Free Study Material from All Lab Experiments

Classical Mechanics for NET/Gate Physical Sciences

> Lagrangian Formulation, Part-2 <

* Poisson's Bracket :- (P.B.)

If A and B are two dynamical variables -

i.e. $A = A(p_i, q_i, t)$, $B = B(p_i, q_i, t)$

Poisson Bracket of A with B is defined as -

$$\boxed{\{A, B\}_{q_i, p_i} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}}$$

Summation over i.

$\{q_i, p_i\} = 1$, $\{x, p_x\} = 1$

$\{q_i, p_j\} = \delta_{ij}$, $\{x, p_y\} = 0$

$\leftarrow$ Fundamental P.B.

Properties :-

* $\{A, B\} = -\{B, A\}$
* $\{A, B+C\} = \{A, B\} + \{A, C\}$
* $\{A, BC\} = \{A, B\}C + B\{A, C\}$ $\rightarrow$ Leibnitz Identity
* $\{A, \{B, C\}\} + \{C, \{A, B\}\} + \{B, \{C, A\}\} = 0$ $\leftarrow$ Jacobi Identity.

* Poisson's Bracket with angular momentum :-

$[x, L_y] = z$, $[y, L_z] = x$

$[p_x, L_z] = -p_y$

$[p_z, L_x] = p_y$

$[L_x, L_y] = L_z$

$[p_y, L_y] = 0$

$[q_y, L_y] = 0$

If we replace $i\hbar \to 1$, then commutator bracket changes into P.B. and vice-versa.
Here in P.B. we can use { } or [ ]

* General Relation :-

$$[A_i, L_j] = \varepsilon_{ijk} A_k$$

$$[A_i, L_j] = \varepsilon_{ijl} A_l$$

Here l is repeated index so we have summation over it, so we can change variable.

Levicivita Tensor.

* A and B are constant of motion (Conserved) then their possion bracket is also constant of motion.

Q. If $A = ap_1 + bq_2$, $B = cq_1 + dp_2$ find the value of P.B. $\{A, B\}$.

Note :- Here A and B depends upon position and momentum co-ordinate so these two are dynamical variable.

Formula $\Rightarrow \{A, B\}_{q_i, p_i} = \left[\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right]$

$$\{A, B\}_{q_i, p_i} = \{\underbrace{ap_1 + bq_2}_{A}, \underbrace{cq_1 + dp_2}_{B}\}_{q_1, p_1} + \{\underbrace{ap_1 + bq_2}_{A}, \underbrace{cq_1 + dp_2}_{B}\}_{q_2, p_2}$$

$$= (0 - a.c) + (bd - 0)$$

$$= bd - ac \quad \text{Ans}$$

$$= \left[\frac{p_x + \frac{qB}{2}y}{m}, \frac{p_y - \frac{qB}{2}x}{m}\right]_{x,p_x} + \left[\frac{p_x + \frac{qB}{2}y}{m}, \frac{p_y - \frac{qB}{2}x}{m}\right]_{y,p_y}$$

Q. Evaluate $\{(\vec{a}\cdot\vec{r}), \vec{p}\}$ where $\vec{a}$ is constant

Sol^n

$$\because \{(\vec{a}\cdot\vec{r}), \vec{p}\} = \{a_i r_i, p_j\}$$

$$= a_i \{r_i, p_j\}$$

$$= a_i \delta_{ij} = a_j \times 1$$

$$= a_j$$

$$= \vec{a}. \quad \text{Ans}$$

Q. $\{(\vec{a}\cdot\vec{r}), \vec{r}\}$ evaluate ?

Sol^n

$$\{(\vec{a}\cdot\vec{r}), \vec{r}\} = 0$$

($\because$ Here no momentum term ~~absent~~)

Q. Evaluate $\{(\vec{a}\cdot\vec{r})^2, \vec{p}\}$

Sol^n $\{(\vec{a}\cdot\vec{r})^2, \vec{p}\} = \{(\vec{a}\cdot\vec{r})(\vec{a}\cdot\vec{r}), \vec{p}\}$

$\because \{AB, C\} = A\{B, C\} + B\{A, C\}$

$\therefore \{(\vec{a}\cdot\vec{r})^2, \vec{p}\} = (\vec{a}\cdot\vec{r})\{(\vec{a}\cdot\vec{r}), \vec{p}\} + (\vec{a}\cdot\vec{r})\{(\vec{a}\cdot\vec{r}), \vec{p}\}$

$= 2(\vec{a}\cdot\vec{r})\{(\vec{a}\cdot\vec{r}), \vec{p}\}$

$= 2(\vec{a}\cdot\vec{r})\vec{a}$ $\quad \{\because \{(\vec{a}\cdot\vec{r}), \vec{p}\} = \vec{a}\}$

Ans

Second Method:-

$\{(\vec{a}\cdot\vec{r})^2, \vec{p}\} = \{a_i^2 r_i^2, p_j\}$

$= a_i^2 \{r_i^2, p_j\}_{r_i, p_i}$

$= a_i^2 \left[2r_i \frac{\partial p_j}{\partial p_i} - 0\right]$

$= a_i^2\, 2r_i\, \delta_{ij}$

$= 2a_j^2 r_j$ $\quad \{\because \delta_{ij} = 1 \text{ when } i=j\}$

$= 2\, a_j\, (a_j r_j)$

$= 2a_j\, (\vec{a}\cdot\vec{r})$

$= 2\vec{a}\,(\vec{a}\cdot\vec{r})$ Ans

Q. Evaluate $[\vec{a}\cdot\vec{r}, \vec{L}]$

angular momentum not Lagrangian becoz Lagrangian is scalar quantity

Sol^n $[a_i r_i, L_j] = a_i[r_i, L_j]$

$= a_i \varepsilon_{ijk} r_k$

$= \varepsilon_{ijk} a_i r_k$

$= -\varepsilon_{jik} a_i r_k$

$= -(\vec{a}\times\vec{r})_j \quad \left\{ \because (\vec{a}\times\vec{b})_i = \varepsilon_{ijk} a_i b_k \right\}$

$= (\vec{r}\times\vec{a})_j$

$= (\vec{r}\times\vec{a})$ Ans

Q. Evaluate $[\vec{a}\cdot\vec{L}, \vec{b}\cdot\vec{L}]$

$\Rightarrow [a_i L_i, b_j L_j]$

$\Rightarrow a_i b_j [L_i, L_j]$

$= a_i b_j \varepsilon_{ijk} L_K$

$= a_i \varepsilon_{ijk} b_j L_K$

$= a_i (\vec{b}\times\vec{L})_i$

$= \vec{a}\cdot[\vec{b}\times\vec{L}] = [\vec{a}\times\vec{b}]\cdot\vec{L}$

Q. Evaluate $[L_x, L_y] = ?$, $[L_x, y] = ?$

$$[yp_z - zp_y, y] = [\quad]_{y,p_y} + [\quad]_{z,}$$

$$= p_z \times 0 + z \times 1 + (-p_y) \times 0 - y = 0$$

$$= z$$

$$[p_x, L_y] = p_z$$
$$[L_x, L_y] = p_z$$
$$[L_x, y] = z$$

Q. Evaluate $[\vec{r}, \vec{a}.\vec{L}]$ where $\vec{a}$ = Constant vector.

$$[r_i, a_j L_j] = a_j [r_i, L_j]$$

$$= a_j \, \varepsilon_{ijk} r_k$$

$$= \varepsilon_{ijk} a_j r_k$$

$$= (\vec{a} \times \vec{r})_i = \boxed{\vec{a} \times \vec{r}}$$ Ans

Q. Evaluate $\{\vec{a}.\vec{r}, \vec{b}.\vec{p}\}$

$$= \{a_i r_i, b_j p_j\}$$

$$= a_i b_j \{r_i, p_j\}$$

$$= a_i b_j \delta_{ij}$$

$$= a_j b_j \times 1 = \vec{a}.\vec{b} \text{ Ans}$$

Q. Evaluate $[|\vec{r}|, |\vec{p}|]$

Sol^n $\because r = \sqrt{x^2+y^2+z^2}$

So $\left[\sqrt{x^2+y^2+z^2}, \sqrt{p_x^2+p_y^2+p_z^2}\right]$

$= [\quad]_{xp_x} + [\quad]_{yp_y} + [\quad]_{zp_z}$

$= \left(\frac{x}{|\vec{r}|} \cdot \frac{p_x}{|\vec{p}|} + 0\right) + \left(\frac{y}{|\vec{r}|} \frac{p_y}{|\vec{p}|} + 0\right) + \left(\frac{z}{|\vec{r}|} \frac{p_z}{|\vec{p}|} + 0\right)$

$= \frac{xp_x + yp_y + zp_z}{|\vec{r}||\vec{p}|} = \frac{\vec{r}\cdot\vec{p}}{|\vec{r}||\vec{p}|}$

$= \frac{\vec{r}}{|\vec{r}|} \cdot \frac{\vec{p}}{|\vec{p}|} = \boxed{\hat{r}\cdot\hat{p}}$ Ans

Q. If $x' = x\cos\theta - p_x\sin\theta$, $p_x' = x\sin\theta + p_x\cos\theta$.
evaluate P.B. $\{x', p_x'\}_{x, p_x}$

Sol^n

$\{x\cos\theta - p_x\sin\theta, x\sin\theta + p_x\cos\theta\}_{x, p_x}$

$= \cos\theta \times \cos\theta + \sin\theta \times \sin\theta$

$= \cos^2\theta + \sin^2\theta$

$= 1.$

* Poission's Equation of Motion :-

$$\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t}$$

If $A = H$

$$\frac{dH}{dt} = [H, H] + \frac{\partial H}{\partial t}$$

$$\boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}}$$

If H does not depends on t explicitly.

$$\frac{\partial H}{\partial t} = 0$$

$$\frac{dH}{dt} = 0 \Rightarrow \boxed{H = \text{Constant or Conserved.}}$$

If H does not depends on time explicitly then H is conserved.

A-11
Net-2012 Dec.

Q. A System is governed by the Hamiltonian $H = \frac{1}{2}(p_x - ay)^2 + \frac{1}{2}(p_y - bx)^2$ where a and b are constants and p_x, p_y are momenta conjugate to x and y respectively. For what values of a and b will the quantities $(p_x - 3y)$ and $(p_y + 2x)$ be conserved.

Sol^n

$$H = \frac{1}{2}(p_x - ay)^2 + \frac{1}{2}(p_y - bx)^2$$

$p_x - 3y$, $p_y + 2x$ conserved.

Let $A = p_x - 3y$ it does not depends on time explicitely

* Imp. So $\frac{\partial A}{\partial t} = 0$ so it is conserved

So $\frac{dA}{dt} = 0$

$\therefore [A, H] = 0$

$\therefore$ $B = p_y + \cancel{2x}$ it is also it does not depends on time explicitely

So $\frac{\partial B}{\partial t} = 0$ so it is also conserved

So $\frac{dB}{dt} = 0$

$\therefore [B, H] = 0$

$$\Big[\underbrace{p_x - 3y}_{A}, \underbrace{\frac{(p_x - ay)^2}{2} + \frac{(p_y - bx)^2}{2}}_{B}\Big] = 0$$

$\Rightarrow [\quad]_{x, p_x} + [\quad]_{y, p_y} = 0$

$0 - 1 \cdot (p_y - bx)(-b) + (-3)(p_y - bx) - 0 = 0$

$\Rightarrow (p_y - bx)(b - 3) = 0$

$b - 3 = 0$

$\boxed{b = 3}$

$$[B,H] = \Big[\underbrace{p_y + 2x}_{A}, \underbrace{\frac{(p_x - ay)^2}{2} + \frac{(p_y - bx)^2}{2}}_{B}\Big] = 0$$

$$[B,H] = \Big[\underbrace{(p_y + 2x)}_{A}, \underbrace{\frac{(p_x - ay)^2}{2} + \frac{(p_y - bx)^2}{2}}_{B}\Big]_{x,p_x} + \Big[\underbrace{(p_y + 2x)}_{A}, \underbrace{\frac{(p_x - ay)^2}{2} + \frac{(p_y - bx)^2}{2}}_{B}\Big]_{y,p_y}$$

$$\left[\frac{\partial A}{\partial x}\cdot\frac{\partial B}{\partial p_x} - \frac{\partial A}{\partial p_x}\cdot\frac{\partial B}{\partial x}\right] \qquad \left[\frac{\partial A}{\partial y}\cdot\frac{\partial B}{\partial p_y} - \frac{\partial A}{\partial p_y}\cdot\frac{\partial B}{\partial y}\right]$$

$$= \Big[\{2\cdot(p_x - ay)\} - \{0\}\Big] + \Big[\cancel{0 + 1\cdot(p_y - b)}\Big]$$

$$\{0 + 1\cdot(p_x - ay)(-a)\}$$

$$\cancel{= 2(p_x - ay) - (p_y - b)}$$

$$2(p_x - ay) + (p_x - ay)(+a)$$

$$(p_x - ay)(2 + a) = 0$$

$$a = -2 \quad \text{Ans}$$

(*) "Conservation Laws :

(1) If L or H is translationally invariant then linear momentum is conserved in that direction.

$$p_x = \frac{\partial L}{\partial \dot{x}} \quad , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}(p_x) - \frac{\partial L}{\partial x} = 0$$

$$\dot{p}_x = \frac{\partial L}{\partial x}$$

If L does not explicitly depend on x

$$\frac{\partial L}{\partial x} = 0$$

$$\dot{p}_x = 0 \Rightarrow p_x = \text{Constant}$$

Hamiltonian eq^n of motion -

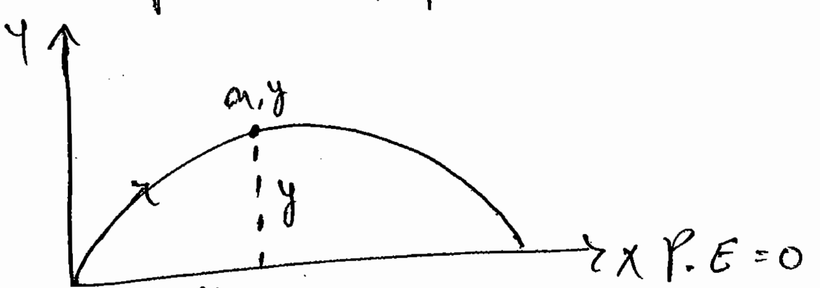
$$\dot{p}_x = -\frac{\partial H}{\partial x}$$

If H is not explicit function of x

$$\frac{\partial H}{\partial x} = 0$$

$$\dot{p}_x = 0 \quad , \quad p_x = \text{Constant}$$

Ex-



$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$

x is cyclic

Translation $x \longrightarrow x + \alpha$, then $L \longrightarrow L$

Here L is invariant w.r.to translation x-direcn

$\therefore$ p_x is conserved.

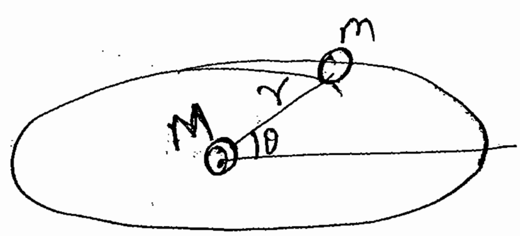
* Homoginity of space leads to conservation of momentum.

Second Theorem:-

If L or H is Rotationally invariant, Angular momentum is conserved in that direction.

$$\theta \longrightarrow \theta + \alpha \ , \quad L \rightarrow L$$

Ex - Sun - Planet System -



$$L = \frac{1}{2} m (\dot{r}^2 + r\dot{\theta}^2) + \frac{GMm}{r}$$

$\theta \rightarrow$ is cyclic , $\frac{\partial L}{\partial \theta} = 0$

$$\dot{p}_\theta = \frac{\partial L}{\partial \theta}$$

$$\dot{p}_\theta = 0$$

$$\boxed{p_\theta = \text{Constant}}$$

So

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r}$$

θ is cyclic $\frac{\partial H}{\partial \theta} = 0$

Hamilton's equation –

$$\dot{p}_\theta = \frac{-\partial H}{\partial \theta} = 0$$

$$\boxed{p_\theta = \text{Conserved}}$$

Angular momentum is Conserved.

$$\boxed{p_\theta = L_z}$$

* Isotropy of space leads to Conservation of angular momentum.

* Third Theorem :-

If L or H does not explicitly depends on time then Hamiltonian is conserved.

$$\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t}$$

$$A = H$$

$$\boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}} \to 0$$

$$\frac{dH}{dt} = 0$$

$$\boxed{H = \text{Constant}}$$

(a) If potential energy does not depends on velocity then H = total energy.

(b) If potential energy depends on velocity then

$$\boxed{H \neq \text{total energy}}$$

Imp.
* If L or H does not depends on t explicitly and potential does not depends on velocity then total energy is conserved.

** If potential depends on velocity then H is conserved but total is not conserved.

*** Homogeneity of time leads to conservation of Hamiltonian (Energy).

Q. $L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} K(x^2 + y^2)$ which of the following is conserved.

p_x, p_y, L_z, E = energy

Sol^n x and y are not cyclic so p_x and p_y are not conserved.

