 (\quad\quad) (3+i) dy$$

Home work

111.1

Q. Evaluate $\int_c (z-a)^n dz$; where c is a circle of radius r, centered at $(a, 0)$

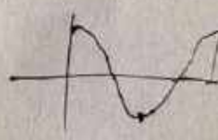
Ans. Circle radius r & center $(a, 0)$

then equ. $|z - a| = r$ ⟸ Imp Step

$z - a = re^{i\theta}$ ⟶ $|x - a + iy| = r$

$dz = i r e^{i\theta} d\theta$ $\quad (x-a)^2 + y^2 = r^2$

$$\int_0^{2\pi} (re^{i\theta})^n (ire^{i\theta}) d\theta = [i r^{n+1}] \int_0^{2\pi} e^{i(n+1)\theta} d\theta$$



$$= [i r^{n+1}] \left[\frac{e^{i(n+1)\theta}}{i(n+1)} \right]_0^{2\pi} = [i r^{n+1}] [e^{i2\pi} - 1]$$

$$(z-a)^n dz = \frac{r^{n+1}}{(n+1)}\left[1 + ai - 1\right]$$

This gives you 0 if $n \neq -1$

If $n = -1$

$$\oint_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}\, d\theta}{re^{i\theta}} = i\int_0^{2\pi} d\theta = \underline{\underline{i2\pi}}$$

Here comes the <u>singularity</u>. (Singular points)

Singular points

- Isolated [$f(z)$ is non-analytical at $z=a$ & there is no other singular point in its neighbourhood. $\boxed{\frac{1}{(z-2)(z-3)}}$
- Non-Isolated [$f(z)$ is non-analytical at $z=a$ & there are singular points in its neighbourhood $\boxed{\frac{1}{\sin \frac{\pi}{z}}}$

Rough Work

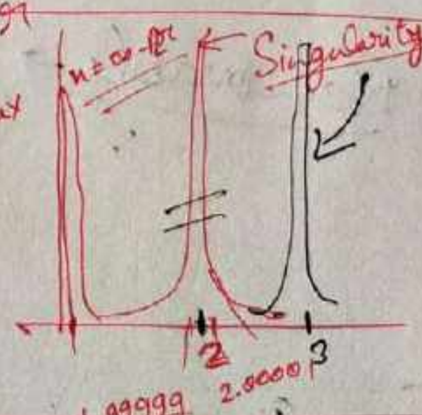
Isolated

$f(z)$ $\frac{1}{(z-2)(z-2.00001)}$ $|z-x| = r$ ↓ complex

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{h\to 0} \left[f(a+h) - f(a)\right] < \varepsilon$

$\varepsilon \to$



$\frac{1}{\sin \frac{\pi}{z}}$ $\sin\frac{\pi}{z} = \sin n\pi$

$\frac{1}{\sin 10^{8000}\pi}$ $\frac{1}{z} = n$ $z = \frac{1}{n}$ | $n = \pm 1$ $n = 0$ --- $n = \infty$; $z = \pm 1$ $z = \infty$ --- $z = 0$

Isolated Singular points

- Essential $\left[e^{\frac{1}{z-a}}\right]$
- Non essential $\left[\frac{1}{(z-a)(z-b)^2}\right]$
- Removable $\left[\frac{\sin z}{z}\right]$

essential →

$$e^{\frac{1}{z-a}} = 1 + \frac{1}{(z-a)} + \frac{1}{(z-a)^2 \lfloor 2} + \cdots \frac{1}{(z-a)^n}$$

