Free Study Material from All Lab Experiments

Mathematical Physics for JAM/NET/Gate Physics

> Matrix <

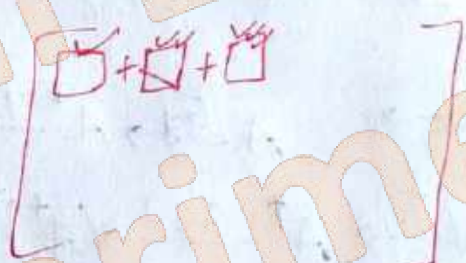
Matrix → An arrangement of numbers in rows & columns.

$m \to$ no. of rows

$n \to$ no. of columns

Matrix multiplication →

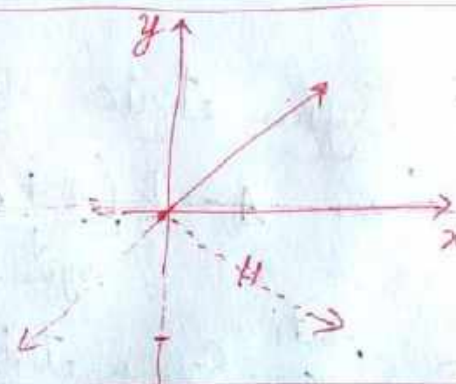
Condition
$n = p$



AB ≠ BA
Matrix multi doesnot commute

Indenty matrix

$$\begin{bmatrix}x\\y\end{bmatrix} \to \begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$



Reflection matrix (X → mirror)

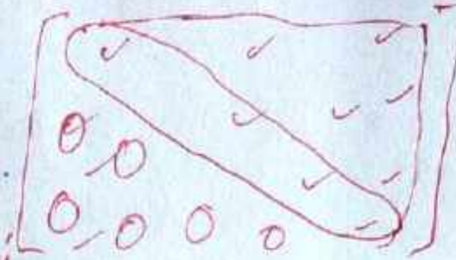
$$\begin{bmatrix}x\\-y\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Inversion matrix

$$\begin{bmatrix}-x\\-y\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Square Matrix = $n \times n$
Rectangular Matrix = $m \times n$ $m \neq n$
Column matrix = $1 \times n$
Row matrix = $n \times 1$

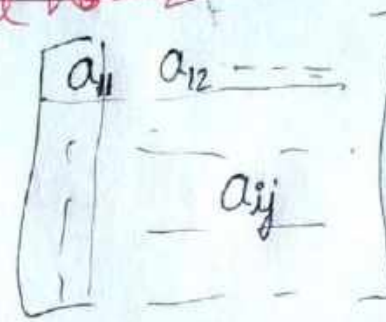
Triangular Matrix →



- Upper tri.
- Lower tri.

~~Diagonal Matrix → only diagonal non zero.~~

Transpose → $a_{ij} \rightarrow a_{ji}$



$$A \xrightarrow{T} A^T \xrightarrow{T} (A^T)^T = A$$

$$(AB)^T = B^T A^T$$

Conjugate → $a+ib \xrightarrow[i \to -i]{\text{Con.}} a-ib$

$$A \xrightarrow{\text{Conj}} A^* \xrightarrow{\text{Conj}} (A^*)^* = A$$

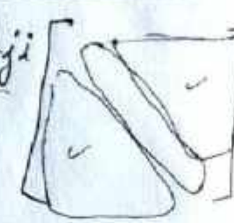
$$\boxed{(AB)^* = B^* A^*}$$

Conjugate Transpose → $A \xrightarrow{\text{Conj}} A^* \xrightarrow{\text{Tran.}} (A^*)^T = A^\dagger$

$$A \xrightarrow{\text{Conj } T.} A^\dagger \xrightarrow{\text{Conj } T} (A^\dagger)^\dagger = A$$

$$\boxed{(AB)^\dagger = B^\dagger A^\dagger}$$

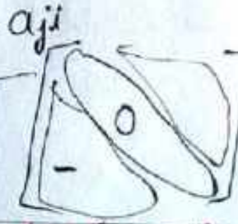
① Symmetric Matrix → $A = A^T$ $\quad a_{ij} = a_{ji}$



$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{Skew-Symmetric}}$$

② Anti- Skew-Symmetric → $A^T = -A$ $\quad a_{ij} = -a_{ji}$

Diagonal elements are all zero.