To know about L_z write L in plane polar co-ordi[nates]

$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}Kr^2$ here θ is not con[tained]

so θ is cyclic.

$\therefore$ p_θ = Constant

$\therefore$ L_z = Constant

L does not explicitly depends on t.

so H = Conserved.

$\because$ Potential does not depend on velocity

$\therefore H = E$

So $\boxed{E = \text{Conserved}}$

* How to identify the K.E. term and P.E. term in L.

$\Rightarrow$ Say $L = \underbrace{\frac{1}{2}m(\dot{x}^2+\dot{y}^2)}_{\text{K.E. term}} - \underbrace{\frac{1}{2}K(x^2+y^2) + \omega(x\dot{y}-y\dot{x})}_{\text{P.E. term.}}$

$\because$ K.E. term is always quadratic in velocity.

like $\dot{x}^2$, $\dot{y}^2$ or $(\dot{x}\dot{y})$ etc.

in $\dot{q}_i$ i.e. $\dot{q}_i^2$ or $\dot{q}_i\dot{q}_j$

$H = \text{Constant}$

$H \neq E$

$\therefore E$ is not conserved.

A-11

Q.11 Lagrangian of a system is $L = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) + qEy - \frac{qB}{c}y\dot{x}$

Which of the following is not correct.

Sol^n

$$L = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) + qEy - \frac{qB}{c}y\dot{x}$$

(a) $m\dot{x} - \frac{qBy}{c} = \text{Const.}$

(b) $m\dot{z} = \text{Const.}$

× (c) $m\dot{y} + \frac{qBx}{c} + qEt = \text{Const}$

✓ (d) $m\dot{y} + \frac{qBy}{c} - qEt = \text{Const.}$

$\therefore$ x and z are constant.

$p_x = \text{Constant}, \quad p_z = \text{Constant}.$

$$p_x = \frac{\partial L}{\partial \dot{x}}$$

$$p_x = m\dot{x} - \frac{qB}{c} y = \text{Constant}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} = \text{Constant}$$

~~If $m\dot{y}$~~ say $A = m\dot{y} + \frac{qBx}{c} + qEt$

If $A = \text{Constant}$

then $\frac{dA}{dt} = 0$

$$\frac{dA}{dt} = m\ddot{y} + \frac{qB\dot{x}}{c} + qE \quad \text{—— (I)}$$

write Lagrangian's eq^n in y –

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$m\ddot{y} - qE + \frac{qB}{c}\dot{x} = 0$$

$$m\ddot{y} + \frac{qB}{c}\dot{x} = qE \quad \text{—— (II)}$$

So from (I)

$$\therefore \frac{dA}{dt} = 2qE \neq 0$$

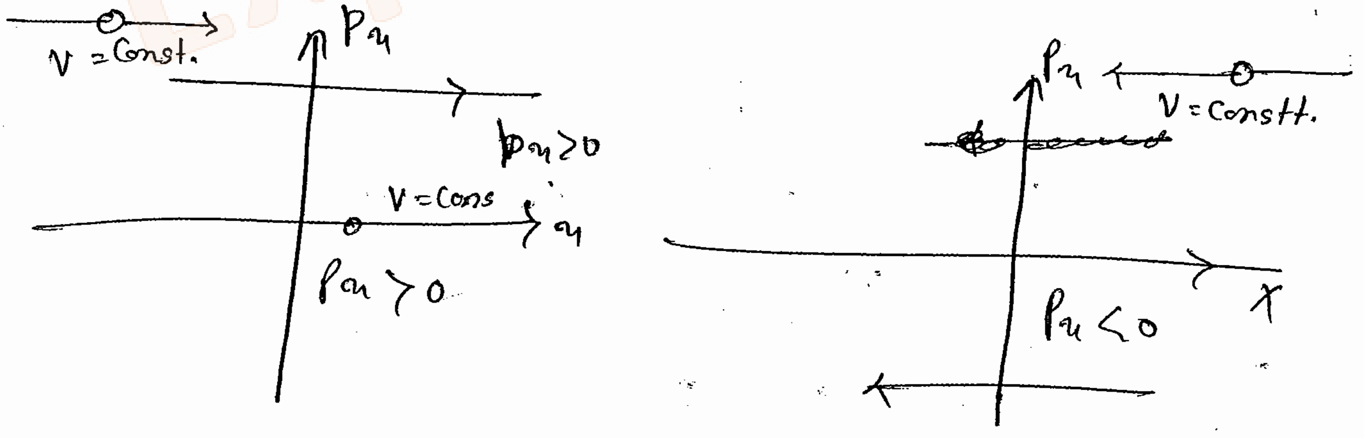
So (c) is wrong. <u>Ans</u>

* Phase Space Dynamics :- How momentum varies with coordinate (graph b/w p_i, q_i)

Phase Space : It consists of Co-ordinates and momenta. for a system having f-D.O.F. phase space is a $2f$ dimensional space (f-Coordinate + f momenta).

How to draw phase space line (phase space trajectory) [for one dimensional problem]

* If particle is moving towards +ve x direc^n then $p_x > 0$ (+ve)
* If particle is moving towards -ve x direc^n $p_x < 0$ (-
* If $p_x > 0$ then arrow should be towards +ve X
* If $p_x < 0$ then arrow should be towards -ve X.



* Cauchy Lipschitz Condition :- The two phase space trajectory can not intersect each other.

* For Conservative System (Energy = Constant):-

If $V(x)$ [P.E.] is increasing then K.E. must decrease [p_x must decrease] and vice-versa.

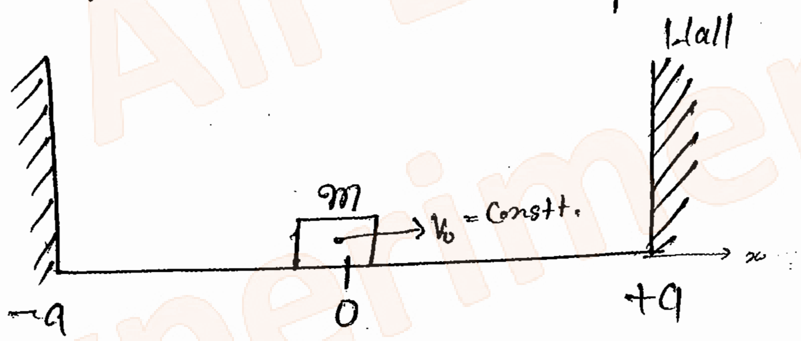
Therefore we first draw $V(x)$ and then using it we draw phase space lines for conservative system.

* Types of questions:-

① Potential Energy is Given

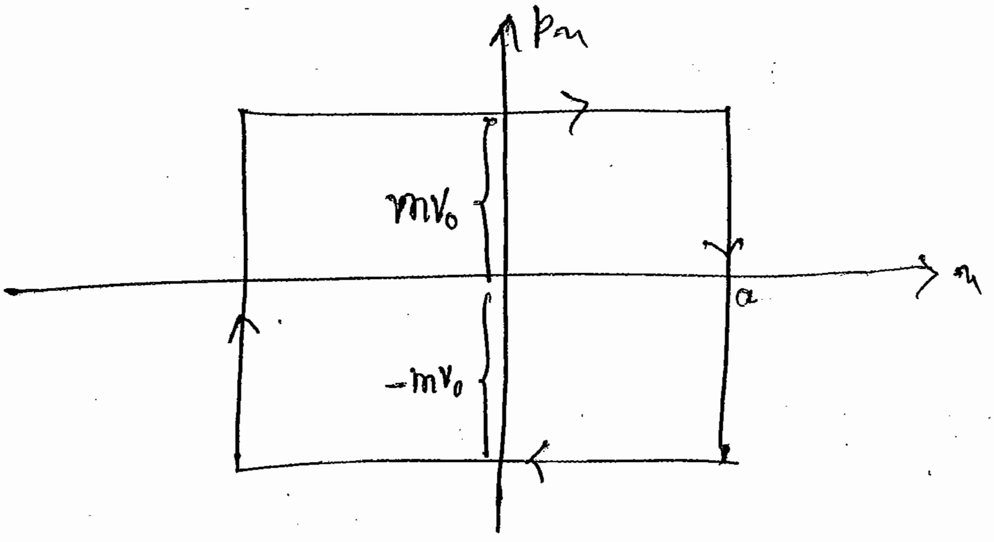
② Some information about dynamics is given.

Q.

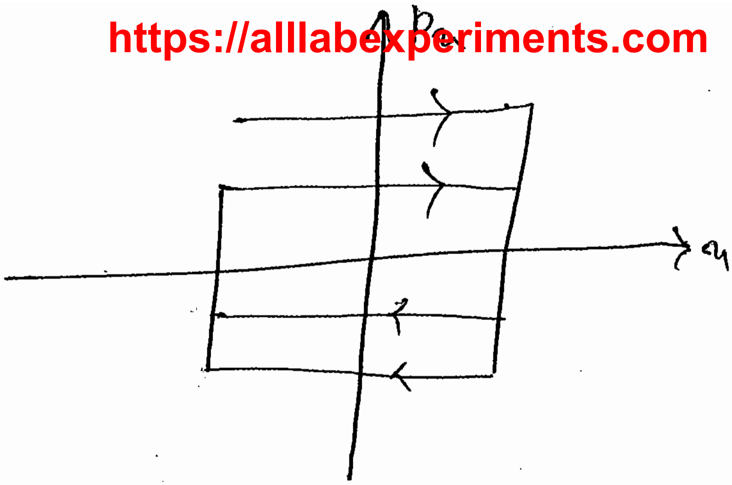


If collisions are elastic draw phase space line.

Sol^n

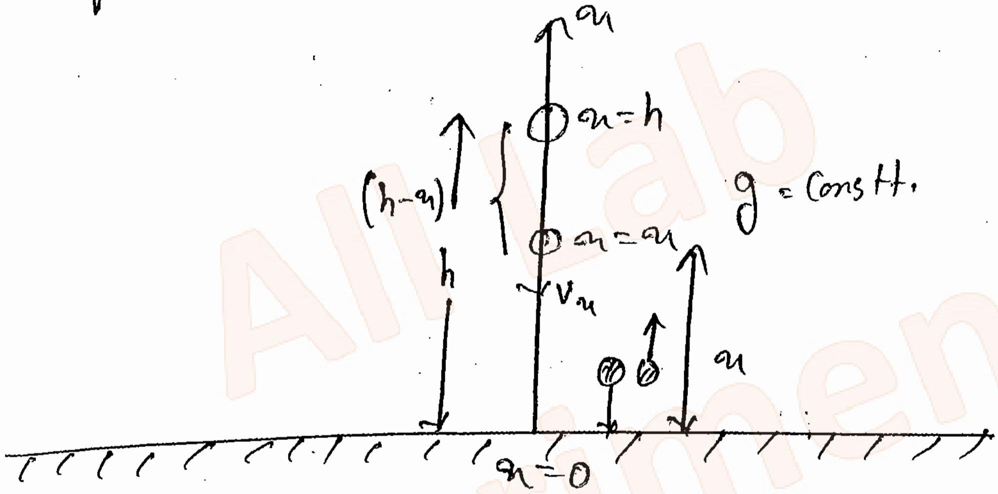


(ii) If Collisions are inelastic :-



P.

Q. A ball dropped from some height on horizontal surface.



$$v^2 = u^2 + 2as$$

$$v_x^2 = 0 + 2g(h-x)$$

$$v_x^2 = 2g(h-x)$$

$$\boxed{\frac{p_x^2}{m^2} = 2g(h-x)} \leftarrow \text{parabola}$$

$$p_x^2 = 2g(h-x)m^2$$

put $h - x = x'$

$$\boxed{p_x^2 = 2g\,m^2 x'}$$

$$v_x^2 = 2g(h-x)$$

$$v_x = \frac{p_x}{m}$$

$$p_x^2 = 2gm^2(h-x)$$

$$p_x \frac{dp_x}{dx} = -2gm^2$$

$$\frac{dp_x}{dx} = -\frac{2gm^2}{p_x}$$

$$\frac{d^2p_x}{dx^2} = \frac{2gm^2}{p_x^2}\cdot$$

$$\frac{d^2p_x}{dx^2} = -\frac{4g^2m^4}{p_x^3}$$

$p_x \not> 0$ so $\frac{d^2p_x}{dx^2} > 0$

So trajectory is +ve.

Rule :-

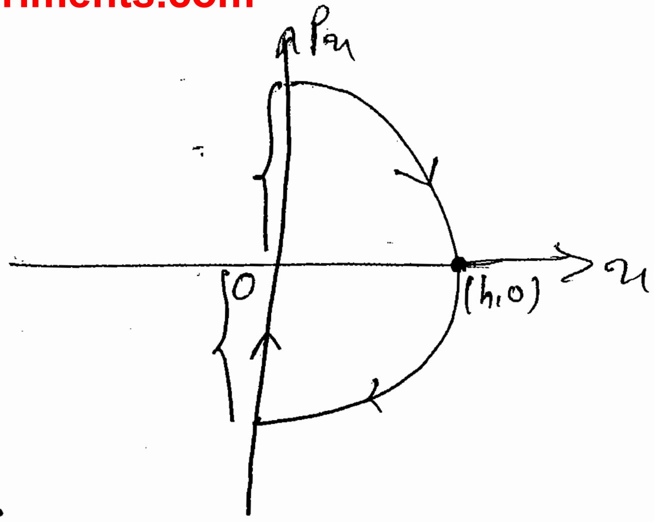
$$E = \frac{p_x^2}{2m} + mgx$$

$p_x \rightarrow -p_x$

E does not change then Symmetry about x-axis.

When $x \rightarrow -x$

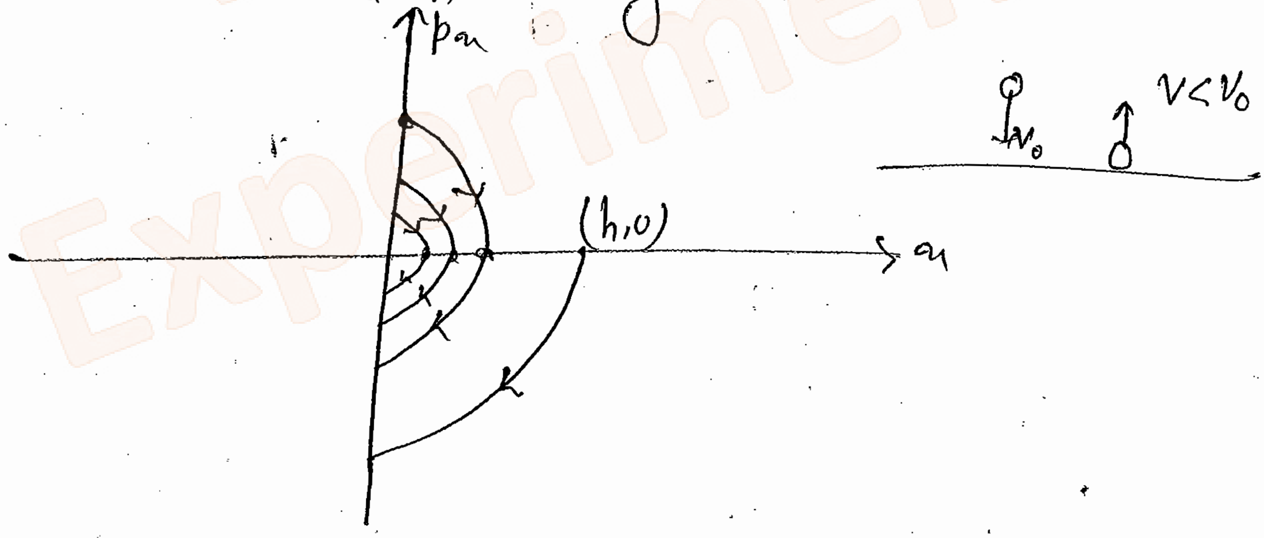
E does not change then symmetry about p_x axis.



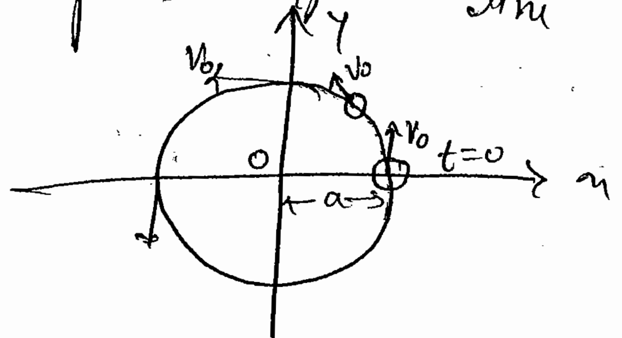
Elastic

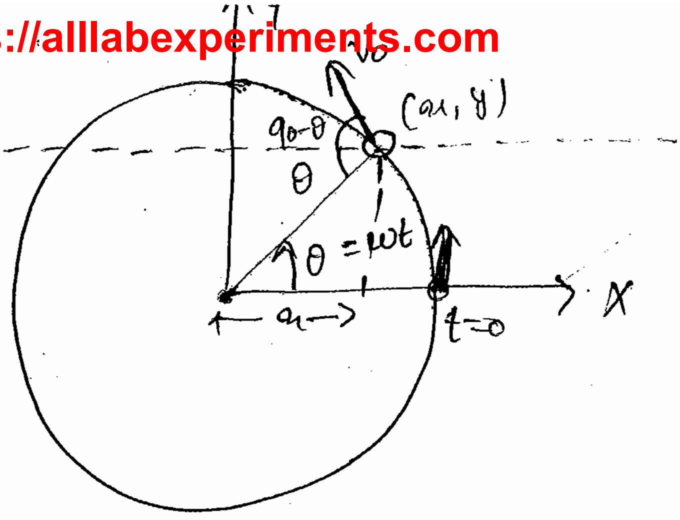
(II) If Collision is inelastic :-

Inelastic collision with ground.