non-essential →

$$\frac{1}{(z-a)(z-b)^2}$$

$$\frac{\square}{(z-a)^1} + \square + \cdots$$

$$\frac{\square}{(z-b)^2} + \frac{\square}{(z-b)^1} + \square + \cdots$$

$a \longrightarrow$ order $\longrightarrow 1$

$b \longrightarrow$ order $\longrightarrow 2$

$$e^{i\theta} = (\cos\theta + i\sin\theta)$$

Removable →

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} - \cdots\right)$$

$$= \left(1 - \frac{z^2}{\lfloor 3} + \frac{z^5}{\lfloor 5} - \cdots\right)$$

$$\frac{\sin z}{z^2} = \frac{1}{z^2}\left(z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} + \cdots\right]$$

$$= \frac{1}{z} - \frac{z}{\lfloor 3} + \frac{z^5}{\lfloor 5} + \cdots$$

Laurent Series →

A function of complex var. $f(z)$ can be explained in terms, ~~around~~ $z=a$ point →

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \text{-----}$$
$$+ \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \text{------}$$

$$f(z) = \sum_{n=0}^{n=\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

If $f(z)$ is not analytic only then these negative powers will exist.

$n \to \infty \longrightarrow$ singularity c essential

$n \to$ finit $\longrightarrow$ non-essential

$n \to 0 \longrightarrow$ removable

$$e^{z-a} = 1 + (z-a) + \frac{(z-a)^2}{\underline{|2}}$$

⇒ Example to express in terms of Laurent Series → around point 1 write Laurent Series.

$$f(z) = \frac{1}{(z-1)(z-2)}$$

$z = 1 \qquad z-1 = t$

$$f(z) = \frac{1}{t(t-1)} = -\frac{1}{t(1-t)} = -\frac{1}{t}(1-t)^{-1}$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

$$f(z) = -\frac{1}{t}(1 + t + t^2 + t^3 + \cdots)$$

$$= -\left(\frac{1}{t} + 1 + t + t^2 + \cdots\right)$$

$$= -\left(\frac{1}{(z-1)} + 1 + (z-1) + (z-1)^2 + \cdots\right)$$

pole at 1 is of order 1 ⟸

write this around point 2

$z - 2 = t$

$$f(z) = \frac{1}{(t+1)t} = \frac{1}{t}(1+t)^{-1}$$

$$(1+x)^{-1} = (1 - x + x^2 - x^3 + \cdots)$$

$$= \left(\frac{1}{t} - 1 + t - t^2 + \cdots\right)$$

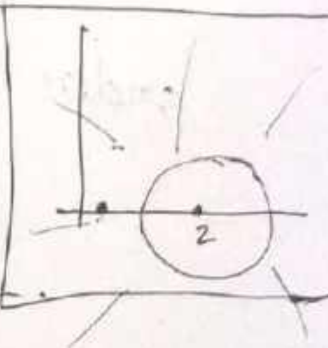
$$= \frac{1}{(z-2)} - 1 + (z-2) - (z-2)^2 + \cdots$$

pole at 2 is of order 1 ⟸

In case, $f(z) = \frac{1}{(z-1)(z-2)^2}$

pole at 1 order = 1

pole at 2 order = 2



Q. Discuss the order of singularity in

$$f(z) = \frac{\sin z}{(z-\pi)^2}$$

$$\frac{\sin z}{z^2} = \frac{1}{z^2}\left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5}\right)$$

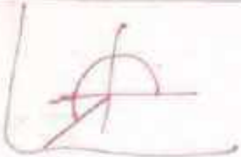
$$= \frac{1}{z} - \frac{z}{\lfloor 3} + \frac{z^3}{\lfloor 5}$$

order = 1

Ans. (1) $z = \pi$ is singular point.

(2) $z - \pi = t \qquad z = (t+\pi)$

$$f(t) = \frac{\sin(t+\pi)}{t^2} = -\frac{\sin(t)}{t^2}$$



(3) $-\frac{1}{t^2}\left(\qquad \right) = \frac{1}{t} + 1 + \cdots$

order of singularity is 1.