What is the symmetric ~~portion~~ part of $A = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a-2 & b \end{pmatrix}$

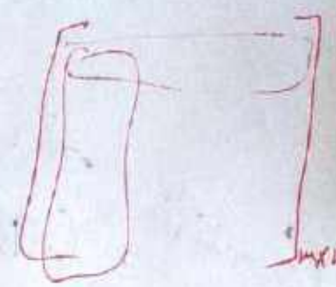
Home work

Matrix of order $\underline{n}$

How many independent no. of elements are there in this matrix.

$$= n \times n = n^2$$

$$= m \times n$$



Consider this matrix is symmetric

$$n + \frac{n^2-n}{2} = \frac{n^2+n}{2}$$

Consider the matrix is skew-symmetric

no. of ind. ele. $= \frac{n^2-n}{2}$

$$\begin{bmatrix} 2 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 4 \end{bmatrix}_{3\times 3}$$

Gate 2010 Here is an anti-symmetric tensor of order 5. What is the no. of independent elements?

Ans. $\rightarrow$ $\underline{10}$

Hermitian Matrix $\rightarrow$ $A^{\dagger} = A$ $\quad a_{ij}^{*} = a_{ji}$

$$a + ib = a - ib$$
$$b = 0$$

& diagonal elements they are real.

Anti-Skew-Hermitian Matrix $\quad A^{\dagger} = -A \quad a_{ij}^{*} = -a_{ij}$

diagonal elements zero.

Q. If Matrix A & Matrix B are Hermitian Then in what condition AB will be hermitian.

a) b) c) $AB = BA$ d)
A & B commute

$$A^\dagger = A \quad \& \quad B^\dagger = B$$

Condition $(AB)^\dagger = AB$ [Required]

$$(AB)^\dagger = B^\dagger A^\dagger = BA$$

∴ In case, $AB = BA$ ✓

Q. If A, B & C are Hermitian Matrix then which of the following is not true?

1) $C = AB - BA$ 2) $C = AB + BA$ 3) $C = ABA$

$$A^\dagger = A \qquad B^\dagger = B \qquad C^\dagger = C$$

$$(ABA)^\dagger = (BA)^\dagger A^\dagger = (A^\dagger B^\dagger)A^\dagger = ABA$$

Home work

Q. Let $B^\dagger AB$ is Hermitian then which of the following statement is true

a) A is Hermitian b) B is Hermitian c) A is skew-Her. d) B is skew-Her.

Ans a)

$$(B^\dagger AB)^\dagger = B^\dagger AB$$

$$B^\dagger A^\dagger B^{\dagger\dagger} = B^\dagger A^\dagger B$$

|| when $A^\dagger = A$

Orthogonal Matrix →

$$AA^T = A^TA = I \qquad A^T = A^{-1}$$

in this case, A is your orthogonal matrix.

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad A^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

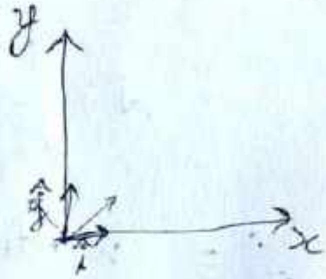
$$AA^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \boxed{A^T = A^{-1}}$$

① $|A| = \pm 1$

② Each row or each column of an orthogonal matrix is a normalized unit vector.

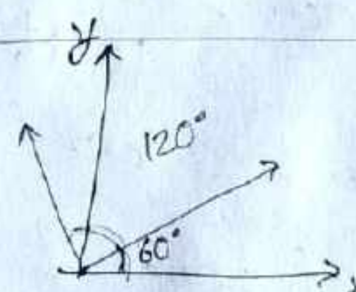
$$\underline{\underline{AA^T}} = \begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = 1.$$

③ $$A_1 A_2^T = \begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$



$$= 0$$

$$\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$$\begin{bmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{bmatrix}$$

$$60^\circ + 30^\circ = 90^\circ$$

$$\theta \times n = n\theta$$

④
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$$

$$\theta \times n = n\theta$$

⑤
$$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}$$

Complex form of orthogonal matrix is Unitary Matrix

$$A^{\dagger}A = I = AA^{\dagger}$$

Homework

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$a+ib = \sqrt{a^2+b^2}$$

① $|A|$ = modulus of its determinant is 1.

② Each row & each column normalized unit vector.

③ two rows / columns orthogonal to each other.

The product of two orthogonal matrix is also an orthogonal matrix.

$A^TA = I \quad B^TB = I \qquad (AB)^T(AB) = I$.