Q. A particle is moving in a circle with constant speed, draw phase space line in $x-p_x$ space.





$\therefore \quad \omega = \frac{d\theta}{dt}$

$$\boxed{\omega = \frac{V_0}{a}}$$

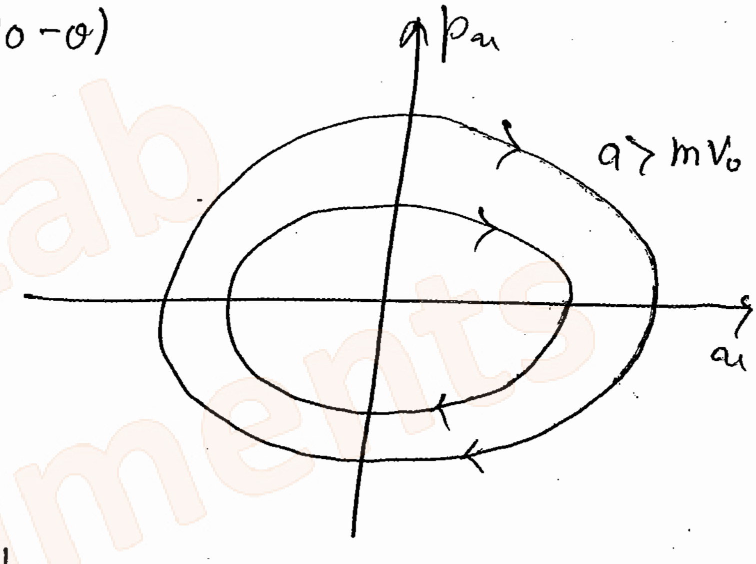
$x = a\cos\theta$

$x = \cos\omega t$

$V_x = -V_0 \cos(90-\theta)$

$\frac{p_x}{m} = -V_0 \sin\theta$

$\frac{p_x}{mV_0} = -\sin\omega t$

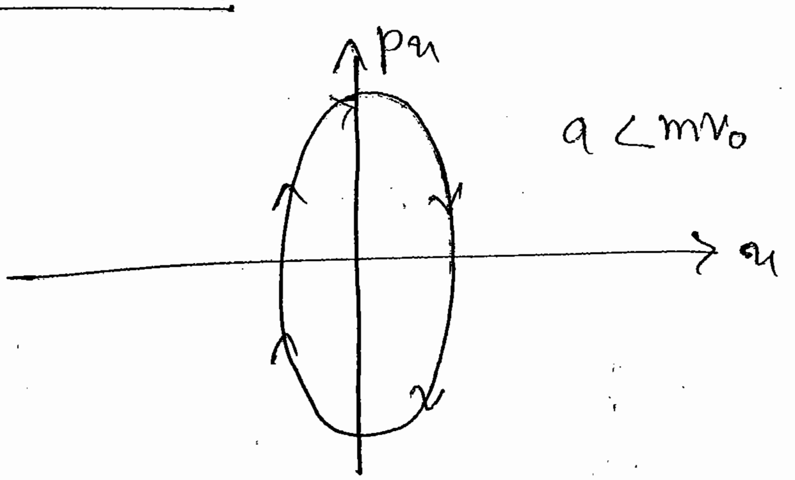


$\frac{x}{a} = \cos\omega t$

$\frac{p_x}{mV_0} = -\sin\omega t$

Square and add

$$\boxed{\frac{x^2}{a^2} + \frac{p_x^2}{(mV_0)^2} = 1}$$ eq^n of ellipse.



* Potential Based Questions:-

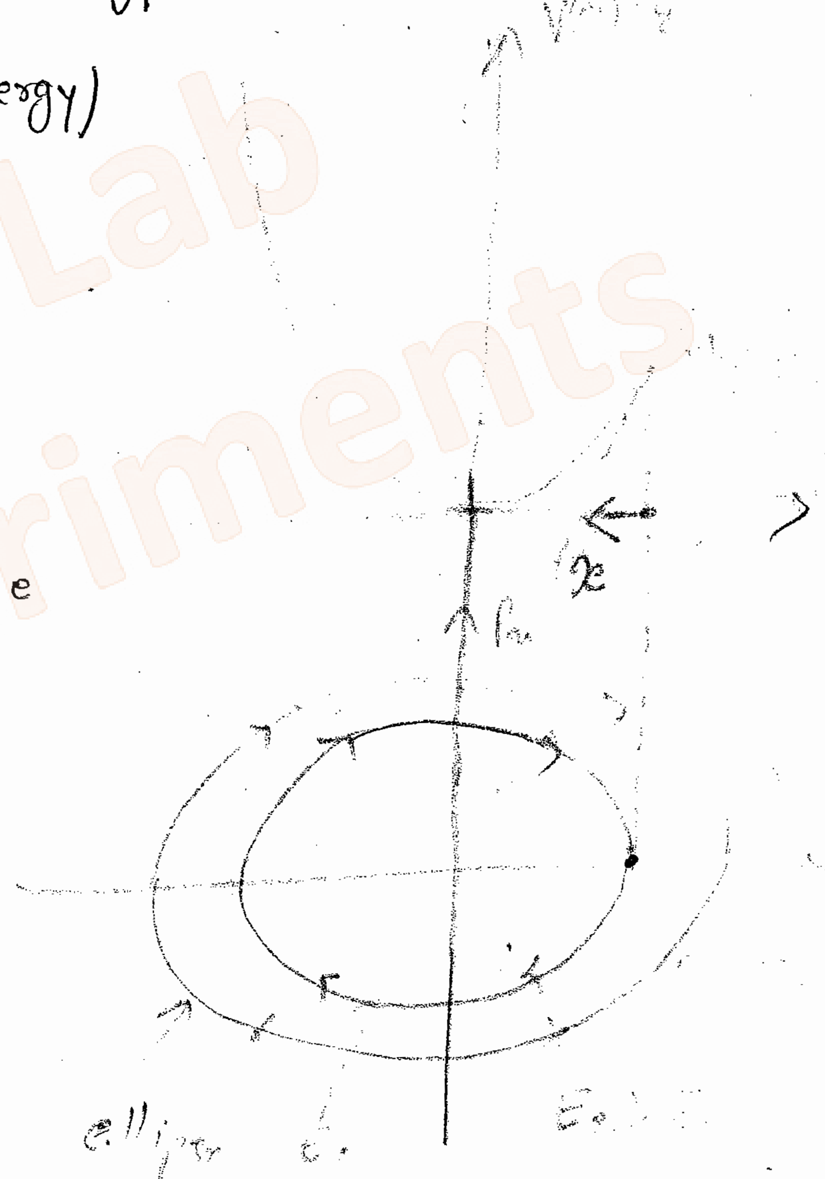
1) Harmonic Oscillator:-

$$H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2 = E \text{ (total Energy)}$$

If potential does not depends on velocity then H represents total energy.

$$H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2 = E \text{ (energy)}$$

$$\frac{p_x^2}{2mE} + \frac{x^2}{\frac{2}{k}E} = 1$$



Here E does not change after

$p_x \to -p_x$

$x \to -x$

$\therefore$ Symmetric about x and p_x both.

Here infinite potential well then phase space lines are closed curve.

* If potential well is infinite then phase space lines <u>inside</u> the well are closed lines if

energy less than height of well. phase space lines are closed lines.

Q. Particle moving under potential $V(x) = ax - bx^2$

Sol^n $H = \frac{p_x^2}{2m} + ax - bx^2 = E = \text{Constant}$

To draw $V(x)$ find zeros

$$V(x) = 0$$

$$ax - bx^2 = 0$$

$$x = 0, \quad x = \frac{a}{b}$$

find extrimum points

$$\frac{dV}{dx} = 0$$

$$a - 2bx = 0$$

$$x = \frac{a}{2b}$$

$$\frac{d^2V}{dx^2} = -2b < 0 \ (\text{max})$$

Q. Draw phase space trajectory for a particle moving under potential.

$$V(x) = ax - bx^3.$$

Sol^n

To draw $V(x)$:-

find zeros :-

$$V(x) = 0$$

$$ax - bx^3 = 0$$

$$x(a - bx^2) = 0$$

$$x = 0 \quad \text{or} \quad a - bx^2 = 0$$

$$x = \pm\sqrt{\frac{a}{b}}$$

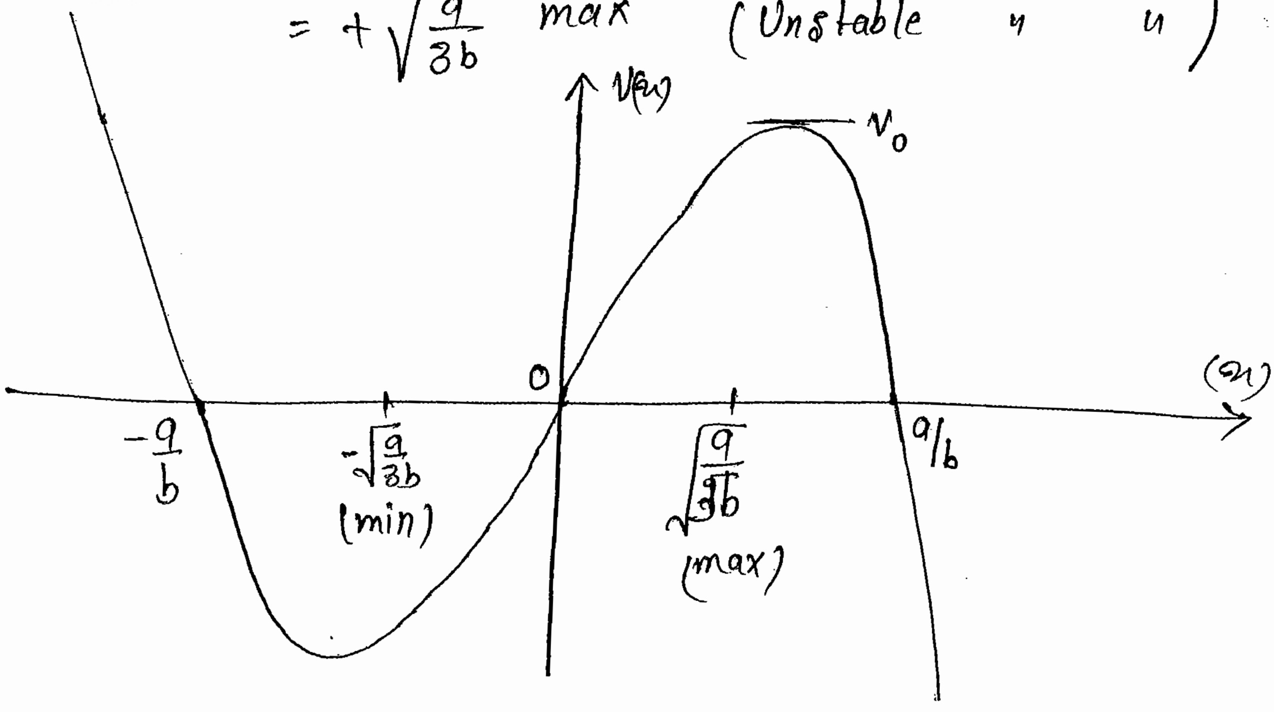
Extrimum points :-

$$\pm\sqrt{\frac{a}{3b}}$$

$$\frac{d^2V}{dx^2} = -6bx$$

$x = -\sqrt{\frac{a}{3b}}$ min (Stable equilibrium point)

$= +\sqrt{\frac{a}{3b}}$ max (Unstable " ")



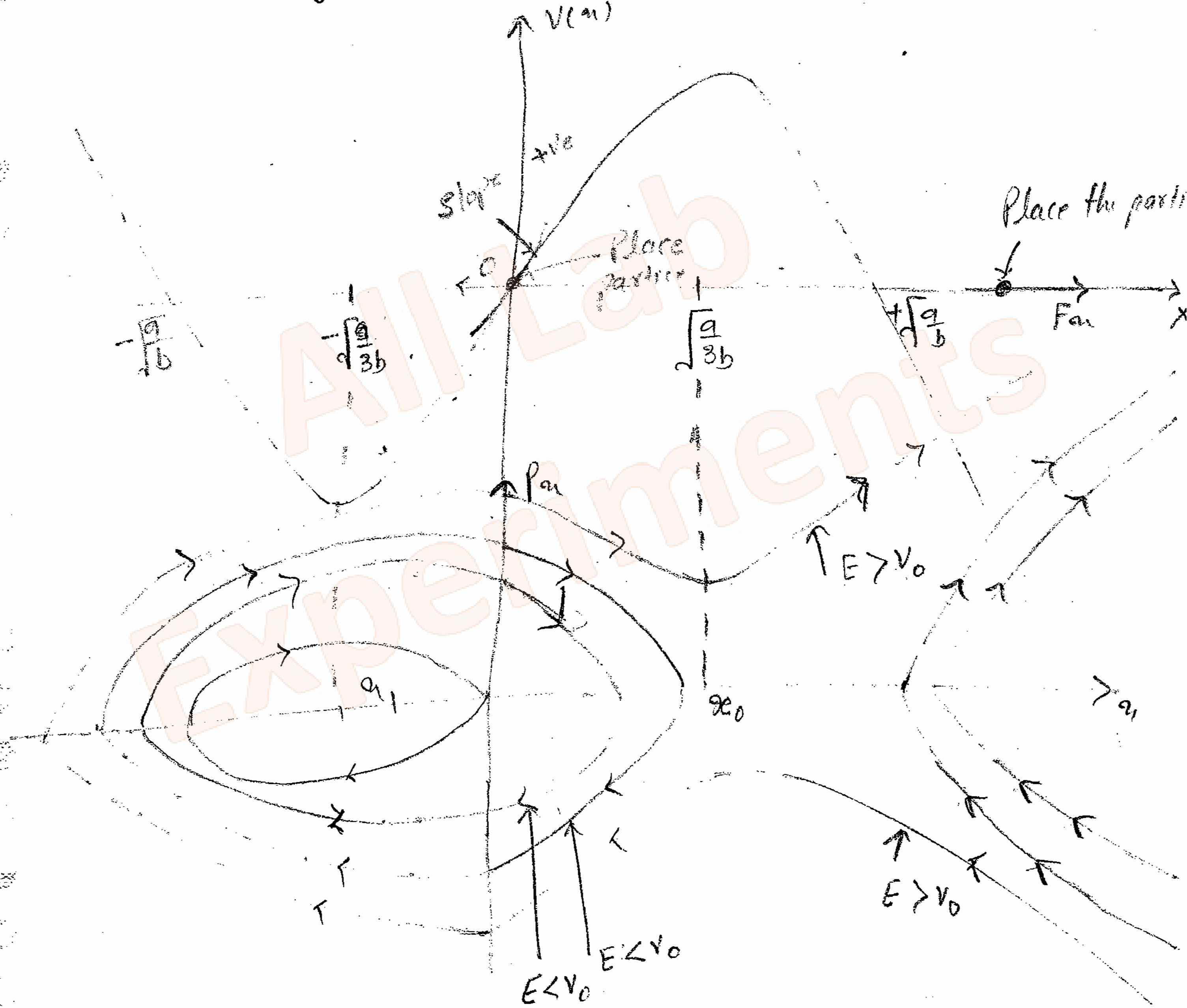
$\therefore \quad H = \frac{p_x^2}{2m} + ax - bx^3 = E$

Here $\quad p_x \to -p_x$

No change so symmetric about x axis

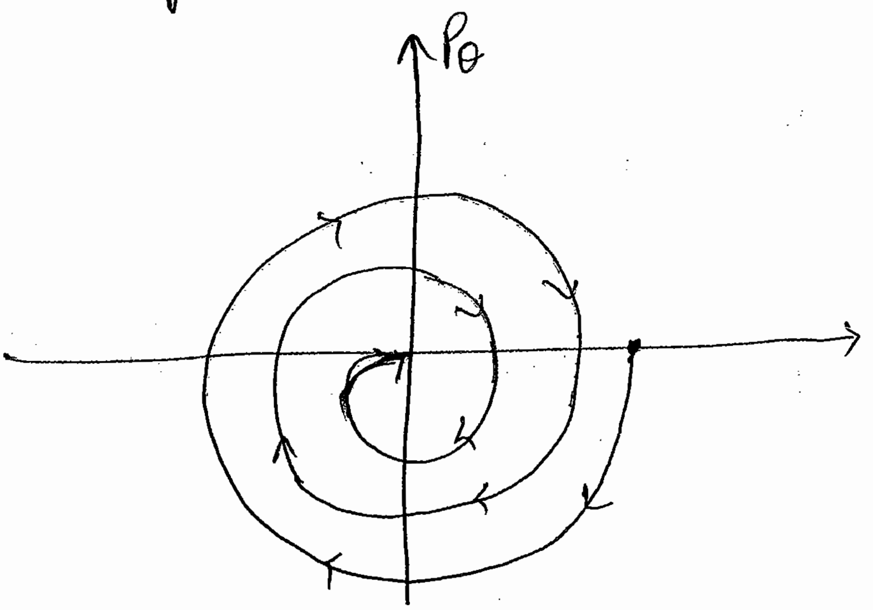
$x \to -x$

Change so does not symmetric about p_x-axis.