Q. $f(z) = \frac{\cos z}{(z-\pi)^2}$ now find the order of singularity.

(1) $z = \pi$ is singular point

(2) $\frac{\cos(\pi + t)}{t^2} = -\frac{\cos t}{t^2}$

(3) $-\frac{1}{t^2}\left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \cdots\cdots\right) \; x \to t$

$= -\frac{1}{t^2} + \frac{1}{\lfloor 2} - \frac{t^2}{\lfloor 4} - \cdots\cdots$

order of singularity is 2.

Residue at a singular point → Residue of a complex function $f(z)$ at a singular point is the measurement of singularity.

Laurent series → Cofficient of $\frac{1}{(z-a)}$ (b_1) is residue

Case 1 → Singularity is of order 1.

$$\text{Res } f(z=a) = \lim_{z\to a} (z-a)\cdot f(z)$$

e.g. $\frac{z^2+2}{(z-1)(z-3)}$

$\Rightarrow$ $z=1, 3$ are singular points of order 1.

$$\text{Res. } f(z=1) = \frac{3}{-2} = \lim_{z\to 1} (z-1)\left(\frac{z^2+2}{\cancel{(z-1)}(z-3)}\right)$$

$$\text{Res } f(z=3) = \frac{11}{2} =$$

Case 2 → Singularity is of order > 1

$$\text{Res } f(z=a) = \frac{1}{\lfloor n-1} \frac{d^{n-1}}{dz^{n-1}}\left((z-a)^n f(z)\right)$$
order n

$$f(z) = (z-a) + 5 + \frac{5}{(z-a)} + \frac{2}{(z-a)^2} + \cdots$$

$(z-a) + (z-a)^2 + 0$

$= (z-a)^n ($

$$= \frac{d^{n-1}}{dz^{n-1}} 5 (z-a)^{n-1} \qquad \frac{(n-1)(n-2)(n-3)\cdots 1}{(n-1)}$$

$f(z) = \boxed{\dfrac{z^2}{(z+1)^2(z-2)}}$ find singularity order & residue

① singularity $\rightarrow -1 \; , \; 2$

② Order $\rightarrow 2 \; , \; 1$

③ $\text{Res}\,[f(z=2)] = \dfrac{4}{9}$

$\text{Res}\,[f(z=-1)] = \dfrac{1}{\lfloor 2-1} \dfrac{d}{dz}\left(\dfrac{z^2}{(z-2)}\right)$

$= \left[\dfrac{(z-2)2z - z^2}{(z-2)^2}\right]_{z=-1} =$

$\oint_C \bigcirc = 2\pi i\left[\quad\right]$

$= \cancel{-}\dfrac{5}{9}$

Q. $z^2 e^{1/z}$ find its residue

$\Rightarrow z^2\left(1 + \dfrac{1}{z} + \dfrac{1}{z^2\lfloor 2} + \dfrac{1}{z^3\lfloor 3} + \text{-----}\right)$

$\Rightarrow z^2 + z + \dfrac{1}{\lfloor 2} + \dfrac{1}{z\lfloor 3} + \dfrac{1}{z^2\lfloor 4} \text{——}$

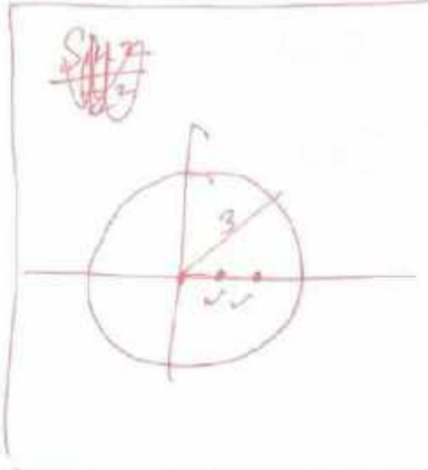
$\text{Res}\,\cancel{f}(\,) = \text{Coef. of } \dfrac{1}{z} = \dfrac{1}{\lfloor 3} = \dfrac{1}{6}$