JAM2005

Which of the following is incorrect for the matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

a) It is its own inverse. b) Its own transpose

c) non-orthogonal d) eigen values are ± 1

$$MM^{\dagger} = I$$
$$\downarrow$$
$$MM = I$$

JAM2011

If P is Hermitian & Q is anti-Hermitian then which of the given term is Hermitian →

a) PQ b) iPQ c) $P+iQ$ d) $P+Q$

$P^{\dagger} = P$ $Q^{\dagger} = -Q$ $(i)^{\dagger} = -i$

Home work

Inverse Matrix → Minor → Co-factor → Adjoint

$$\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ a_{31} & \cdots & \cdots \\ \vdots & \vdots & \\ \vdots & \vdots & \end{bmatrix}$$

~~Minor~~

Minor → Leave that Row or column

Minor. $\Big| \quad \Big|$

Co-factor $\rightarrow$ $C_{ij} = (-1)^{i+j} M_{ij}$

Adjoint $\rightarrow$ $C_{ij}^{T} = C_{ji}$

2×2 $A = \begin{bmatrix} 2_{11} & 3_{12} \\ 5_{21} & 4 \end{bmatrix}$

(2) $\rightarrow$ Minor $\rightarrow$ $4 \xrightarrow{\text{co-factor}} (-1)^2 \cdot 4 = 4$

(3) $\rightarrow$ Minor $\rightarrow$ $5 \xrightarrow{\text{Co-factor}} (-1)^3 \cdot 5 = -5$

Co-factor $\begin{bmatrix} 4 & -5 \\ -3 & 2 \end{bmatrix} \xrightarrow[\text{Adjoint}]{\text{Transpose}} \begin{bmatrix} 4 & -3 \\ -5 & 2 \end{bmatrix}$

$A.(\text{Adj}\,A) = |A|\,I$

$A \cdot \underbrace{\boxed{\frac{1}{|A|}\text{Adj}\,A}}_{A^{-1}} = I \implies AA^{-1} = I$

$A^{-1} = -\frac{1}{7}\begin{bmatrix} 4 & -3 \\ -5 & 2 \end{bmatrix}$

$-\frac{1}{7}\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -5 & 2 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Determinat of A^{-1} →

$$A A^{-1} = I$$
$$\det(A A^{-1}) = \det(I)$$
$$\det(A) \cdot \det(A^{-1}) = 1$$
$$\boxed{Det(A^{-1}) = \frac{1}{Det(A)}}$$

Properties of Determinant →

① $Det(AB) = Det(A) \cdot Det(B)$

② $|A| = |A^T|$

③ If two rows or colums of a matrix are equal or have equal ratio then their elements the determinant is Zero.

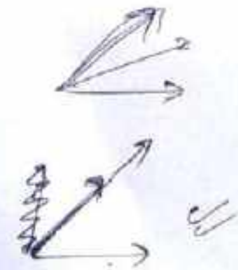
④ If two rows or colums are interchaged then determinant changes its sign.

⑤ $$\begin{vmatrix} a_1+b_1 & c_1 & d_1 \\ a_2+b_2 & c_2 & d_2 \\ a_3+b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

⑥ for Diagonal / triangular Matrix. The determinant is the product of diagonal elements.

$$\begin{vmatrix} 2 & 2 & 1 \\ 3 & 4 & 1 \\ 4 & 4 & 2 \end{vmatrix} = 0$$

$-|A|$



$$\begin{vmatrix} 2 & 3 & 6 \\ 0 & 7 & 3 \\ 0 & 0 & 8 \end{vmatrix} = 112$$

Common Errors → Matrix & determinant are not the same

$$\begin{bmatrix} \; \end{bmatrix} \qquad \begin{vmatrix} \; \end{vmatrix}$$

$$k\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \qquad k\begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix}$$

$$\begin{bmatrix} 2k & 3k \\ 5k & 4k \end{bmatrix} \qquad \begin{vmatrix} 2k & 3 \\ 5k & 4 \end{vmatrix}$$

$k^2 \cancel{|A|}\ A \quad A(\text{Adj } A) = \underbrace{|A| I} \longrightarrow |A|\begin{bmatrix} 1 & 0 & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$

$A\,(\mathrm{Adj}\,A) = |A|\, I \rightarrow |A| \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$

Take determinant both sides

$$\mathrm{Det}(A \cdot \mathrm{Adj}A) = \mathrm{Det}(|A| I) = \mathrm{Det}\begin{bmatrix} |A| & & & 0 \\ & |A| & & \\ & & |A| & \\ 0 & & & |A| \end{bmatrix}_{n\times n} = |A|^n$$

$$\mathrm{Det}(A) \cdot \mathrm{Det}(\mathrm{Adj}A) = (\det A)^n$$

$$\boxed{\mathrm{Det}(\mathrm{Adj}A) = (\mathrm{Det}\,A)^{n-1}}$$

JAM2012 $M = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$ find that its eigen values are

a) real & positive b) imaginary with modulus 1

c) complex with mod. 1. d) real & negative.