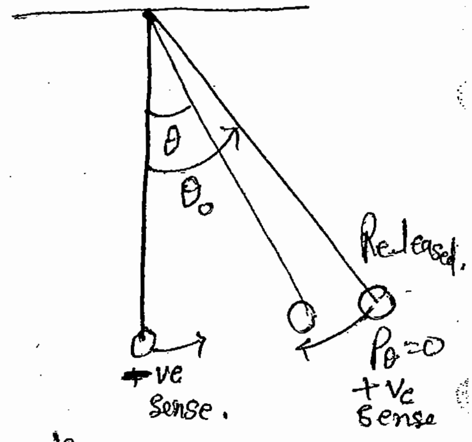


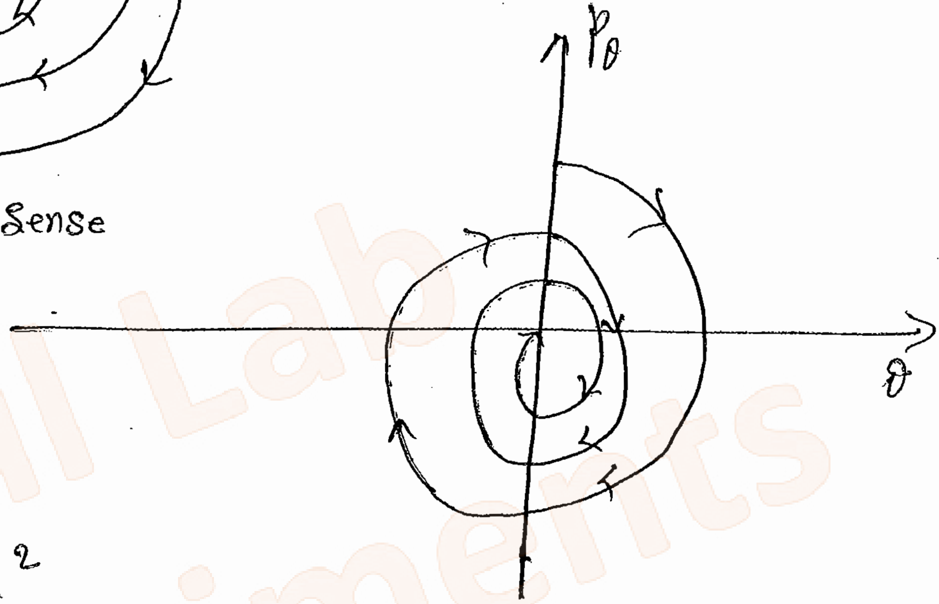
If a particle place a particle outside the potential well.

* Damped Oscillator (Pendulum):



For +ve Sense

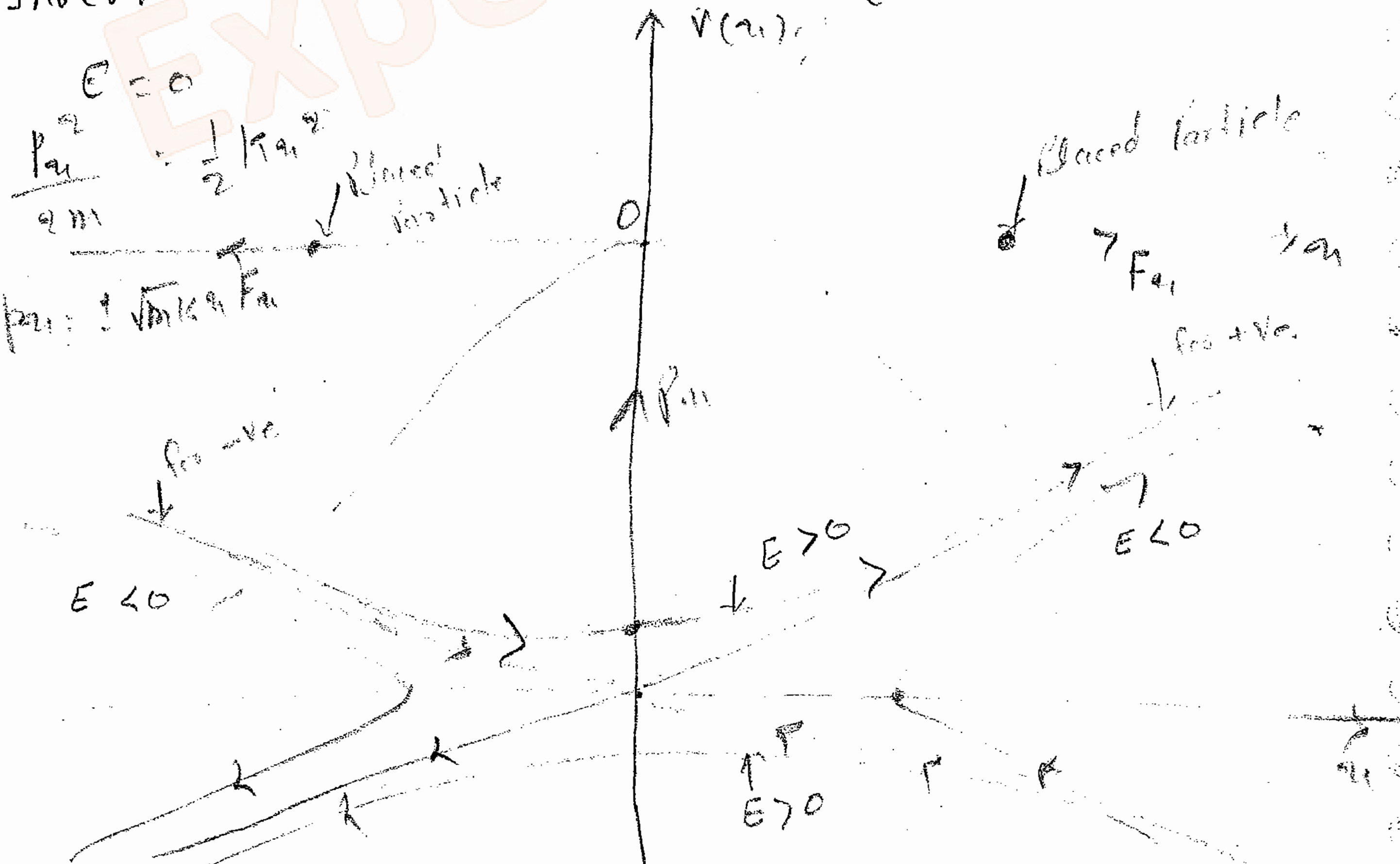




for -ve Sense

$$H = \frac{P_x^2}{2m} - \frac{1}{2}Kx^2$$

Inverted harmonic oscillator (No oscillation).



* Transformation :-

Point Transformation :- It is done in co-ordinate space.

$$(x, y, z) \longrightarrow (r, \theta, \phi)$$

It changes the form of Lagrangian but leave the form of Lagrangian's equation unchanged.

$$x, y, z \longrightarrow r, \theta, \phi$$

$$L \longrightarrow L'$$

$$\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \text{P.E. term} \longrightarrow \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2) - \text{P.E. term.}$$

Eq^n of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \longrightarrow \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{r}}\right) - \frac{\partial L'}{\partial r} = 0$$

* Gauge Transformation of Lagrangian :-

If Lagrangian is changed in following manner

$$L \longrightarrow L' = L \pm \frac{dF}{dt}$$

, where F is function of q_i and t only, Then, Lagrangian equation of motion does not change.

Example :-

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}Kx^2$$

Thus Lagrangian equation of motion.

$$m\ddot{x} + Kx = 0$$

Let $L' = L + \frac{dF}{dt}$ and let $F = \beta x^2$

So $L' = \frac{1}{2} m\dot{x}^2 - \frac{1}{2}kx^2 + 2\beta x\dot{x}$

Equation of motion.

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}(m\dot{x} + 2\beta x) + kx - 2\beta\dot{x} = 0$$

$$m\ddot{x} + \cancel{2\beta\dot{x}} + kx - \cancel{2\beta\dot{x}} = 0$$

$$\boxed{m\ddot{x} + kx = 0}$$

In Guage transformation if we add any term which is total derivative of position and time, the equation of motion remains unchanged.

Note :

Guase transformation of electromagnetic potentials does not change equation of motion because Lagrangian changes by total time derivative of function of $(\vec{r}, t)$.

Consider a particle (non-relativistic) moving in e.m. field (A, ϕ), So Lagrangian is –

$$\boxed{L = \frac{1}{2} mv^2 - q\phi + q\vec{A}\cdot\vec{V}}$$

Guage transformation of e.m. potential.

$$\vec{A}' = A + \vec{\nabla}\lambda(\vec{r}, t)$$

$$\phi' = \phi - \frac{\partial\lambda}{\partial t}$$

$$L' = \frac{1}{2}mv^2 - q\phi + q\vec{A}\cdot\vec{v}$$

$$L' = \frac{1}{2}mv^2 - q\phi + q\frac{\partial \lambda}{\partial t} + q\vec{A}\cdot\vec{v} + q(\vec{\nabla}\lambda)\cdot\vec{v}$$

$$L' = L + q\left[\frac{\partial \lambda}{\partial t} + (\vec{\nabla}\lambda\cdot\vec{v})\right]$$

$\because$ $f(x, y, z, t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial t}$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot(\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}) + \frac{\partial f}{\partial t}$$

$$\boxed{\frac{df}{dt} = \vec{\nabla}f\cdot\vec{v} - \frac{\partial f}{\partial t}}$$ $\longrightarrow$ Total time derivative of any function.

So $$L' = L + q\frac{d\lambda}{dt}$$

$$\boxed{L' = L + \frac{d}{dt}(q\lambda)}$$

So equation of motion will not change.

B.A.2
Q.36

Sol^n $$L = \frac{1}{2}mv^2 + e\vec{A}\cdot\vec{v} - e\phi$$

$$= \frac{1}{2}mv_i^2 + eA_i v_i - e\phi$$

$$p_i = \frac{\partial L}{\partial v_i} = mv_i + eA_i$$

$$\boxed{\vec{p} = m\vec{v} + e\vec{A}}$$

So first option is wrong.

Option (4) is correct. by previous relation.

A-11
Q.10

Sol^n

$L_1 = f(x, \dot{x})$, $L_2 = f(x, \dot{x}) + A(x\dot{y} - y\dot{x})$

$L_2 = L_1 + A(x\dot{y} - y\dot{x})$

$L_3 = f(x, \dot{x}) + A(x\dot{x} + y\dot{y})$

$L_4 = f(x, \dot{x}) + A(x\dot{y} + y\dot{x})$

$L_3 = L_1 + A\frac{d}{dt}(x^2 + y^2)$

Here we can not write this term in the form of total time derivative So L_1 and L_2 can't have same eq^n of motion.

Here we can write total time derivative then L_3 and L_4 have same eq^n of motion.

* Canonical Transformation :- {CT} :-

It is a point transformation in phase space.

CT : $(q_i, p_i) \longrightarrow (Q_i, P_i)$

(q_i, p_i) ↓ Old co-ordinate & momentum

(Q_i, P_i) ↓ New Cordinate & momentum.

$H \longrightarrow H'$

H ↑ Old Hamiltonian

H' ↑ New Hamiltonian.

* In canonical transformation form of Hamiltonian's eqn does not change.

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \;,\; \dot{p}_i = \frac{-\partial H}{\partial q_i} \longrightarrow \dot{Q}_i = \frac{\partial H'}{\partial P_i} \;,\; \dot{P}_i = -\frac{\partial H'}{\partial Q_i}$$

* If form of Hamilton's equation does not change then form of Lagranges eqn of motion will also not change and this happens if Gauge condition is satisfied.

$$\boxed{L' = L - \frac{dF}{dt}(q_i, Q_i, t)}$$

$$\because L = p_i \dot{q}_i - H$$

so

$$\boxed{P_i \dot{Q}_i - H' = p_i \dot{q}_i - H - \frac{dF}{dt}(q_i, Q_i, t)}$$

This is condition for Canonical Transformation

Use $\frac{d}{dt}F(q_i, Q_i, t) = \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial Q_i}\dot{Q}_i + \frac{\partial F}{\partial t}$

$$P_i\frac{dQ}{dt} - H' = p_i\frac{dq_i}{dt} - H - \frac{\partial F}{\partial q_i}\frac{dq_i}{dt} - \frac{\partial F}{\partial Q_i}\frac{dQ_i}{dt} - \frac{\partial F}{\partial t}$$

$$P_i dQ - H'dt = p_i dq_i - H dt - \frac{\partial F}{\partial q_i}dq_i - \frac{\partial F}{\partial Q_i}dQ - \frac{\partial F}{\partial t}dt$$

Equating the coefficients of dq_i, dQ_i, dt from both side we get.

$$\left[p_i - \frac{\partial F}{\partial q_i}\right]dq_i = 0 \Rightarrow p_i - \frac{\partial F}{\partial q_i} = 0$$

$$\Rightarrow \boxed{p_i = \frac{\partial F}{\partial q_i}}$$

and

$$\left[P_i + \frac{\partial F}{\partial Q_i}\right]dQ_i = 0 \Rightarrow P_i + \frac{\partial F}{\partial Q_i} = 0$$

$$\Rightarrow \boxed{P_i = -\frac{\partial F}{\partial Q_i}}$$

and $\boxed{H' = H + \frac{\partial F}{\partial t}}$ —— ⑧

$$\left.\begin{aligned} P_i &= -\frac{\partial F}{\partial Q_i} \\ H' &= H + \frac{\partial F}{\partial t} \\ p_i &= \frac{\partial F}{\partial q_i}(q_i, Q_i, t) \end{aligned}\right\} \quad \text{—— (1)}$$

If $q_i, p_i \longrightarrow Q_i$ is C.T. then we can write a function $F(q_i, Q_i, t)$ such that such that eq^n

(1) will be satisfied. Here $F(q_i, Q_i, t)$ is a generating function.

Now we can use eqⁿ (1) to show that in C.T. Poisson bracket

$$\boxed{\{Q_i, P_i\}_{q_i p_i} = 1 \quad \text{and} \quad \{q_i, p_i\}_{Q_i, P_i} = 1}$$

← for checking function is C.T. or not, we satisfies this poisson bracket. if (1) bracket is satisfies then second also satisfies so we can check easily.

* Jacobian of Transformation:-

Jacobian is defined for all type of function.

Here, in this case - Jacobian of Transformation is one. but it is valid only for canonical form.

$$\boxed{dp_i dq_i = dP_i dQ_i}$$

$$dp_1 dp_2 \ldots dq_1 dq_2 \ldots = dP_1 dP_2 \ldots dQ_1 dQ_2 \ldots$$

For Ex - $x, y, z \longrightarrow r, \theta, \phi$

$$dx, dy dz \longrightarrow dr\, d\theta\, d\phi$$

$$dx\, dy\, dz \longrightarrow J\, dr\, d\theta\, d\phi$$

When $J = r^2 \sin\theta$ is Jacobian of transformation

Q. Consider a transformation $x, p_x \longrightarrow x', p_x'$ such that

$$x' = x\cos\theta - p_x \sin\theta$$

$$p_x' = x\sin\theta + p_x\cos\theta$$

is this transformation a C.T?

P.B. $\{x', p_x'\}_{x, p_x} = 1$

$\Rightarrow (\cos\theta - 0)(0 + \cos\theta) - (-\sin\theta)\sin\theta = 1$

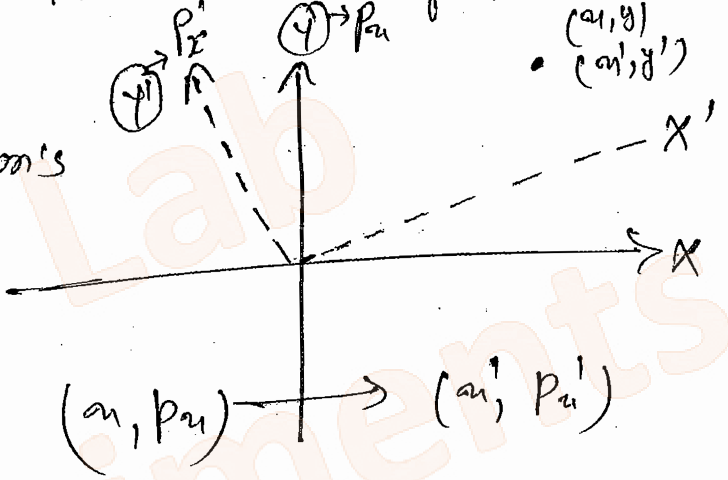
$\cos^2\theta + \sin^2\theta = 1$

$\boxed{1 = 1}$

So given transformation is canonical transformation

Meaning of C.T. is it is a transformation in phase space.