$\oint z^2 e^{1/z} = 2\pi i\left[\dfrac{1}{6}\right]$

$\dfrac{1}{\lfloor 3} \cdot \dfrac{1}{z} \qquad \dfrac{b_1}{z}$

$(1/z) \times \dfrac{1}{\lfloor 3} \qquad \dfrac{(1/\lfloor 3)}{z}$

Q. Solve $\oint_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)}$ where $C: |z| = 3$



① $\Rightarrow$ Singular points $1, 2$

② $\Rightarrow$ order $1, 1$

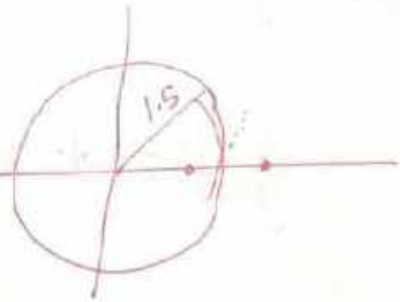
③ $\Rightarrow$ Residue

$$\text{Res}(z=1) = \frac{\sin\pi + \cos\pi}{-1} = \frac{-1}{-1} = 1$$

$$\text{Res}(z=2) = \frac{\sin 4\pi + \cos 4\pi}{1} = \frac{1}{1} = 1$$

$$\oint \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} = 2\pi i[1+1] = 4\pi i$$

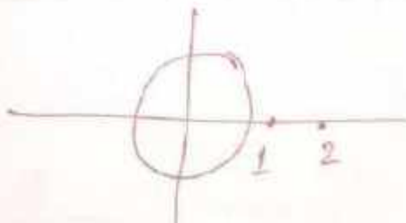
If $C: |z| = 1.5$



$$\oint \boxed{} = 2\pi i[1]$$

$$= 2\pi i$$

If $C: |z| = 0.8$



$$\oint \boxed{} = 0$$

5 June, 2020

The Protocols →

1. find out the singularity
2. find the order of singularity
3. find out the residue

order 1 Simple pole	$\lim_{z \to a} (z-a) f(z)$
order > 1 order = m	$\frac{1}{\underline{\lvert n-1}} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]$
for all	Laurent Series the coefficient of $\frac{1}{z-a}$ term (b_1)
$f(z) = \frac{\phi(a)}{\psi(a)}$	Res is $\frac{\phi(a)}{\psi'(a)}$

$f(z) = \cot(z) = \frac{\cos z}{\sin z}$

(1) $\rightarrow$ $\sin z = \sin n\pi$

$z = n\pi \quad n \rightarrow 0, \pm 1, \pm 2$

$\text{Res}\left[f(z)\right]_{z=n\pi} = \frac{\cos z}{\cos z} = 1 \Big|_{z=n\pi} = 1$

$\oint_c \cot(z)$

$= 2\pi i\left[\text{Sum of res.}\right]$

$C: |z| = 1.5$

$z = 0, \pm 3.14, \pm 6.28$

$= 2\pi i[1]$

$\text{Res}\left[\frac{\sin z}{z}\right] = 0$

$\cancel{\text{Res}}\ \frac{\sin z}{1}\Big|_{z=0} = 0$

$f(z) = \frac{z}{\sin z}$

$\Rightarrow \sin z = \sin n\pi$

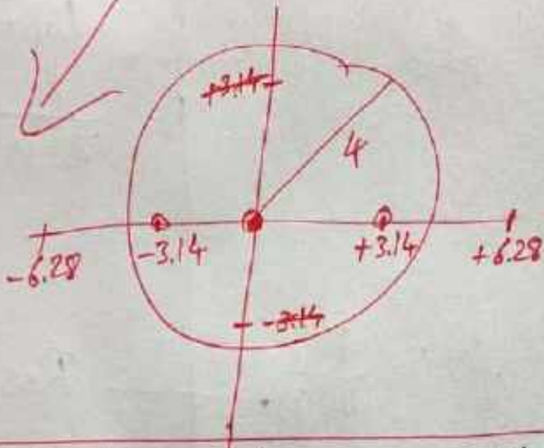
$z = n\pi \quad n \rightarrow 0, \pm 1, \pm 2$