Home work

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\frac{1}{\sqrt{2}}\begin{pmatrix} \square \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} \square \end{pmatrix}$

Eigen Value & Eigen Vector →

$AX = \lambda X$

λ → eigen value

X → Eigen Vector

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 4 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

⇓ Matrix (tensor) (transformer)

→ Eigen Vector

$[AX - \lambda X] = 0$

$[A - \lambda I]X = 0$

Characteristic Eqn. →

$A - \lambda I = 0$

Roots → Characteristic Roots

The roots are obtain.

Characteristic / Eigen vector.

corresponding Vectors

$A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ $\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$A - \lambda I = 0$

$$\begin{bmatrix} 2-\lambda & 3 \\ 5 & 7-\lambda \end{bmatrix} = 0$$

$(2-\lambda)(7-\lambda) - 15 = 0$

find its roots

Home work

$\lambda = \frac{9 \pm \sqrt{77}}{2}$

Conjugate pairs

$\frac{9+\sqrt{77}}{2}, \frac{9-\sqrt{77}}{2}$

$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

If roots are complex

$a+ib$ then $a-ib$

$\lambda^3 + \lambda^2 + \lambda + \_ = 0$

$\lambda = 2$ $(\lambda-2)(\lambda^2 + \_ \_) = 0$

$(\lambda-a)(\lambda-b) = 0 \Rightarrow a, b$

$\boxed{\lambda^2 - (a+b)\lambda + ab = 0}$

$\lambda^2 + (-S)\lambda + P = 0$

$\lambda^3 + (-S)\lambda^2 + \left(\begin{matrix}\text{Sum} \\ \text{set of 2}\end{matrix}\right)\lambda + (-P) = 0$

$a+b+c$ abc

$ab+bc+ca$

(a, b, c)

$(\lambda-a)(\lambda-b)(\lambda-c) = 0$

$(\lambda^2 - (a+b)\lambda + ab)(\lambda - c) = 0$

$\lambda^3 - (a+b)\lambda^2 + ab\lambda$

$-c\lambda^2 + (ac+bc)\lambda - abc = 0$

$\lambda^3 - (a+b+c)\lambda^2 + (ab+bc+ca)\lambda$

$-abc = 0$

Home work

$\lambda^4 \rightarrow$

$\lambda^n \rightarrow$

⇒ Roots of an equ come in ~~the~~ form of conjugate pairs.

$\begin{pmatrix} a+ib \\ a-ib \end{pmatrix}$ $\begin{pmatrix} a+\sqrt{b} \\ a-\sqrt{b} \end{pmatrix}$

Single variable equation + Real coefficients

$x^3 + \_ x^2 + \_ x + \_ = 0$

Caley - Hamilton Theorem → A matrix satisfies its own characteristic equation.

$$|A-\lambda I| = \begin{vmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{vmatrix} \Rightarrow (\qquad\qquad) = 0$$

Char. eqn.

$$\left.\begin{aligned} &\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0 \\ \Rightarrow\ &A^3 + C_1A^2 + C_2A + C_3I = 0 \end{aligned}\right]$$ Caley Hamilton Theorem.

① If you solve char. eqn. → eigen values.
→ roots of this eqn → ??

② finding inverse →

$$A^3 + C_1A^2 + C_2A = -C_3I$$

multi by A^{-1}

$$A^2 + C_1A + C_2 = -C_3A^{-1}$$

Q. If a matrix A satisfies the relation → ②

$A^2 + A - I = 0$ then find its inverse.

Ans. →

$A^2 + A = I$

$$\boxed{A + I = A^{-1}}$$

JL-2013 The inverse of the matrix $M = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is –

① Make its char. equ.

Home work.

② ~~find its i~~ Apply Caley-Ham. Thm. →

③ find inverse →

$$\boxed{M^{-1} = M^2 - I}$$

⇒ $AX = \lambda X$

$AX - \lambda X = 0$

$|A - \lambda I| = 0$

↓

you get λ

↓

2, 3, 4

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

___ =

___ =

___ =

Properties of Eigen values & Eigen vectors

1. Square matrix A & A^T will have same E-values.
2. Sum of E-values is equal to the trace of matrix
3. Product of E-values of A is equal to the det.(A).
4. If A has E-values $\lambda_1\ \lambda_2\ \lambda_3$ ---
   (i) then kA will have $k\lambda_1, k\lambda_2, k\lambda_3$ ----
   (ii) ~~if~~ then A^m will have $\lambda_1^m, \lambda_2^m, \lambda_3^m$ ---
   (iii) then A^{-1} will have $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$
5. The E-values of a real & symmetric matrix or Hermitian matrix are real.
6. E-values of skew-symm or skew-Herm. are either zero or imaginary.
7. Eigen values of Orthogonal or Unitary matrix are of unit modulus.
8. E-values of an upper Δ, Lower Δ or diagonal matrix are same as that of diagonal elements
9. E-vectors corresponding to different E-values in the case of Hermitian or Unitary matrix are orthogonal to ~~eq~~ each other.
10. An equatiation of single variable & real coefficients will have roots in conjugate pairs.
    $a \pm ib$ $\quad$ $a \pm \sqrt{b}$