That need the hamilton's eqn of motion remain's unchange.



$x' = x\cos\theta - p_x \sin\theta$

$p_x' = x\sin\theta + p_x\cos\theta$

Note ÷ Why we do this transformation.

$\cancel{H \neq \frac{p^2}{2m} + \frac{1}{2}kx^2 - bx}$

Say $V(x) = \frac{1}{2}kx^2 - bx$

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 - bx$$

$$H = \frac{p^2}{2m} + \frac{1}{2}k\left[\left(x^2 - \frac{2b}{k}x\right) + \left(\frac{b}{k}\right)^2 - \left(\frac{b}{k}\right)^2\right]$$

$$\Rightarrow H = \frac{p^2}{2m} + \frac{1}{2}k\left(x - \frac{b}{k}\right)^2 - \frac{b^2}{2k}$$

$$q_x' = \left(q_x - \frac{b}{K}\right)$$
$$p_x' = p_x$$

C.T. is a general transformation

So $\boxed{E = \hbar\omega\left(n+\frac{1}{2}\right) - \frac{b^2}{2K}}$

We can also solve this ~~transfor~~ problem by perturbation theory but by simplisity we can do this transformation to solve prob. easily. So ~~Th~~ this is a canonical transformation in phase space.

Q. If transformation

$$P = aq + bp$$
$$Q = cq + dp$$

is a C.T. then what is relation b/w constants a, b, c & d ?

Sol^n

$$\{Q, P\}_{q,p} = 1$$

$$cb - da = 1 \Rightarrow \boxed{bc - ad = 1} \text{ Ans.}$$

Q. If transformation $Q = q^\alpha \sin\beta p$ and $P = q^\alpha \cos\beta p$ is a C.T. What is the relation of α and β ?

Sol^n

$$\{Q, P\}_{q,p} = 1$$

$$\{q^\alpha \sin\beta p,\ q^\alpha \cos\beta p\}_{q,p} = 1$$

$$-\alpha q^{\alpha-1}\sin\beta p \cdot \beta q^\alpha \sin\beta p - \beta q^\alpha \cos\beta p \cdot \alpha q^{\alpha-1}\cos\beta p = 1$$

$$-\alpha\beta q^{2\alpha-1} = 1 \cdot q^0 \longrightarrow (1)$$

"If we change q then Right side is not change so left side should also equating the

$$2\alpha - 1 = 0$$

$$\boxed{\alpha = \frac{1}{2}}$$

from ①

$$-\alpha \beta q = 1$$

$$-\frac{1}{2}\beta = 1$$

$$\boxed{\beta = -2}$$

* Generating function ⇒

If is a function of one new variable and one old variable which puts a condition on transformation in phase space to make it canonical.

If $F(q_i, Q_i)$ then it is related to co-ordinates as -

$$p_i = \frac{\partial F}{\partial q_i}(q_i, Q_i)$$

$$P_i = -\frac{\partial F(q_i, Q_i)}{\partial Q_i}$$

$$\boxed{\begin{matrix} q, Q \\ p, Q \\ p, P \\ q, P \end{matrix}}$$

* Legendre Transformation (Mathematics):

This is a most general transformation.

It changes one variable into other variable due to which form of the function changes.

If there is a function which depends on the m.

$x \longrightarrow s$

$$F(x) \longrightarrow G(s)$$

$$\boxed{G(s) = F(x) \pm xs}$$ ← This is Legendree transformation.

Here in mathematics + or −ve sign are taken by maxima or minima. we want $G(s)$ is maximum. So when $F(x)$ is −ve so xs is also −ve when $F(x)$ is +ve then xs is taken as positive.

But in Physics the consideration of +ve or −ve sign's concepts are different from mathematics. which is specified below.

* Generating functions :-

$$q_i \longrightarrow p_i$$

$$F_1(q_i, Q_i) \longrightarrow F_2(p_i, Q_i) \quad \{\text{Legendre Trans.}\}$$

$$F_2(p_i, Q_i) \longrightarrow F_1(q_i, Q_i) \pm q_i p_i$$

Diff. w.r. to q_i which is not present in F_2

$$0 = \frac{\partial F_1}{\partial q_i} \pm p_i$$

we will get previous relation $\left(p_i = \frac{\partial F}{\partial q_i}\right)$ if we use (−)ve sign.

i.e. ⇒ $$\boxed{F_2(p_i, Q_i) \longrightarrow F_1(q_i, Q_i) - q_i p_i}$$

Similarly, $\boxed{F_2(p_i, Q_i) = \underline{F_1(q_i Q_i)} - q_i p_i}$

If we put $F_1(q, Q) = F_2(P_i, Q_i) + q_i p_i$ in the relation.

$$P_i \dot{Q}_i - H' = p_i \dot{q}_i - H + \frac{dF_1(q, Q)}{dt}$$

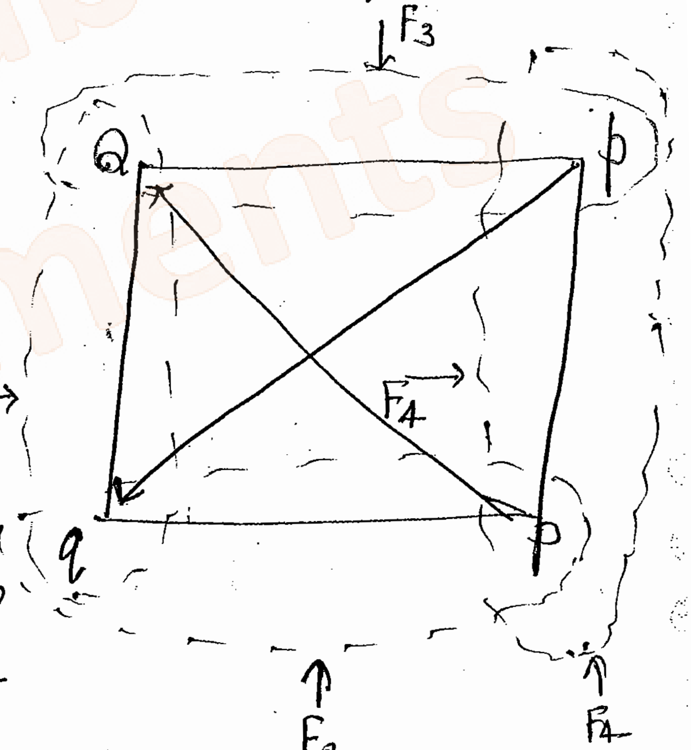
then we get relation for F_2.

Short Trick :- (No physical background)

It is a method to guess relation for different Generating function.

Steps :-

(I) Draw a rectangle $Q\,q\,P\,p$



(II) Choose any two variable side wise.

(III) Relation for Generating function can be obtained as follows -

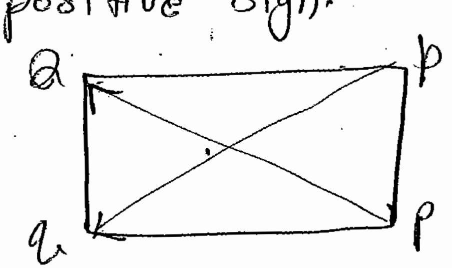
(a) Draw an arrow from the variable which is to be obtained towards diagonally opposite variable.

If arrow points up use negative sign

If arrow points down use positive sign.

Ex → $F_1 = F_1(q, Q)$

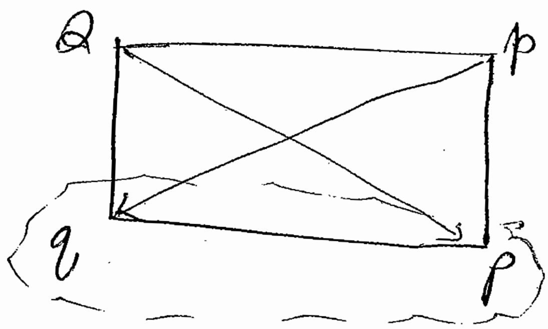
$$p_i = \frac{\partial F_1}{\partial q_i} \quad , \quad P = -\frac{\partial F_1}{\partial Q_i}$$



* Relation for $F_2(q,P)$

$$p = \frac{\partial F_2}{\partial q}$$

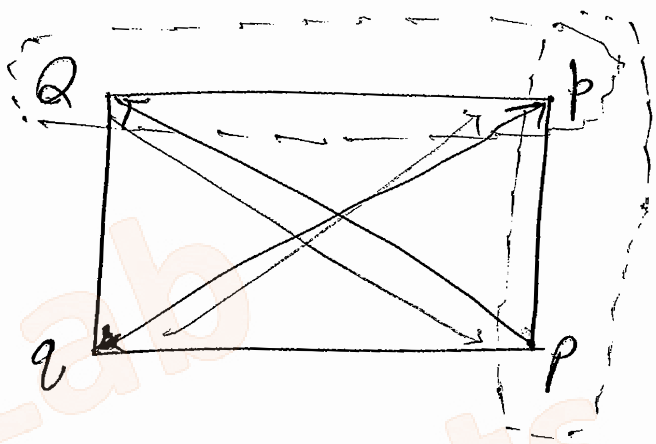
$$Q = \frac{\partial F_2}{\partial P}$$



(*) Relation for $F_3(p, Q)$:-

$$q = -\frac{\partial F_3}{\partial p}$$

$$P = -\frac{\partial F_3}{\partial Q}$$



$$Q = \frac{\partial F_4}{\partial P}$$

$$q = -\frac{\partial F_4}{\partial p}$$

---

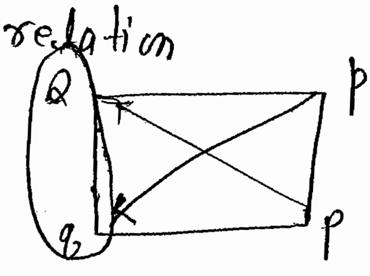
*Imp How to obtain generating function from a given transformation :-

① If we have to obtain $f(q, Q)$

② Then use its relation

$$p = \frac{\partial F}{\partial q}$$

$$P = \frac{-\partial f}{\partial Q}$$



③ Express L.H.S. of above relation in trems of chosen variables by using given transformation relation.

④ Integrate the relation for F.

⑤ Combine the two value of f obtain (write common ter once only).

BA-3

Q.46

Sol^n

$$(q, p) \longrightarrow (Q, P)$$

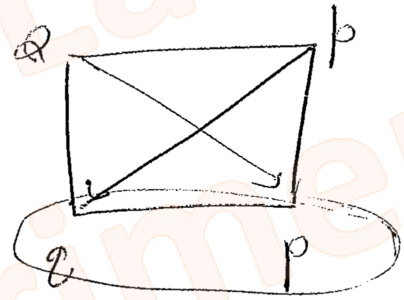
$$Q = q^2$$

$$P = \frac{p}{2q}$$

We have to find $F(q, P)$

$$Q = \frac{\partial F}{\partial P} \quad \text{—(I)}$$

$$p = \frac{\partial F}{\partial q} \quad \text{—(II)}$$



from (I)

$$q^2 = \frac{\partial F}{\partial P}$$

$$F = q^2 P + G_1(q) \quad \text{—(III)}$$

from (2)

$$p = \frac{\partial F}{\partial q}$$

$$2Pq = \frac{\partial F}{\partial q}$$

Integrate :-

$$F = q^2 P + G_2(P) \quad \text{— (IV)}$$

Using (III) and (IV)

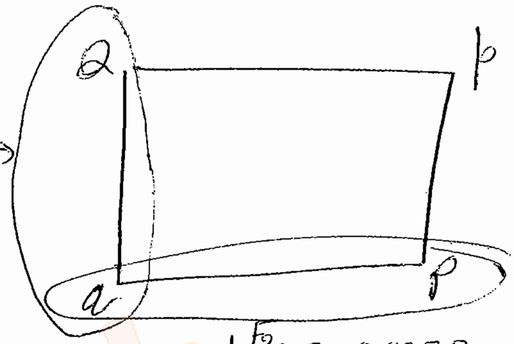
$$\boxed{F = q^2 P}$$

Ans

Q. 2

Sol^n Identity Transformation -

$$Q = q$$
$$P = p$$



$f_1(q, Q)$

$$p = \frac{\partial F(q,Q)}{\partial q}$$

$$P = \frac{-\partial F}{\partial Q}$$

Here we can not express F_2 in terms of choosen variable.

So we can not solve it -

$$p = \frac{\partial F_2(q,P)}{\partial q}$$

$$Q = \frac{\partial F_2}{\partial P}$$

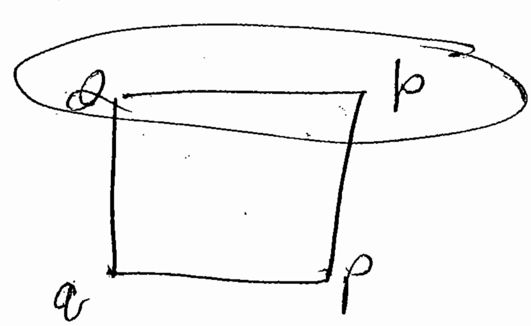
Q. Which of the following G.F. gives identity transformation

$F_1 = qP$, $F_2 = -Qp$

(a) Only F_1 (b) only F_2 (c) both (d) non

Sol^n
$$P = \frac{-\partial F_2}{\partial Q_1} = p$$

$$q = -\frac{\partial F_2}{\partial p} = Q$$



Q.5

Sol^n Given $H = \frac{p^2}{2m} + mgq$

$F(q, Q, t) = \frac{1}{3m^2g}\left[2m^2g(Q-q)\right]^{3/2} = \frac{1}{3m^2g}(2m^2g)^{3/2} \times (Q-q)^{3/2}$

Here F is not depends on time explicitely.

$H'(P,Q) = H(p,q) + \underbrace{\left(\frac{\partial F}{\partial t}\right)}_{0}$ ['F' is not explicit function of t]

So $\boxed{H' = H}$

$H = (P,Q) = \frac{p^2}{2m} + mgq$

Express p & q in new variable

Relation of G.F. :-

$p = \frac{\partial F}{\partial q} = \frac{-1}{3m^2g}(2m^2g)^{3/2}\frac{3}{2}(Q-q)^{1/2}(-1)$

$P = \frac{-\partial F}{\partial q} = +\frac{1}{3m^2g}(2m^2g)^{3/2}\frac{3}{2}(Q-q)(1)$

So $\boxed{p = P}$

$P = \frac{1}{\cancel{3}m^2g}(2m^2g)^{3/2}\cancel{\frac{3}{2}}(Q-q)^{1/2}$

Squairing on both side :-

$p^2 = \frac{1}{4a^2g^2} \cdot 8m^6g^3(Q-q).$

$$\boxed{q = Q - \frac{p^2}{2m^2 g}}$$

So New hamiltonian is -

$$H'(PQ) = \frac{p^2}{2m} + mgq$$

$$\boxed{H'(PQ) = \frac{p^2}{2m} + mg\left(Q - \frac{p^2}{2m^2 g}\right)}$$

A-12

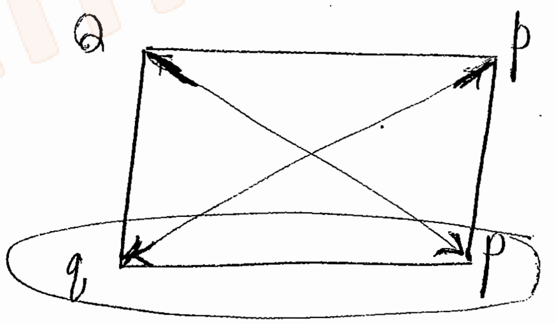
Q.11 Find generating function $F(P, q)$ for transformation $p = \frac{1}{Q}$, $q = PQ^2$ is -?

(a) $\sqrt{Pq}$ (b) $-\sqrt{Pq}$ (c) $2\sqrt{Pq}$ (d) $-2\sqrt{Pq}$.

Sol^n Given transformation relation is -

$$p = \frac{1}{Q}$$

$$q = PQ^2$$



$$p = \frac{\partial F(q, P)}{\partial q} \quad \text{—(I)}$$

$$Q = \frac{\partial F(q, P)}{\partial P} \quad \text{—(II)}$$

$$Q = \sqrt{\frac{q}{P}}$$

$$p = \frac{1}{Q}$$

$$p = \sqrt{\frac{P}{q}}$$

from (I) relation -

$$\sqrt{\frac{P}{q}} = \frac{\partial F}{\partial q}$$

$$\sqrt{P}\, 2\sqrt{q} + c_1(P) = F \quad \text{—(a)}$$

from (II)

$$\sqrt{\frac{q}{P}} = \frac{\partial F}{\partial P} \Rightarrow F = \sqrt{q}\,.\, 2\sqrt{P} + c_2(q) \quad \text{—(b)}$$

So from (a) & (b) $\boxed{F = 2\sqrt{Pq}}$ Ans

Q9 Values of a and b for which following transformation $Q=(2q)^a \cos^b p$, $P=(2q)^a \sin p$ is canonical are –

(a) $a=\frac{1}{2}, b=1$ (b) $a=2, b=\frac{1}{2}$ (c) $a=1, b=1$, (d) $a=\frac{1}{2}, b=\frac{1}{2}$

Sol^n for canonical possion bracket should be one.