$\text{Res}\ f(z)\Big|_{z=n\pi} = \frac{z}{\cos z}\Big|_{z=n\pi} = \frac{n\pi}{\cos n\pi} = \frac{n\pi}{(-1)^n}$

$C: |z| = 4$

$z = 0, \pm 3.14$

$\oint \cot(z) = 2\pi i[1 + 1 + 1]$



$C: |z| = 1.5$

$z = 0$

$\oint \frac{z}{\sin z} = 2\pi i[0]$

$= 0$

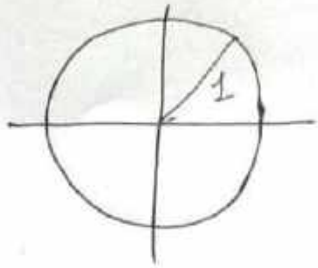
$C: |z| = 4$

$z = 0, \pm 3.14 \nearrow n = 0, \pm 1$

$\oint \frac{z}{\sin z} = 2\pi i\left[0 + (-\pi) + \pi\right]$

$= 2\pi i[0] = 0$

Evaluate $\displaystyle\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$



Consideration $\rightarrow$ $C: |z| = 1$

$z = re^{i\theta} \Rightarrow \underline{z = e^{i\theta}}$ $\Leftarrow$

$dz = ie^{i\theta} d\theta$

$d\theta = \dfrac{dz}{iz}$ $\Leftarrow$

$2\cos\theta = z + \dfrac{1}{z}$

$\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$

$2i\sin\theta = z - \dfrac{1}{z}$

$\sin\theta = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)$

$\Rightarrow \displaystyle\int_0^{2\pi} \frac{\sin^2\theta - 2\cos\theta}{2 + \cos\theta}$ $\rightarrow$ Change it to linear term

$\boxed{\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}}$

$$\int_0^{2\pi} \frac{\frac{1-\cos 2\theta}{2} - 2\cos\theta}{2 + \cos\theta}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1 - \cos 2\theta - 4\cos\theta}{2 + \cos\theta}\, d\theta$$

$\boxed{\begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ Re[e^{i\theta}] = \cos\theta \\ Im[e^{i\theta}] = \sin\theta \end{array}}$

$\Rightarrow$ R.P. of $\displaystyle\frac{1}{2}\int_0^{2\pi} \frac{1 - e^{i2\theta} - 4e^{i\theta}}{2 + \cos\theta}\, d\theta$

R.P. of $\displaystyle\frac{1}{2}\int_0^{2\pi} \frac{1 - z^2 - 4z}{2 + \frac{1}{2}\left(z + \frac{1}{z}\right)}\, d\theta$ $\Downarrow$

$Re[z_1 + z_2)$
$= Re[z_1] + Re[z_2]$

$Re[(z_1 + z_2)^2]$

$Re\left[\dfrac{1}{z_1} + \dfrac{1}{z_2}\right) = \alpha$

R.P. of $\displaystyle\oint_C \frac{1 - z^2 - 4z}{z^2 + 4z + 1}\, \frac{dz}{i}$

1 → $z^2+4z+1=0$

Singular points

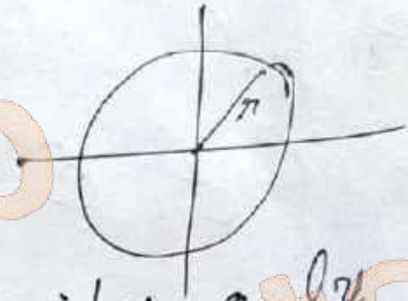
$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=-2\pm\sqrt{3}$$

2 order = 1

$$z^2+4z+1=\left(z-(-2+\sqrt{3})\right)\left(z-(-2-\sqrt{3})\right)$$

$-2+1.732=-0.268$

$-2-1.732=-3.732$



This point will come in this circle.
So we will look at residue only because of this root.