$A A^{-1}$

$\det(A^{-1}) = \frac{1}{\text{Det}|A|}$

$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

$\det(A^{-1}) = \frac{1}{\lambda_1} \cdot \frac{1}{\lambda_2} \cdot \frac{1}{\lambda_3}$

2 2 5 11

1 3 5 2

$A = \begin{bmatrix} a^{2} & & & \\ & b^{2} & & \\ & & c^{5} & \\ & & & d^{2} \end{bmatrix}$ 32

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = Tr(A) = a+b+c+d.$

In case of
upper Δ
lower Δ
Diagonal

$\begin{bmatrix} 2 & & \\ 1 & 2 & \\ & 1 & 5 \end{bmatrix}$

det = product of diagonal elements

$\Rightarrow$ Product of E-values = det

$\Rightarrow$ ur diagonal elements are your E-values.

$\begin{vmatrix} 1-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} =$

$(1-\lambda)(2-\lambda) = 0$

$\lambda = 1, 2$

$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$

Q. find the eigen values of $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

a) 1 i 1 i

b) 1 −1 0 0

c) 1 1 1 1

d) 1 1 −i i

e) 2 −1 −1 0

$Tr(A) = 0$

Q. The Trace & Det of a 2×2 matrix is 1 & 1, resp. Then which statement is true?

a) one of the E-values is 0 b) one of the E-values is 1

c) both E-values are 1 d) neither E-value are 1.

$\lambda_1 \; \lambda_2 \longrightarrow \lambda_1 + \lambda_2 = 1$

$\lambda_1 . \lambda_2 = 1$

JAM2015 Trace of a 2×2 matrix is 4 & Det. is 8 one of it eigen value is 2(1−i), the other one is—

a) 2(1−i) b) 2(1+i) c) (1+2i) d) 1−2i

Home work.

$\lambda_1 + \lambda_2 = 4 \qquad \lambda_1 \lambda_2 = 8$

$\lambda_2^2 \quad \lambda_2^2 \qquad \lambda_1 = \frac{8}{\lambda_2}$

~~2(1−i) 2(1~~

$\lambda_2 = \frac{8}{2(1-i)} \times \frac{(1+i)}{(1+i)}$

JAM2016 E-values of the matrix representing the following pairs of eqns.

$x + iy = 0$
$ix + y = 0$ $\Rightarrow \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Home work.

Q. $A = \begin{pmatrix} 1 & \sqrt{8} & 0 \\ \sqrt{8} & 1 & \sqrt{8} \\ 0 & \sqrt{8} & 1 \end{pmatrix}$ Then find the E-vector corresponding to the E-value 5.

a) $(1 \ \ \sqrt{2} \ \ 1)$ b) $(1 \ \ -\sqrt{2} \ \ 1)$ c) $(\sqrt{2} \ \ 1 \ \ 1)$ d) $(-\sqrt{2} \ \ 1 \ \ 1)$

$AX = \lambda X$, $\lambda \to 5$

$$\begin{pmatrix} 1 & \sqrt{8} & 0 \\ \sqrt{8} & 1 & \sqrt{8} \\ 0 & \sqrt{8} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = 5\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \Leftarrow \begin{pmatrix} 5 \\ 5\sqrt{2} \\ 5 \end{pmatrix}$$

$1+4=5$

$2\sqrt{8}+\sqrt{2}$

$4\sqrt{2}+\sqrt{2}=5\sqrt{2}$

5

Q. If one of the E-vector of matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ corresponding to -1 is $(-2 \ \ 1 \ \ 1)$ then the other orthogonal E-vector for the same E-value is-

a) $(0 \ \ 1 \ \ -1)$ b) $(-4 \ \ 2 \ \ 2)$ c) $(1 \ \ 0 \ \ 1)$ d) $(1 \ \ -1 \ \ 0)$

(Gate 2011) In a 3X3 matrix the trace is 11 & the determinant is 36. Then the largest E-value is -

a) 18 b) 12 c) 9 ~~d)~~ 6

$$\lambda_1 + \lambda_2 + \lambda_3 = 11 \qquad \lambda_1 \lambda_2 \lambda_3 = 36$$

(JNU 2013) if λ_i $(i = 1, 2, 3)$ are the eigen-values of a matrix A. Then find out the $\sum_{i=1}^{3} \lambda_i^2$ when $A = \begin{pmatrix} 2 & -1 & -3 \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{pmatrix}$

a) 14 b) 42. c) 6 d) 0

Target $\rightarrow$ $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 =$

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{Sum of E-v} = Tr(A)$$

$$Tr(A^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

Home-work

$$A^2 = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}$$

$\Rightarrow$ Rank of a Matrix $\rightarrow$ Let a matrix A (n×n)

① Rank is r if at least one minor of this order is non-zero

② Every minor of order r+1 or higher order must be zero.