So $\{Q,P\}_{q,p}=1$

$\Rightarrow \{(2q)^a \cos^b p, (2q)^a \sin^b p\}_{q,p}=1$

$2a(2q)^{a-1}\cos^b p \cdot (2q)^a b \sin^{b-1} p \cos p + (2q)^a b \cos^{b-1} p \sin p \times 2a(2q)^a \sin^b p = 1$

$\Rightarrow 2ab(2q)^{2a-1}\cos^{b-1} p \sin^{b-1} p\,[\cos^2 p+\sin^2 p]=1$

$2ab\,(2q)^{2a-1}(\cos p \sin p)^{b-1}=1$

$\because$ R.H.S is co-ordinate independent $\therefore$ L.H.S. should also be co-ordinate independent.

So $2a-1=0$ and $b-1=0$

$\boxed{a=\frac{1}{2}}$ $\boxed{b=1}$

* Hamilton Jacobi theory:

In this theory a canonical transformation is done in such a way that new hamiltonian becomes zero. And G.F. for this C.T. is chosen to be $F[q, P]$

$$q, p \xrightarrow{C.T.} Q, P$$

$$H \xrightarrow[F(q,P)]{} H' = 0$$

Hamiltons equation in new co-ordinate -

$$\dot{Q} = \frac{\partial H'}{\partial P} = 0 \qquad \{\because H' = 0$$

$$\dot{P} = -\frac{\partial H'}{\partial Q} = 0$$

$\Rightarrow$ Q = constant

& P = constant

Importance of $F(q, P, t)$ in H.J. theory.

Total time derivative of F.

$$\frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial P}\underbrace{\dot{P}}_{=0} + \frac{\partial F}{\partial t}$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial t}$$

relation for $f(q, P, t)$

$$p = \frac{\partial F}{\partial q}$$

$$Q = \frac{\partial F}{\partial P}$$

$$\underset{0}{\overset{\parallel}{H'}} = H + \frac{\partial F}{\partial t}$$

$$\frac{\partial F}{\partial t} = -H$$

$$\frac{dF}{dt} = p\dot{q} - H$$

$$\boxed{\frac{dF}{dt} = L} \leftarrow \text{Lagrangian}$$

$$\boxed{F = \underbrace{\int L\,dt}_{\text{Action }(S)}} = S$$

In H-J theory F is equal to action (S).

Imp. H-J Equation :-

In Hamilton every place of p we will write -

$$\boxed{H + \frac{\partial S}{\partial t} = 0} \qquad \boxed{p = \frac{\partial S}{\partial q}}$$

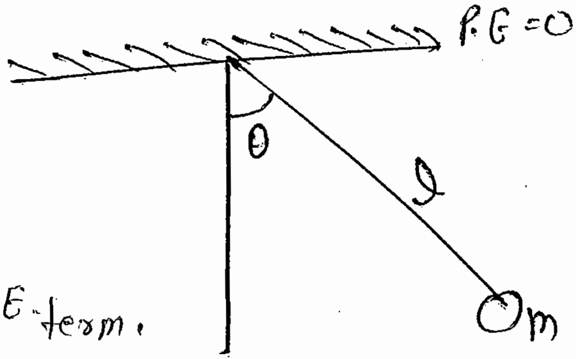
Q. Write hamilton-Jacobi eq^n for simple pendulum.

(r, θ)

$r = l$

$\dot{r} = 0$

$p_r = 0$



$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \text{P.E. term.}$$

$$= \frac{p_\theta^2}{2ml^2} - mgl\cos\theta$$

$$p_\theta = \left(\frac{\partial S}{\partial \theta}\right)$$

So H.J. eq^n for simple pendulum

$$H + \frac{\partial S}{\partial t} = 0$$

$$\boxed{\frac{1}{2ml^2}\left(\frac{\partial S}{\partial \theta}\right)^2 - mgl\cos\theta + \frac{\partial S}{\partial t} = 0}$$ *

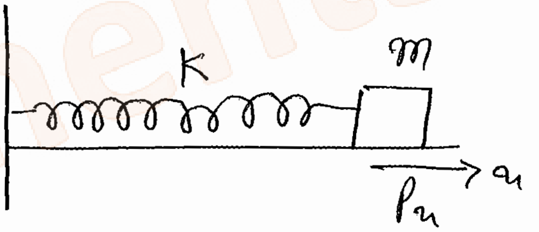
⇓

Non-linear P.D.E.

(Solution of Non-linear partial diff. eq^n can be found by variable separation on sumation not multiplication)

* H-J Equation for S.H.O. :-

$$H + \frac{\partial S}{\partial t} = 0 \quad , \quad p = \frac{\partial S}{\partial q}$$



$$H = \frac{p_x^2}{2m} + \frac{1}{2}Kx^2$$

$$= \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}Kx^2$$

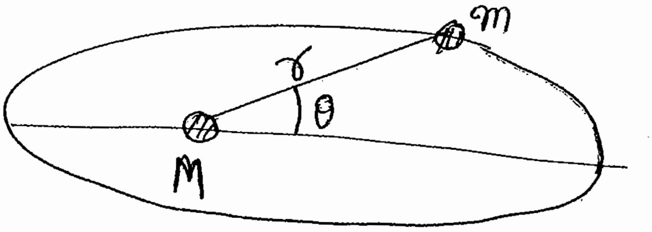
Here H-J eq^n -

$$\boxed{\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}Kx^2 + \frac{\partial S}{\partial t} = 0}$$

Ques: H-J equation for planatary motion.

Sol^n:-

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r}$$



$$p_r = \left(\frac{\partial S}{\partial r}\right), \quad p_\theta = \frac{\partial S}{\partial \theta}$$

$$\boxed{\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta}\right)^2 - \frac{GMm}{r} + \frac{\partial S}{\partial t} = 0}$$

Non-Linear P.D.E.

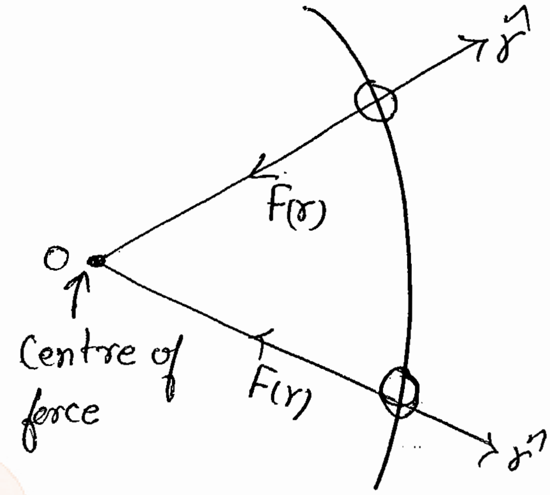
$$S = S(r, \theta, t)$$

$$S = S_1(r) + S_2(\theta) + S_3(t).$$

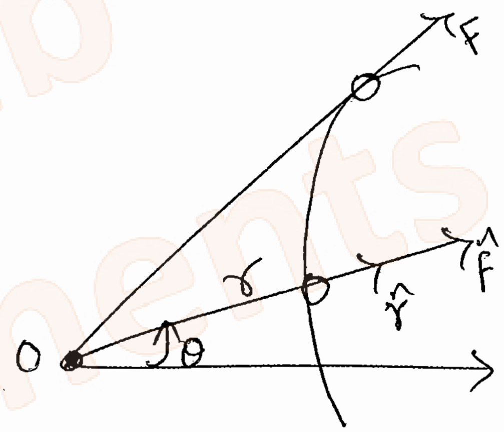
∴ Central forces Motion ∴

force is directed towards or away from a point is central force.

$$\boxed{F = \pm f(r)\,\hat{r}}$$



- When $\vec{F}$ and $\hat{r}$ is in same direction then we use +ve sign.
- When $\vec{F}$ and $\hat{r}$ is in opposite direction then we use −ve sign.



* Properties :-

$$\vec{\nabla} \times \vec{F} = 0$$

⇒ Central force is conservative.

⇒ Total energy is conserved.

$\boxed{\vec{\tau}_0 = 0}$, $\boxed{\vec{L}_0 \text{ is conserved.}}$

When angular momentum is conserv then it must be two directional problem.

$\vec{F}$ is conservative. So we can define P.E. (V(r))

$$\boxed{V(r) = -\int f(r)\,dr}$$

, f(r) is +ve for repulsive force

, f(r) is −ve for attractive force

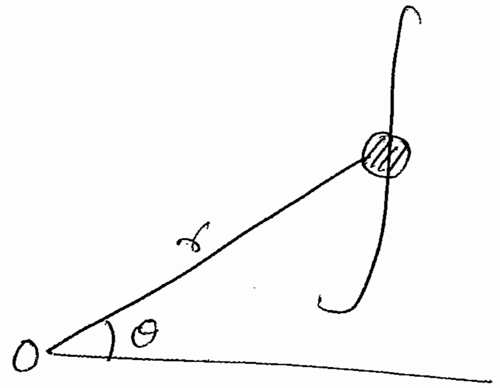
$$\vec{f} = -\vec{\nabla} V(r)$$

$$\boxed{f = -\frac{\partial}{\partial r} V(r)}$$

Equation of motion under central force :-

$$L = T - V$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$



Here L is independent of t so total energy is conserved and L is independent of θ so P_θ (angular momentum) is conserved.

Equation of motion :-

r- equation :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$m\ddot{r} - mr\dot{\theta} + \frac{\partial V}{\partial r} = 0$$

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 - f(r) = 0}$$ Imp. ← radial eqⁿ of motion.

θ- equation :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}(mr^2\dot{\theta}) - 0 = 0$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$$\boxed{mr^2\dot{\theta} = \text{Constant}}$$ Imp

$$\boxed{2\dot{r}\dot{\theta} + r\ddot{\theta} = 0}$$ Imp.

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \boxed{mr^2\dot{\theta} = l}$ — Angular momentum.

$\boxed{\dot{\theta} = \frac{l}{mr^2}}$

Q. Transverse velocity of a particle under central force varies with r as –

(a) $\frac{1}{r}$ (b) $\frac{1}{r^2}$ (c) r (d) $\frac{1}{r^2}$

Sol^n Transverse velocity $= r\dot{\theta}$ $\left\{ \because \dot{\theta} = \frac{l}{mr^2} \right\}$

$= r \cdot \frac{l}{mr^2}$

$\left\{ K.E. = \frac{1}{2} m (\underbrace{\dot{r}^2}_{\text{Radial Velocity}} + \underbrace{r^2\dot{\theta}^2}_{\text{Trans velocity}}) \right.$

$= \frac{l}{m} \cdot \frac{1}{r}$

Hence transverse velocity $\propto \frac{1}{r}$

* Equation of path in central force motion :-

↓

is independent of ~~path~~ time.

remove t from equation of motion.

Introduce $u = \frac{1}{r}$

This leads to following eq^n

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{mf}{l^2u^2}}$$

⇓

Diff. eq^n of path

$u = \frac{1}{r} \Rightarrow r = \frac{1}{u}$

$\dot{r} = -\frac{1}{u^2}\frac{du}{dt}$

$= -\frac{1}{u^2}\frac{du}{d\theta} \cdot \frac{d\theta}{dt}$

$= -r^2 \frac{du}{d\theta} \cdot \dot{\theta}$

$= -r^2 \frac{du}{d\theta} \cdot \frac{l}{mr^2}$

$= -l \frac{du}{d\theta}$

Use :- It is use to find $f(r)$ when relation b/w r and θ is given.

Q. If $r = a\cos\theta$ for a particle moving under central force what is force law?

Sol^n

$$u = \frac{1}{r}$$

$$u = \frac{1}{a}\sec\theta$$

$$\frac{du}{d\theta} = \frac{1}{a}\sec\theta\tan\theta$$

$$\frac{d^2u}{d\theta^2} = \frac{1}{a}\left[\sec\theta\tan^2\theta + \sec^3\theta\right]$$

Differential equation of path -

$$\frac{d^2u}{d\theta^2} + u = \frac{-mf}{l^2u^2}$$

$$u\tan^2\theta + u\sec^2\theta + u = \frac{-mf}{l^2u^2}$$

$$2u\sec^2\theta = \frac{-mf}{l^2u^2}$$

$$f = -\frac{2l^2u^3\sec^2\theta}{m}$$

$$f = -\frac{2l^2a^2}{m}u^5$$

$$f = \frac{-2l^2a^2}{m}\frac{1}{r^5}$$

$$\boxed{f \propto \frac{1}{r^5}} \leftarrow \text{Force Law}$$

Note (i) If $r = a\sin\theta$

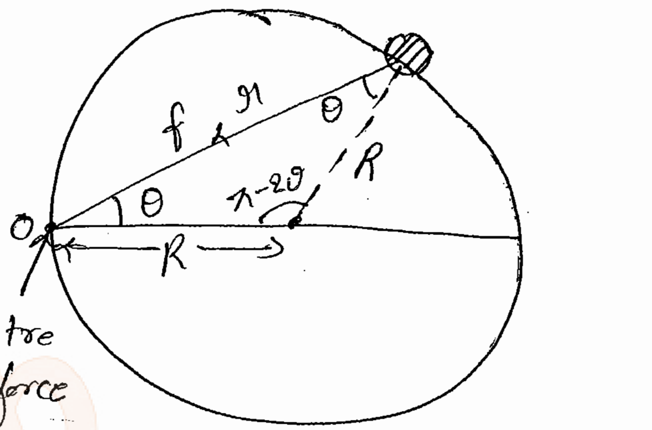
$$\Rightarrow \boxed{f \propto \frac{1}{r^5}}$$

(ii) if $r^n = a\cos n\theta$ or $r^n = a\sin n\theta$ $\boxed{f \propto \frac{1}{r^{2n+3}}}$

Imp.

Q. A particle is moving in circle of radius R under a force which is always directed towards a point on periferi on circle (i) What is the force low. (ii) What is total energy.

Sol^n



By sine low

$$\frac{r}{\sin(\pi-2\theta)} = \frac{R}{\sin\theta}$$

$$\frac{r}{\sin 2\theta} = \frac{R}{\sin\theta} \Rightarrow \frac{r}{2\sin\theta\cos\theta} = \frac{R}{\sin\theta}$$

$$\boxed{r = 2R\cos\theta}$$

Q. A particle is moving under a central force if eq^n of path of the particle is $r = Ae^{K\theta}$ What is Potential Energy of the particle.

Sol^n

$r = Ae^{K\theta}$ is eq^n of path becoz this relation b/w two co-ordinate and also it is time independent. when it is written in differential form then it is Kla differential eq^n of path.

Ans:- $f \propto \frac{1}{r^3}$

$$V(r) = \frac{-(K^2+1)l^2}{2mr^2}$$

T.E. = 0

For total energy:-

T.E. = K.E. + P.E.

$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + P.E.$

$r = Ae^{K\theta}$

$\dot{r} = KAe^{K\theta}\dot{\theta}$

$\dot{r} = Kr\dot{\theta}$

$\dot{\theta} = \frac{l}{mr^2}$

* Effective Potential – [Potential Energy]:-

It is introduce to convert 2-D problem (in central force motion) in to 1-D problem.