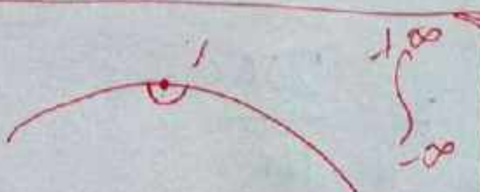
3

Residue at $(-2+\sqrt{3})$ $\lim\limits_{z\to(-2+\sqrt{3})} \left[\cancel{z-(-2+\sqrt{3})}\right]\left[\dfrac{1-z^2-4z}{\left[\cancel{z-(-2+\sqrt{3})}\right]\left[z-(-2-\sqrt{3})\right]}\right]$

Residue at $(-2+\sqrt{3})$ $=\dfrac{1}{\sqrt{3}}$

R.P. of $\dfrac{1}{i}\oint\boxed{}=\dfrac{1}{i}\left[2\pi i\times\dfrac{1}{\sqrt{3}}\right]$

R.P. of $\dfrac{2\pi}{\sqrt{3}}=\dfrac{2\pi}{\sqrt{3}}$



Limiting case → $2\pi i$ πi

Q. $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta$

R.P. of $\int_0^{2\pi} e^{\cos\theta} \cdot e^{i(\sin\theta - n\theta)}\, d\theta$

R.P. of $\int_0^{2\pi} \underline{e^{\cos\theta} \cdot e^{i\sin\theta}} \cdot e^{-in\theta}\, d\theta$

R.P. $\int_0^{2\pi} e^{z} \cdot e^{-n \ln z}\, d\theta$

R.P. $\int_0^{2\pi} e^{z} z^{-n}\, d\theta = $ R.P. $\oint_{c:|z|=1} \frac{e^z}{z^n} \frac{dz}{iz}$

R.P. $\frac{1}{i} \oint \frac{e^z}{z^{n+1}}\, dz$

$\boxed{z = e^{i\theta}}$
$= \cos\theta + i\sin\theta$

$z = e^{i\theta}$
$\ln z = i\theta$

(1) Singularity $\quad z = 0$

(2) order $\quad n+1$

(3) Residue $\quad \frac{1}{\boxed{(n+1)-1}} \frac{d^n}{dz^n} (z^{n+1}) \frac{e^z}{z^{n+1}}$

$$= \frac{1}{\lfloor n} \cdot e^z \Big|_{z=0} = \frac{1}{\lfloor n}$$

R.P. of $\frac{1}{i} \oint \frac{e^z}{z^{n+1}}\, dz = $ R.P. of $\frac{1}{\cancel{i}}\left[2\pi \cancel{i} \times \frac{1}{\lfloor n}\right]$

R.P. of $\frac{2\pi}{\lfloor n} = \frac{2\pi}{\lfloor n}$

6 June, 2020

Evaluation of real improper integral using complex Integration

$$\int_{-\infty}^{+\infty} f(x)\,dx$$

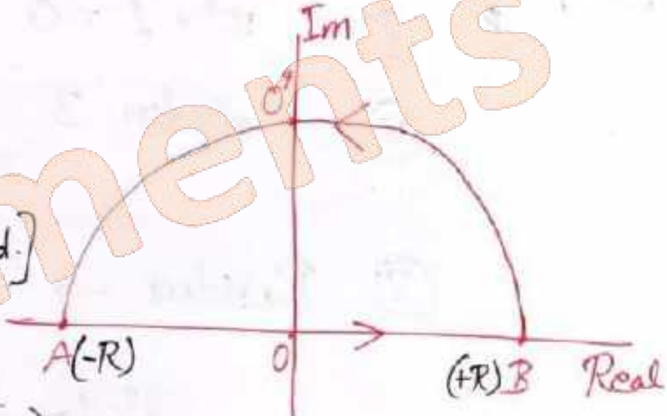
Improper Int.