$\Rightarrow$ The biggest matrix with non-zero determinant.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \vdots \\ a_{31} & \vdots & \ddots & \vdots \\ \vdots & & (n-1)\times(n-1) & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}_{n \times n}$$

Q. find the Rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}_{3\times3}$

$Det(A) = 0$ Rank of this matrix is 2

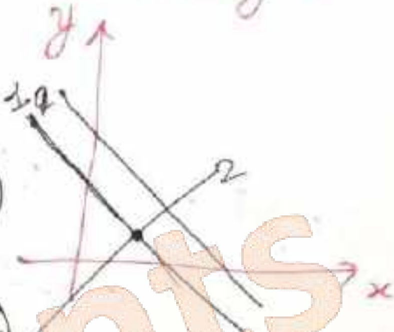
$$a_1x + a_2y = a_3$$
$$b_1x + b_2y = b_3$$

no-solution

unique sol.

∞ many sol.

$$2x + 3y = 4$$
$$4x + 6y = 5$$



Cond. 1 $\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ (∞ many sol.)

Cond 2 $\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} \neq \frac{a_3}{b_3}$ (no solution)

Cond 3 $\Rightarrow \frac{a_1}{b_1} \neq \frac{a_2}{b_2} \neq \frac{a_3}{b_3}$ (unique sol.)

Matrix form $\rightarrow \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_3 \\ b_3 \end{bmatrix}$

$$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix}$$

Q. $x_1 + 2x_2 - x_3 = 1$
$3x_1 - 2x_2 + 2x_3 = 2$
$7x_1 - 2x_2 + 3x_3 = 5$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 2 \\ 7 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}.$$

$\Delta = 1(-6+4) - 2(9-14) - 1(-6+14)$
$= -2 + 10 - 8 = 0$

Rank $\neq 3$ but no. of variables $= 3$
in that case you will have ∞ many sol.

Soln. of Linear Eqns.

(1) Consistent — unique: Rank A = Rank C = n
— ∞ many: Rank A = Rank C = $r < n$

(2) inconsistent — no-solution. Rank A $\neq$ Rank C

$AX = B \qquad C \rightarrow [A : B]$

$$C = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & -2 & 2 & 2 \\ 7 & -2 & 3 & 5 \end{bmatrix}_{3\times4}$$

$AX = B \qquad B = 0.$

$$\left.\begin{array}{l} a_1x + a_2y + a_3z = 0 \\ b_1x + b_2y + b_3z = 0 \\ c_1x + c_2y + c_3z = 0 \end{array}\right.$$

Null Solution
or trivial $\rightarrow$ $x=0 \quad y=0 \quad z=0.$

If rank A = no. of variables / only trivial solution exists

If rank A < no. of variables non-trivial solution exists.

Linear dep. of vectors $\rightarrow$ $X_1 \; X_2 \; X_3 \; \ldots \; X_n$

$\lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_n X_n = 0$

$\rightarrow$ where λ_{1-n} not all are zero.

Check if $(2,2,1)$ $(1,3,1)$ & $(1,2,2)$ are linearly dep./ ind.

$$\lambda_1\begin{pmatrix}2\\2\\1\end{pmatrix} + \lambda_2\begin{pmatrix}1\\3\\1\end{pmatrix} + \lambda_3\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Home work

$$\begin{pmatrix}2 & 1 & 1\\2 & 3 & 2\\1 & 1 & 2\end{pmatrix}_3 \begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix}_3 = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

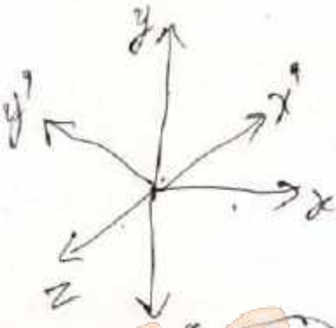
$\Delta \neq 0$ $\quad \lambda_1 = \lambda_2 = \lambda_3 = 0$ Linearly Indep.

$\Delta = 0$ Linearly Dep.