Total Energy –

$$E = K.E + P.E. \Rightarrow E = \frac{1}{2}mv^2 + V(r)$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \underbrace{V(r)}_{\text{Actual P.E.}}$$

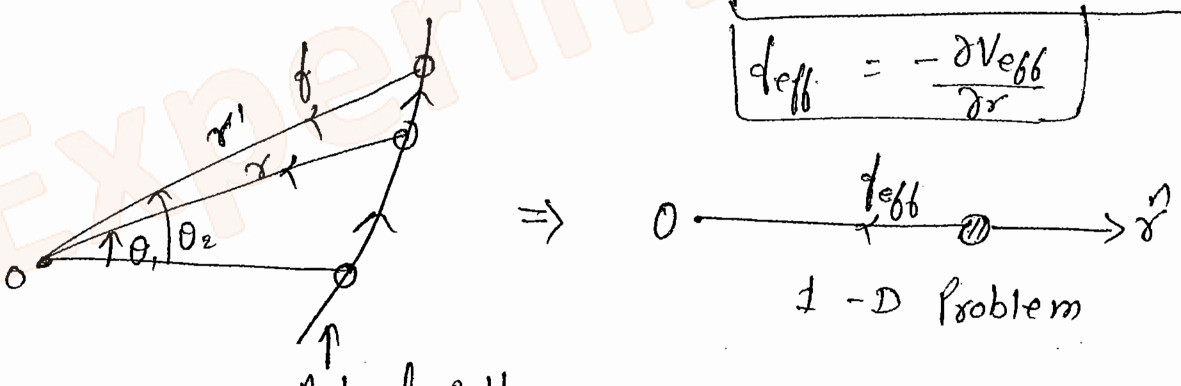
Put $\dot{\theta} = \frac{l}{mr^2}$

$$E = \underbrace{\frac{1}{2}m\dot{r}^2}_{K.E.} + \underbrace{\frac{l^2}{2mr^2} + V(r)}_{\text{P.E. or Eff. Pot.}}$$

$$\boxed{E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)}$$

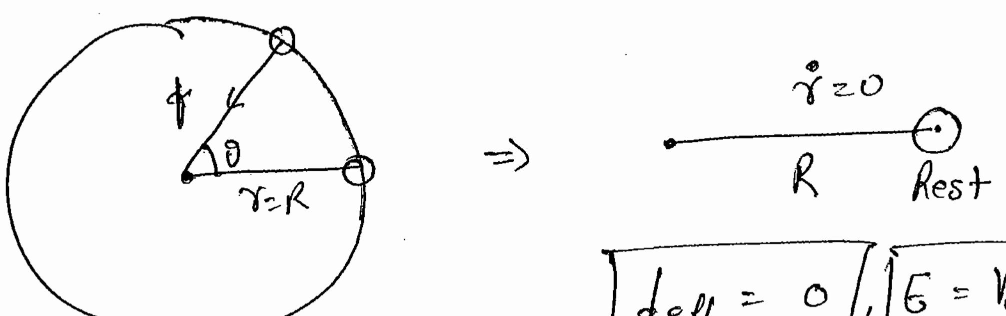
$$\boxed{V_{eff} = V(r) + \frac{l^2}{2mr^2}}$$

$$\boxed{f_{eff} = -\frac{\partial V_{eff}}{\partial r}}$$



2-D Problem

1-D Problem

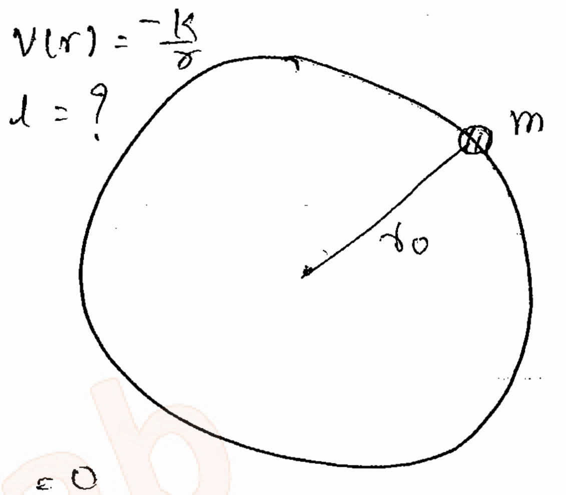


2-D circular motion

$$\boxed{f_{eff} = 0}, \quad \boxed{E = V_{eff}}$$

Q.34

Sol^n It is circular motion



$\therefore \quad f_{eff} = 0$

$$\frac{\partial V_{eff}}{\partial r} = 0$$

$$\frac{\partial}{\partial r}\left(\frac{-K}{r} + \frac{l^2}{2mr^2}\right)_{r=r_0} = 0$$

$$\left[\frac{+K}{r^2} + \left(-\frac{l^2}{mr^3}\right)\right]_{r=0} = 0$$

$$\frac{K}{\cancel{r_0^2}} = \frac{l^2}{mr_0^{\cancel{3}}}$$

$$l^2 = Kmr_0$$

$$\boxed{l = \sqrt{Kmr_0}}$$ Ans

B.A.5
Q.20

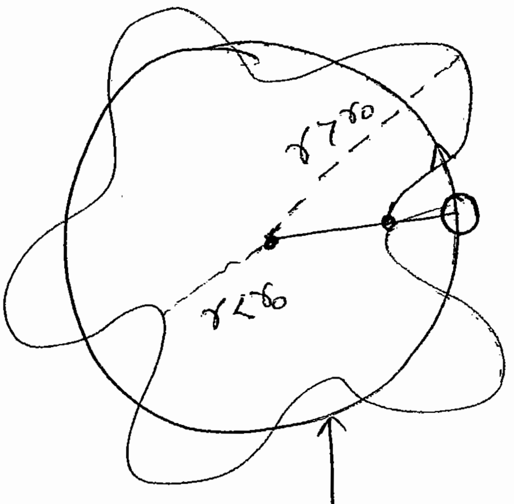
Sol^n

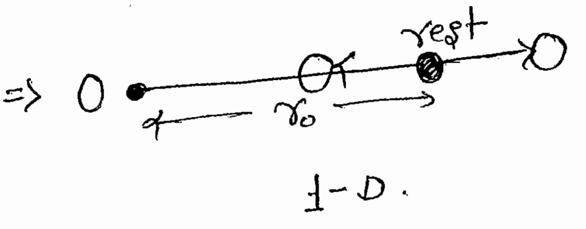
Given $\Rightarrow \frac{l_1}{l_2} = 2$ $\qquad V(r) = \frac{1}{2}kr^2$

$$\frac{r_1}{r_2} = ?$$

* Oscillation about Stable State :-

A stable orbit is circular orbit because $(f_{eff} = 0)$





2-D. stable circular Orbit

Frequency of Oscillation :-

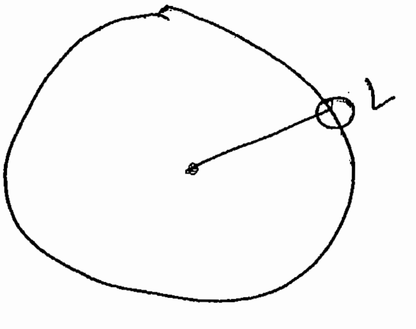
$$\boxed{\omega = \sqrt{\frac{\text{force Constant}}{m}}}$$

Where ω = Radial freq. of angular oscillation.

$$\boxed{\text{Force Constant} = \frac{d^2}{dr^2} V_{eff} \Big|_{r=r_0}}$$

Where r_0 = radius of stable circular orbit.

8.A-5
Q.32



Sol^n $\quad V(r) = \frac{-K}{r}$

$$V_{eff} = V(r) + \frac{L^2}{2mr^2}$$

$$V_{eff} = \frac{-K}{r} + \frac{L^2}{2mr^2}$$

Let r_0 be radius of stable circular orbit.

$$\therefore \quad f_{eff} = 0$$

$$\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_0} = 0$$

$$\frac{K}{r_0^2} - \frac{L^2}{mr_0^3} = 0$$

$$\boxed{r_0 = \frac{L^2}{mK}}$$

Hence force constant $= \left.\frac{d^2 V_{eff}}{dr^2}\right|_{r=r_0}$

$$= \frac{-2K}{r_0^3} + \frac{3L^2}{mr_0^4}$$

$$= \frac{1}{r_0^3}\left(-2K + \frac{3L^2}{mr_0}\right)$$

$$= \frac{(mK)^3}{L^6}\left(-2K + \frac{3\cancel{L^2} \times \cancel{m}K}{\cancel{m}\cancel{L^2}}\right)$$

$$= \frac{m^3K^3}{L^6}(-2K + 3K) = \frac{m^3K^4}{L^6}$$

Hence angular frequency :-

$$\omega = \sqrt{\frac{\text{force Constant}}{m}}$$

$$= \sqrt{\frac{m^{8} K^{4}}{L^{6}\, m}} = \frac{mK^2}{L^3}$$

$$\boxed{\omega = \frac{mK^2}{L^3}}$$ Ans

Q. Graph of 'V_{eff}' V/s 'r' :-

$$V_{eff} = V(r) + \frac{l^2}{2mr^2}$$

Consider following two limits

$r \to 0$, $r \to \infty$ and draw line in these two limits.

in most of the cases ✓

[If $r \to 0$ then higher power in denomenator dominates
[If $r \to \infty$ " Lower " " " "

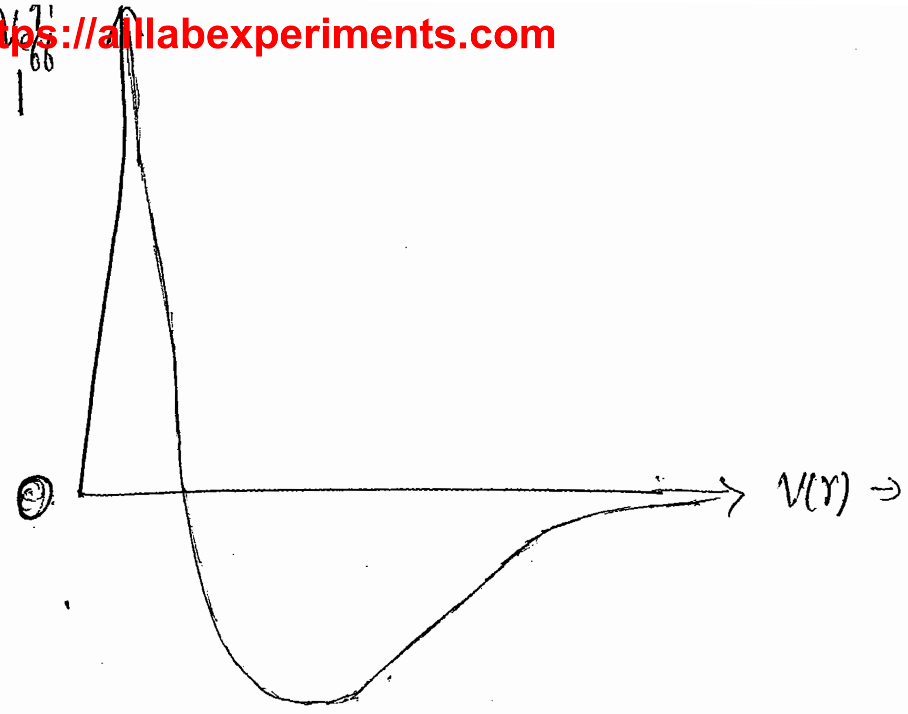
× [If $r \to 0$ Lowe power " " numerator "
[If $r \to \infty$ higher " " " "

Ques Plot V_{eff} for $V(r) = \frac{-K}{r}$

$$V_{eff} = \frac{-K}{r} + \frac{l^2}{2mr^2}$$

$r \to 0$, $V_{eff} \to \infty$

$r \to \infty$, $V_{eff} \to 0$ (from $-$ve side)



Q. V_{eff} for $V(r) = \frac{-K}{r^3}$

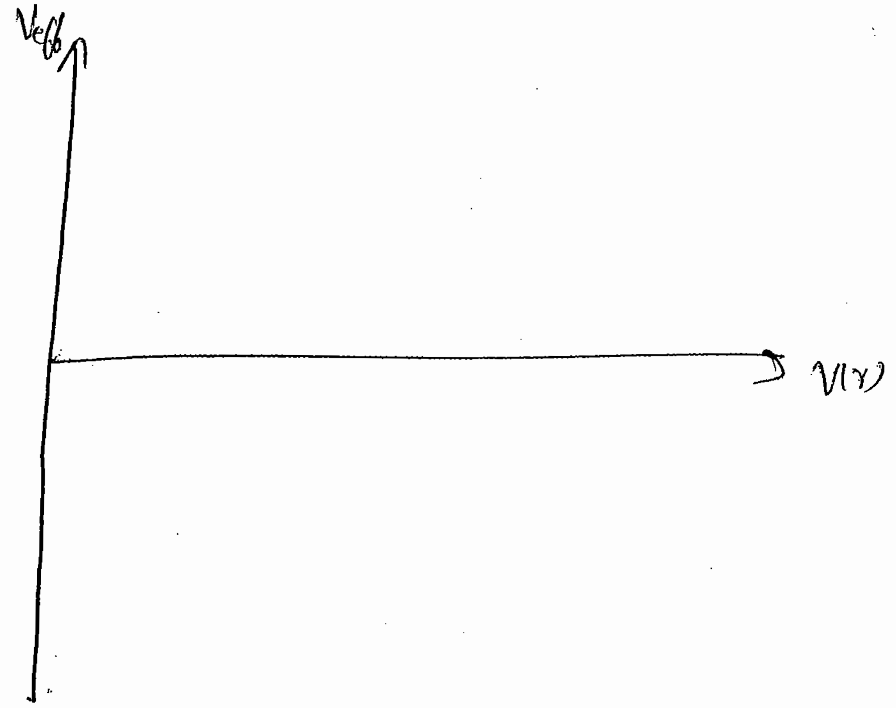
Soln

$$V_{eff} = V(r) + \frac{L^2}{2mr^2}$$

$$V_{eff} = -\frac{K}{r^3} + \frac{L^2}{2mr^2}$$

$r \to 0$, $V_{eff} \to -\infty$

$r \to \infty$, $V_{eff} \to 0$ (from +ve side

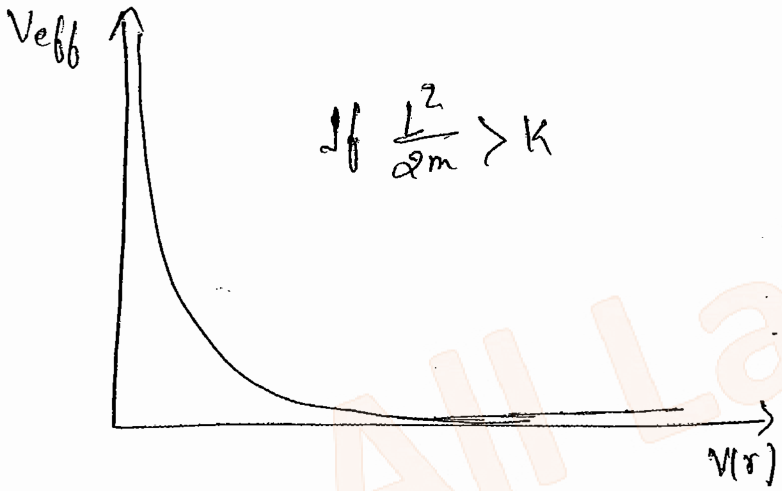


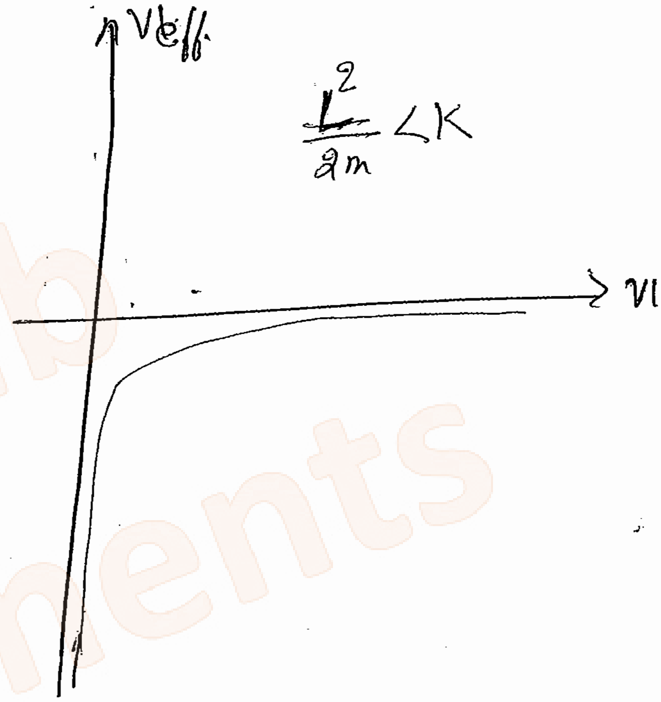
Q. $V(r) = \frac{-K}{r^2}$

$$V_{eff} = -\frac{K}{r^2} + \frac{L^2}{2mr^2}$$

$$= \frac{1}{r^2}\left[\frac{L^2}{2m} - K\right]$$

When $\frac{L^2}{2m} > K$, when $\frac{L^2}{2m} < K$





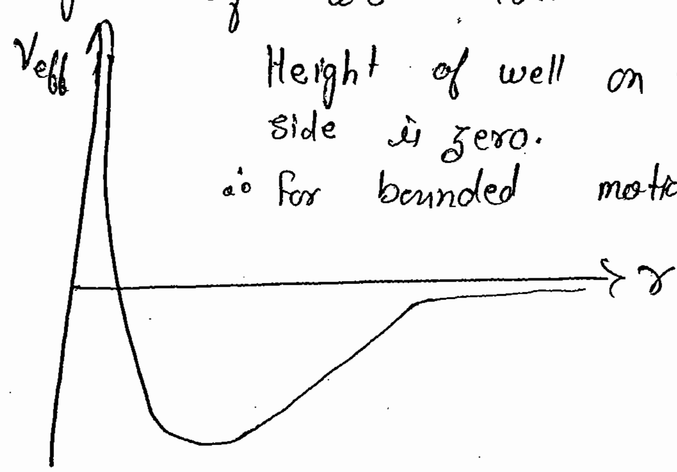
Bounded Motion

If there is a potential well in V_{eff} graph and energy of particle lying inside the well is less than minimum hight of well then motion is bounded.

Example :-

$V(r) = -\frac{K}{r}$

$$V_{eff} = \frac{-K}{r} + \frac{L^2}{2mr^2}$$



Height of well on right side is zero.