Real function

Real variable

Consideration →

$$f(x) \longrightarrow f(z)$$



$$\oint_C f(z)\,dz = 2\pi i\,[\text{Sum of Resid.}]$$

$$\int_{AOB} f(z)\,dz + \int_{BO'A} f(z)\,dz = \oint_C f(z)\,dz$$

$$\Downarrow$$

$$\int_{-R}^{+R} f(z)\,dz = \oint_C f(z)\,dz - \int_{BO'A} f(z)\,dz$$

$R \to \infty$

$$\boxed{\int_{-\infty}^{+\infty} f(z)\,dz} = \boxed{\oint_C f(z)\,dz} - \boxed{\int_{BO'A} f(z)\,dz}$$

$\Downarrow$

① ② ③ Residue

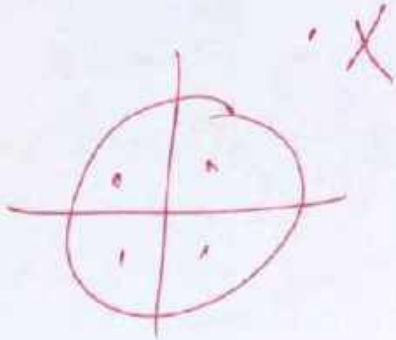
$2\pi i\,[\text{residues}]$

(1) → Singularity

(2) → Order

(3) → Residue

(4) → $2\pi i$ [Sum of Res.] include in ~~surfa~~ curve C

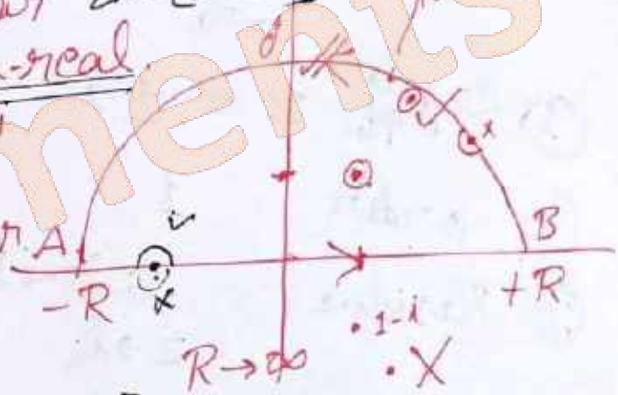


Case 1 → when singular points are non-real $2\pi i$ [Sum]

$\underline{1+i}$, i, $100+100i$

Case 2 → when singular point is real

πi [Sum of residue]



$1+i$ ✓

$1-i$ ×

→ only include points in upper half.

Mix Singular points → 2, $1 \pm i$

$$\oint_C f(z) = 2\pi i\left[\text{Res}\,(1+i)\right] + \pi i\left[\text{Res}\,(2)\right]$$

③

Q. $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^3}\,dx$

$f(x) \to f(z)$

Step 1 $\to$ $\displaystyle\int_{-\infty}^{+\infty} \frac{dz}{(1+z^2)^3}\,dz = \oint_C \frac{dz}{(1+z^2)^3}\,dz - \int_{BO'A} f(z)\,dz$

Step 2 $\to$

① $z^2+1=0$ $\quad z=\pm i$ [Singular points]