Gate 2018 Given $V_1 = i - j$ $\quad V_2 = -2i + 3j + 2k$
Then which of the following vector will make a vector basis?

a) $i + j + 4k$ b) $2i - j + 2k$ c) $i + 2j + 6k$ d) $2i + j + 4k$

$$\begin{vmatrix} 1 & -1 & 0 \\ -2 & 3 & 2 \\ & & \end{vmatrix}$$



Similarity transform → Two matrix

Home work

A & B are similar if
$B = P^{-1}AP$ $\quad$ P → any invertible matrix

Gate 2011 If $B = P^{-1}AP$ where P is any inver. matrix
Then which is ~~not~~ true.

a) Det A = Det B (b) Tr(A) = Tr(B) (c) E-vectors are same
(d) E-values are same.

$\text{Det } B = \text{Det}(P^{-1}AP)$
$\text{Det}(P^{-1})\,\text{Det}(A)\,\text{Det}(P)$

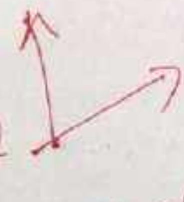
$\text{Det}(P^{-1})\,\text{Det}(P)\ \text{Det}A$

$\text{Det } B = \text{Det } A$

$(ai + bj) = \underline{m}(ci + dj)$

$a = mc$
$b = md$ → same m

Indep. orthogonal



$aX_1 - bX_2 = 0$

TIFR 2010 The matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ can be related by a similarity transform to the matrix- ⑥

a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ c) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ d) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Home work

Diagonalized form $\rightarrow$ A

$$B = P^{-1} A P$$

A $\rightarrow$ eigen vectors $\quad \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}$

$$A\underline{X} = \lambda \underline{X}$$

B = Diagonal

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} =$$

Q. Jest 2019 Let A be a Hermitian Matrix & C & D are Unitary Matrix then which of the following matrix is Unitary?

a) $C^{-1}AC$ b) $C^{-1}DC$ c) $C^{-1}AD$ d) $A^{-1}CD$

$A^{\dagger}=A$ $CC^{\dagger}=I$ $DD^{\dagger}=I$ $C^{\dagger}=C^{-1}$ $D^{\dagger}D^{-1}$

a) $(C^{-1}AC)(C^{-1}AC)^{\dagger} \Rightarrow C^{-1}AC(C^{\dagger}A^{\dagger}(C^{-1})^{\dagger})$ not a unitary transfor.

$\Rightarrow C^{-1}AA^{\dagger}(C^{-1})^{\dagger} \Rightarrow C^{-1}A^{2}(C^{-1})^{\dagger}$

b) Home work

c)

d)

JAM 2016 for a given set of eqn. $x+y=1$, $y+z=1$, $z+x=1$

which of the following statement is true?

a) Eqns are inconsistant b) consistant with non-trivial solution

c) consistant with many sol. d) consistant with trivial sol.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

A X B

$[A:B]$

$2x+2y+2z=3$

$x+y+z=3/2$

$z=\frac{1}{2}$ $x=\frac{1}{2}$ $y=\frac{1}{2}$

70

E-vectors →

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

E-values are 3, 2, 5

$$|A - \lambda I| X = 0$$

$\lambda = 2$

$$\begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Downarrow$$

$$x_1 + x_2 + 4x_3 = 0$$
$$6x_3 = 0$$
$$3x_3 = 0$$

$x_3 = 0$

$x_1 = -x_2$

$$\boxed{\begin{aligned} x_1 &= k \\ x_2 &= -k \\ x_3 &= 0 \end{aligned}}$$

$$\begin{bmatrix} k \\ -k \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$\Downarrow$ normalized form

$$R = \sqrt{(1)^2 + (-1)^2 + (0)^2} = \sqrt{2}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

In case of repeated E-values → 3, 2, 2

Either matrix is non-symmetric ⇒ Linearly Independent

Or matrix is symmetric ⇒ These E-vectors are orthogonal

$\lambda = 3$	$\lambda = 5$
$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} \xRightarrow{\text{E-vectors}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Home work

$$D = P^{-1}AP$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$PP^T = I$

BHU 2018 find the E-vectors of $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \equiv A$

a) $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ~~(b)~~ $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ ~~(c) $\frac{1}{\sqrt{2}}$~~

c) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_{X_1} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}_{X_2}$ d) $\frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$

$$AX = \lambda X$$

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -i \end{pmatrix} \quad \times$$

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \\ \end{pmatrix} \quad 1\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