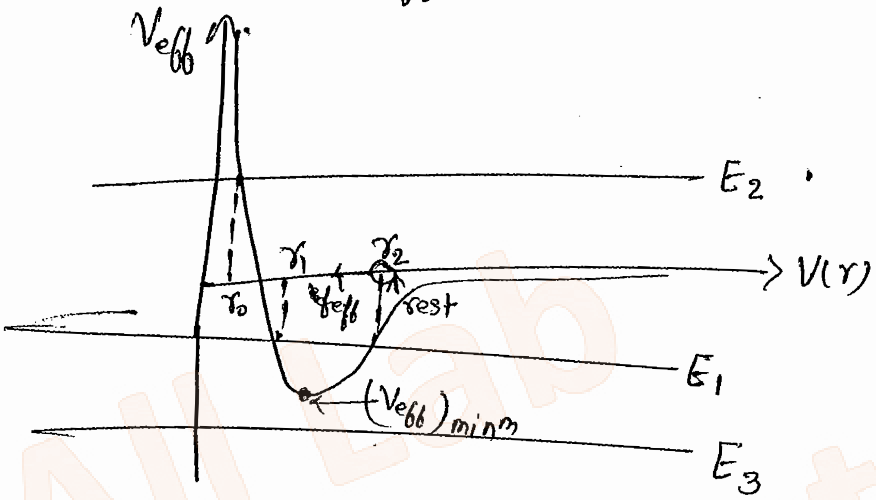
∴ For bounded motion $E<0$

$$E = \frac{1}{2} m\dot{r}^2 + V_{eff}$$

$$E - V_{eff} = \frac{1}{2} m\dot{r}^2 \longleftarrow +ve \text{ or zero.}$$

$$E - V_{eff} \geq 0$$

$E \geq V_{eff} \Rightarrow$ Energy line will be above V_{eff} line.



- E_3 is not allowed.
- For energy E_1 particle moves b/w r_1 and r_2 (these are called terning point).
- for $E = E_2$ particle moves between r_0 and ∞

Minimum allowed Energy :-

$$\boxed{E_{min} = (V_{eff})_{min}}$$

Condition for stable orbit :-

$$\boxed{\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_0} = 0} \quad \text{——— (I)}$$

$$\boxed{\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_0} > 0} \quad \text{——— (II)}$$

In General :-

If $F = kr^n$

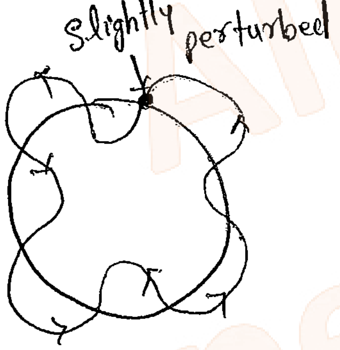
$$V(r) = -\int f(r)\,dr = \frac{-kr^{n+1}}{n+1}$$

$$V_{eff} = V(r) + \frac{l^2}{2mr^2}$$

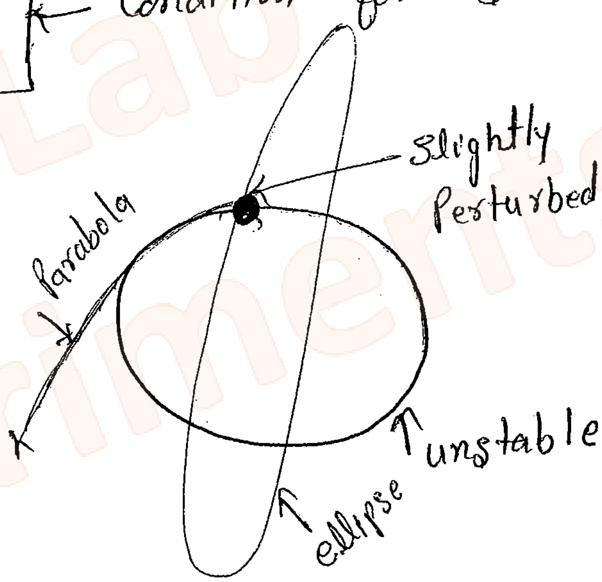
$$V_{eff.} = -\frac{kr^{n+1}}{n+1} + \frac{l^2}{2mr^2}$$

From the above conditions ① & ② for above orbit, we get -

$\boxed{n > -3}$ ← Condition for Stable orbit.



Stable



* Condition for closed Orbit :-

After perturbation slightly the object retrace it's path after some turns. (Ultimately path should closed).

If $F = kr^n$ then closed orbit is possible only for $n = 1$ and $n = -2$.

[Forces should be attractive]

① Hook's Law

② Gravitational law. }

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Sol^n

$$f = \frac{K}{r^{3-n}} = K r^{n-3}$$

$n - 3 = 1 \Rightarrow n = 4$

$n - 3 = -2 \Rightarrow n = 1$

$\boxed{n = 4, 1}$ Ans

⊛ Polar Equation of orbit under $V(r) = \frac{-K}{r}$

m_1 ——— r ——— m_2

$$V(r) = \frac{-G m_1 m_2}{r} = \frac{-K}{r} \qquad K = G m_1 m_2$$

$$F = \frac{-G m_1 m_2}{r^2}$$

$$\boxed{V(r) = -\int f(r)\,dr} = \frac{-K}{r}$$

$-q_1$ ——— r ——— q_2

$$V(r) = \frac{-q_1 q_2}{4\pi\epsilon_0 r} = \frac{-K}{r}$$

$$\Rightarrow \boxed{K = \frac{q_1 q_2}{4\pi \epsilon_0}}$$

Differential Equation of orbit :-

$$\frac{d^2u}{d\theta^2} + u = \frac{-mf}{l^2u^2}$$

$$f(r) = \frac{-\partial V}{\partial r} = \frac{-K}{r^2} = -Ku^2$$

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = +\frac{mK}{l^2}}$$

Sol^n of this eq^n gives -

$$\boxed{r = \frac{l^2/mK}{1 \pm \left(\sqrt{1+\frac{2El^2}{mK^2}}\right)\cos\theta}} \longrightarrow \text{Polar equation of orbit}$$

* Polar form of conic section :-

$$r = \frac{A}{1 \pm e\cos\theta}$$

Compare to get -

$$\boxed{e = \sqrt{1+\frac{2El^2}{mK^2}}} = \text{eccentricity}$$

e depends on E (energy)

$E > 0$, $e > 1$, orbit is hyperbola
$-\frac{mK^2}{2l^2} < E < 0$, $e < 1$, orbit is ellipse.
$E = 0$, $e = 1$, orbit is parabola
$E = -\frac{mK^2}{2l^2}$, $e = 0$, orbit is circle.

only for $V(r) = -\frac{K}{r}$

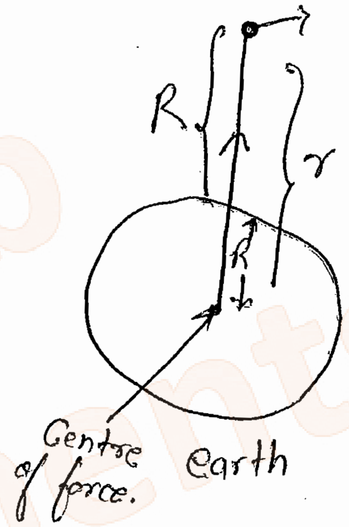
Q. A particle of mass m is projected from a height $h = R$ with speed $v = \sqrt{\frac{GM}{R}}$ where M is mass of earth, m is mass of particle. What is the path of the particle? in some direcⁿ?

Solⁿ "Note
"If a particle is thrown in radial direction it will always be straight line."

$$E = \frac{1}{2}mv^2 + V(r)$$

$$= \frac{1}{2}m\frac{GM}{R} - \frac{K}{r}$$

$$= \cancel{\frac{GMm}{2R}} - \cancel{\frac{GMm}{2R}}$$



$\therefore$ $\boxed{E = 0}$

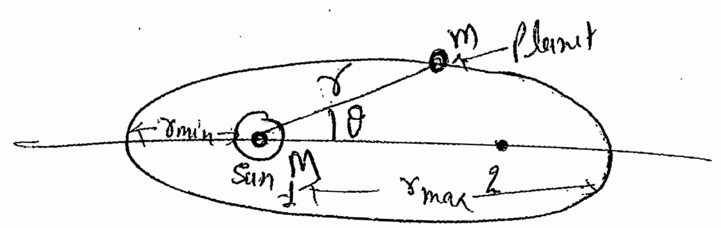
So path is parabola.

Polar Equation of orbit :-

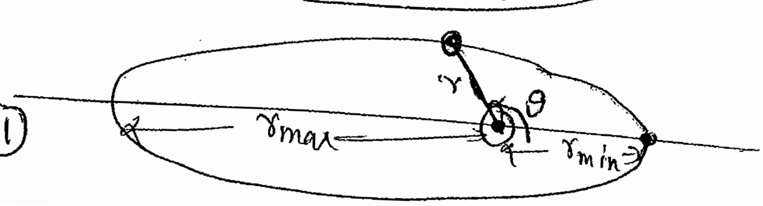
$$\boxed{r = \frac{l^2/mk}{1 \pm e\cos\theta}}$$

Elliptical Orbit :-

$$\boxed{r = \frac{l^2/mk}{1 - e\cos\theta}} \quad \text{—(I)}$$



$$\boxed{r = \frac{l^2/mk}{1 + e\cos\theta}} \quad \text{—(II)}$$



r_{max} when $\theta = 0$ } in eqⁿ (I)
r_{min} when $\theta = \pi$

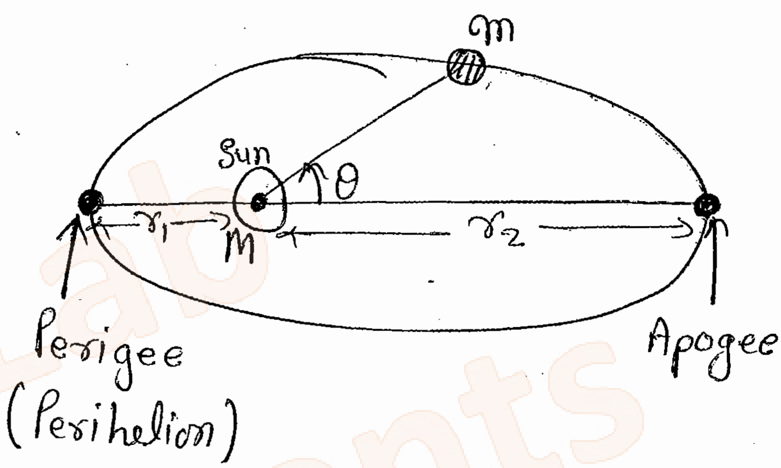
r_{max} when $\theta = \pi$ } in eqⁿ (II)
r_{min} when $\theta = 0$

* Energy and angular momentum of planet:-

Say:- r_1 and r_2 are given

$$V(r) = -\frac{GMm}{r}$$

Polar equation of orbit-



$$r = \frac{l^2/mk}{1-\left(\sqrt{1+\frac{2El^2}{mk^2}}\right)\cos\theta}$$

$$r_1 = \frac{l^2/mK}{1+\sqrt{1+\frac{2El^2}{mk^2}}} \quad \text{—— (I)}$$

$$r_2 = \frac{l^2/mK}{1-\sqrt{1+\frac{2El^2}{mk^2}}} \quad \text{—— (II)}$$

$$1+\sqrt{1+\frac{2El^2}{mk^2}} = \frac{l^2}{mk}\cdot\frac{1}{r_1} \quad \text{—— (III)}$$

$$1-\sqrt{1+\frac{2El^2}{mk^2}} = \frac{l^2}{mk}\cdot\frac{1}{r_2} \quad \text{—— (IV)}$$

(III) + (IV)

$$2 = \frac{l^2}{mK}\left(\frac{r_1+r_2}{r_1 r_2}\right)$$

$$l = \sqrt{\frac{2 r_1 r_2}{(r_1+r_2)}}\; mk$$

$$\boxed{l = m\sqrt{\frac{2GM r_1 r_2}{r_1+r_2}}}$$

Put l in eq^n 1 to get Energy.

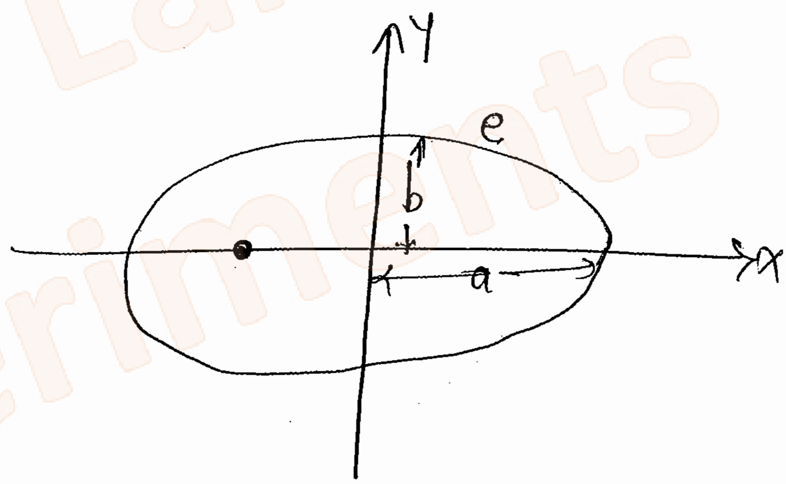
$$\boxed{E = \frac{-GMm}{r_1+r_2}}$$

In terms of ellipse parameter :-

$r_1 = (1-e)a$

$r_2 = (1+e)a$

$e = \sqrt{1-\frac{b^2}{a^2}}$



$$\boxed{r_1+r_2 = 2a}$$

$$\boxed{E = \frac{-GMm}{2a}}$$

$$r_1 r_2 = (1-e^2)a^2$$

$$= \frac{b^2}{a^2}\cdot a^2$$

$$\boxed{r_1 r_2 = b^2}$$

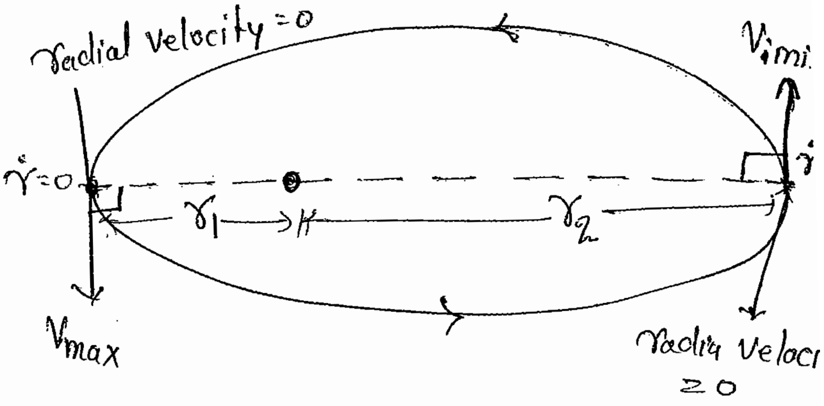
$$L = m\sqrt{\frac{\cancel{2}GMb^2}{\cancel{2}a}}$$

$$\boxed{L = m\sqrt{\frac{GMb^2}{a}}}$$ dependent on a and b.

* Maximum and Minimum Speed in elliptical orbit:-

$L = m v_{max} r_{min} = m v_{max} r_1$

$L = m v_{min} r_{max} = m v_{min} r_2$



$$L = m\sqrt{\frac{2GMr_1r_2}{(r_1+r_2)}}$$

$$\boxed{v_{min} = \sqrt{\frac{GMr_1}{r_2(r_1+r_2)}}}$$

$$\boxed{v_{max} = \sqrt{\frac{2GMr_2}{r_1(r_1+r_2)}}}$$

$r_1 = (1-e)a$

$r_2 = (1+e)a$

$r_1 + r_2 = 2a$

$$\boxed{v_{min} = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}}$$

$$\boxed{v_{max} = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}}$$

Q. Planet is moving around the sum if ratio of maximum to minimum speed is 2. what is accentricity of orbit?

Soln

$$\frac{v_{max}}{v_{min}} = \frac{\sqrt{\frac{G\cancel{M}}{\cancel{a}}\left(\frac{1+e}{1-e}\right)}}{\sqrt{\frac{G\cancel{M}}{\cancel{a}}\left(\frac{1-e}{1+e}\right)}} = 2$$

$$\frac{1+e}{1-e} = 2$$

$1 + e = 2 - 2e$

$3e = 2 - 1$

$$\boxed{e = \frac{1}{3}}$$ Ans

* Virial Theorem :-

It relates average value of Potential Energy and Kinetic Energy.

If $f = kr^n$

$$\boxed{\langle K.E.\rangle = \frac{n+1}{2}\langle P.E.\rangle} \leftarrow \text{Virial Theorem}$$

Ex-

for Gravitation :- $n = -2$

$$\boxed{\langle K.E.\rangle = -\frac{1}{2}\langle P.E.\rangle}$$

for Harmonic Oscillator :-

$$f = -kr^1$$

$$n = 1$$

$$\boxed{\langle K.E.\rangle = \langle P.E.\rangle}$$

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Q.7 Potential energy $V = kr^n$ then Relation b/w K.E & P.E is

(a) $\langle T\rangle = \langle V\rangle$ (b) $\langle T\rangle = \frac{n}{2}\langle V\rangle$ (c) $\langle T\rangle = \frac{3}{2}\langle V\rangle$

(d) $\langle T\rangle = 2\langle V\rangle$

Soln -

$$V = kr^n$$

$$f = -\frac{\partial V}{\partial r} = -knr^{(n-1)}$$