② order 3 $\quad \dfrac{dz}{(z-i)^3(z+i)^3}$

③ Residue $\to$ $+i$ $\quad$ $-i$ X

$$\operatorname{Res}(i) = \frac{1}{\lfloor 2}\,\frac{d^2}{dz^2}\,\cancel{(z-i)^3}\,\frac{1}{\cancel{(z-i)^3}(z+i)^3} = \frac{3}{16i}$$

$$\int_{-\infty}^{+\infty} \boxed{}\,dz = \oint 2\pi i\left[\frac{3}{16i}\right] - \int_{BO'A} f(z)\,dz$$

$$= \frac{3\pi}{8} - i\pi\left[\lim_{z\to\infty} z f(z)\right]$$

$$i\pi \lim_{z\to\infty} z\,\frac{1}{z^6\left(1+\frac{1}{z^2}\right)^3} = 0$$

$$= \frac{3}{8}\pi$$

Q $\displaystyle\int_0^{\infty} \frac{\cos mx}{x^2+1}\,dx =$ (4)

$x \to -x$

$f(x) = f(-x)$ [Even func.]

$$\int_{-\infty}^{\infty} \boxed{}\, dx = 2\int_0^{\infty} \frac{\cos mx}{x^2+1}\,dx$$

$$= \text{R.P. of} \int_{-\infty}^{+\infty} \frac{e^{imz}}{z^2+1}\,dz$$

(1) Singular pt. $z = \pm i$

(2) order 1

(3) Residue $\displaystyle\lim_{z\to i} \cancel{(z-i)} \frac{e^{imz}}{(z+i)\cancel{(z-i)}} = \frac{e^{-m}}{2i}$

$$\oint_C \boxed{}\, dz = 2\pi i \left[\frac{e^{-m}}{2i}\right] = \pi e^{-m}$$

$$i\pi \lim_{z\to\infty} z \frac{\cos mz}{z^2\left(1+\frac{1}{z^2}\right)} = 0$$

$$\int_{-\infty}^{+\infty} \boxed{}\, dz = \pi e^{-m}$$

$$\boxed{\int_0^{\infty} \boxed{}\, dz = \frac{1}{2}\pi e^{-m}}$$

Jest 2018

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+1}\,dx$$

$$= \frac{\pi}{e}$$

Jest 2012 $\int_0^\infty \frac{\ln x}{(x^2+1)^2}\,dx$

$$\int_{-\infty}^{+\infty} \frac{\ln x}{(x^2+1)^2}\,dx = \int_{-\infty}^{0} \frac{\ln x}{(x^2+1)^2}\,dx + \int_{0}^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx$$

$x \to -x$

$$\int_{\infty}^{0} \frac{\ln(-x)}{(x^2+1)^2}(-dx)$$

$\Downarrow$

$$+\int_{0}^{\infty} \frac{\ln(-x)}{(x^2+1)^2}(+dx)$$

$$\int_{0}^{\infty} \frac{\ln(e^{i\pi}x)}{(x^2+1)^2}\,dx = \int_{0}^{\infty} \frac{\ln e^{i\pi}}{(x^2+1)^2} + \int_{0}^{\infty} \frac{\ln(x)}{(x^2+1)^2}$$

$$\int_{-\infty}^{+\infty} \boxed{}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{i\pi}{(x^2+1)^2} + 2\int_{0}^{\infty} \frac{\ln x}{(x^2+1)^2}$$

πi

$\frac{1}{z(z+1)^2}$

$(z=0)$ $(z=-1)$

$-i\pi$

$|z+1|=1$

Jest 2016 $\int_0^\infty \frac{\ln x}{(x^2+1)}\,dx$

$\boxed{-\frac{\pi}{4}}$

$-0, -1$

Jest 2017 → $\displaystyle\int_1^\infty \frac{\sqrt{x-1}}{(1+x)^2}\,dx$ (6)

Subst. $x-1 = z^2 \Rightarrow x = z^2+1$

$dx = 2z\,dz$

$$\int_0^\infty \frac{z}{(z^2+2)^2}\,\cancel{dx}\,2z\,dz$$

$$= \int_0^\infty \frac{2z^2}{(z^2+2)^2}\,dz$$

$\text{Res}\,i = \frac{i}{2\sqrt{2}}$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \boxed{}\,dz$$

$\frac{\pi}{2\sqrt{2}}$