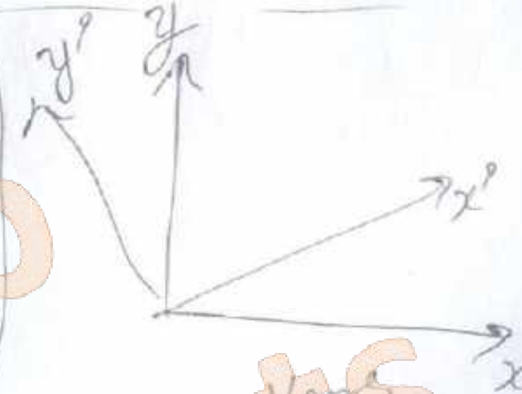
Gate 2019 find the axis of rotation of matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

a) $\frac{1}{3}(2i - j + 2k)$ b) $\frac{1}{\sqrt{3}}(i + j - k)$

c) $\frac{1}{\sqrt{3}}(i - j - k)$ d) $\frac{1}{3}(2i + 2j + k)$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \\ \\ \\ \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$



Jest 2018 $\begin{pmatrix} a & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow$ E-values $1, 1, ?$

Home work

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

① Trace

$1 + (-1) + \lambda_3 = a + 2 + 1$

② Determinant

$2a - 9 = (1)(-1)(\lambda_3)$

Home work

Jest 2012 for an $N \times N$ matrix contains all 1.

a) all E-values 1. ~~b)~~ all E-v. are 0

~~c)~~ $1, 2, 3, \ldots N$ d) $N, 0, 0, 0 \ldots$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & & & \\ 1 & & & \\ 1 & & & \end{bmatrix}_{N \times N}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2, 0 =$$

Tr $\in N$

Test 2017

(5)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix} \quad M = \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix}$$

The transformation from M to A can be performed by -

(a) $\frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix}$ (b) $\frac{1}{\sqrt{9}}\begin{pmatrix} 1 & -3i \\ 3i & 11 \end{pmatrix}$ (c) $\frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3i \\ -3i & 11 \end{pmatrix}$

(d) $\frac{1}{\sqrt{9}}\begin{pmatrix} 1 & 3i \\ -3i & 1 \end{pmatrix}$

$$A = P^{-1} M P$$

$$\begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3i \end{pmatrix} = 1\begin{pmatrix} 1 \\ 3i \end{pmatrix}$$

$$\begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix}\begin{pmatrix} 3i \\ 1 \end{pmatrix} = \begin{pmatrix} 33i \\ 11 \end{pmatrix} = 11\begin{pmatrix} 3i \\ 1 \end{pmatrix}$$

DU 2017 $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ Then find e^A.

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots$$

$$e^A = I + A + \frac{A^2}{\underline{|2}} + \frac{A^3}{\underline{|3}} + \cdots$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2A$$

$$A^3 = 2A.A = 2.(2A) = 2^2A$$

$$A^4 = 2^3A \qquad \boxed{A^n = 2^{n-1}A}$$

$$e^A = I + A + \frac{2A}{\underline{|2}} + \frac{2^2A}{\underline{|3}} + \frac{2^3A}{\underline{|4}} + \cdots$$

$$= I + A\left(1 + \frac{2}{\underline{|2}} + \frac{2^2}{\underline{|3}} + \frac{2^3}{\underline{|4}} + \cdots\right)$$

$$= I + \frac{A}{2}\left(1 + 2 + \frac{2^2}{\underline{|2}} + \frac{2^3}{\underline{|3}} + \frac{2^3}{\underline{|4}} + \cdots - 1\right)$$

$$= I + \frac{A}{2}(e^2 - 1)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^2-1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

JNU 2014

$S = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $S^{-1}SAS^{-1}S = S^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}S$

find the Det of e^A

$$A = S^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}S = \begin{pmatrix} & \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} & \end{pmatrix}$$

Home work

TIFR 2011, 2017, 2018

A A^2 A^3 $tr(e^A)$ $A = \begin{bmatrix} 0 & 0 & \pi/4 & \pi/4 \\ 0 & 0 & \pi/4 & 0 \\ \pi/4 & \pi/4 & 0 & 0 \\ \pi/4 & 0 & 0 & 0 \end{bmatrix}$

A 2X2 matrix whose E-values are $e^{i\pi/5}$ & $e^{i\pi/6}$ Then for what mini value of n

$A^n = I$

find the eigen values of $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$A \rightarrow e^{i\pi/5}$ & $e^{i\pi/6}$ a) 20 b) 30 ~~c) 60~~ d) 120

$A^n \rightarrow e^{in\pi/5}$ & $e^{in\pi/6}$

$\frac{11}{30}n = 2m$

$Det(A^n) = Det(I)$

$e^{in\pi/5} \cdot e^{in\pi/6} = e^{i2m\pi}$

$n\left(\frac{\pi}{5} + \frac{\pi}{6}\right) = 2m\pi$

$\frac{11\cancel{\pi}n}{30} = 2m\cancel{\pi}$