Free Study Material from

All Lab Experiments

Classical Mechanics

for NET/Gate Physical Sciences

> Lagrangian Formulation, Part-1 <

* Lagrangian formulation of Classical Mechanics:-

Degree of freedom :- {DOF} :- It is the number of independent coordinates required to describe dynamics of system. It is represented by 'f'.

for N-particles moving freely in 'd' dimensional world.

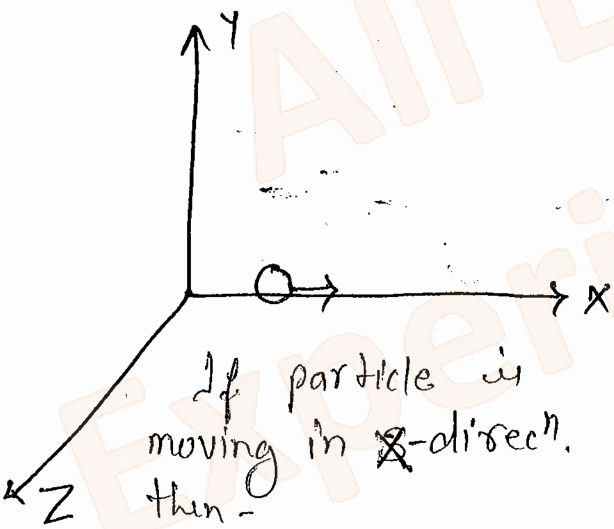
$$\boxed{f = Nd}$$

N = No. of particle
d = dimension of particle.

If there is some constraints then -

$$\boxed{f = Nd - K}$$

where K = No. of constraints.



If particle is moving in x-direcⁿ. then -

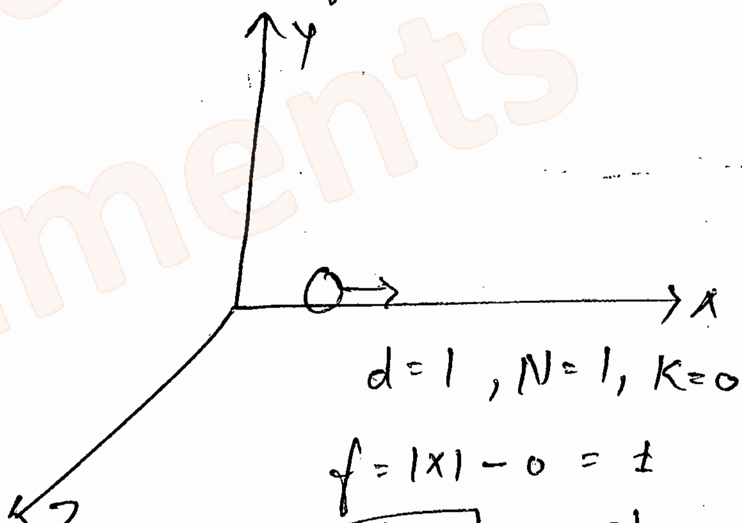
$d = 3,\ N = 1,\ K = 2$ (means it can't travel in y and z)

so $f = 1 \times 3 - 2$

$= 3 - 2$

$\left\{ \begin{matrix} y = 0 \\ z = 0 \end{matrix} \right\}$

$\boxed{f = 1}$ ← IInd way of description.



$d = 1,\ N = 1,\ K = 0$

$f = 1 \times 1 - 0 = 1$

$\boxed{f = 1}$ ← Ist way of description

If 2 particle is moving freely in xy-plane.

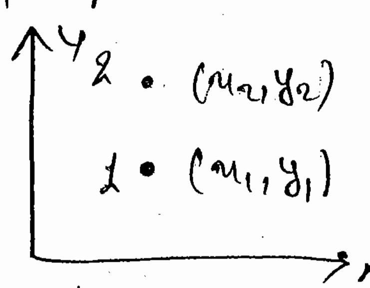
$N = 2,\ d = 2$

$K = 0$

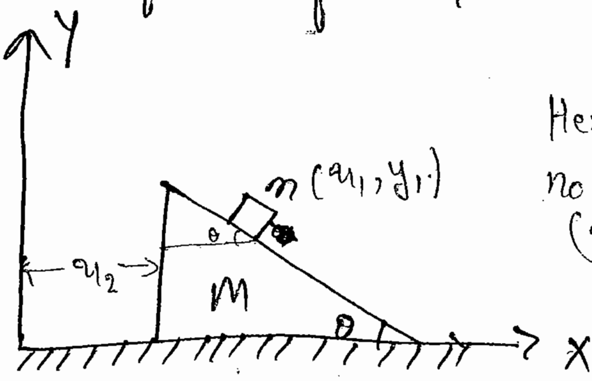
$f = Nd - K$

$f = 2 \times 2 - 0$

$\boxed{f = 4}$



Q. Block and Wedge move in vertical plane. what is D.O.f. of System is ?



Here (x_1, y_1) and x_2 are not independent becoz Here (x_1, y_1) are dependent on each other

Sol^n

$$f = Nd - K$$

$N = 2,\ d = 2,\ K = 2$

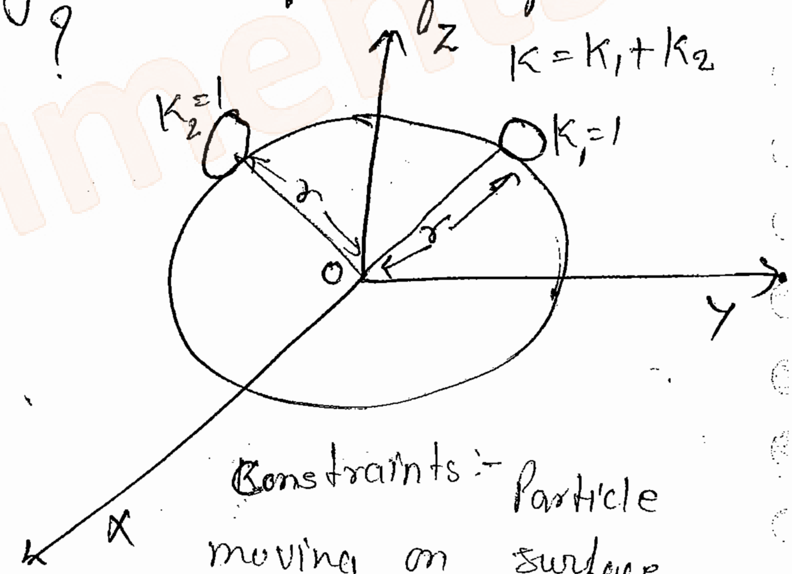
$$f = 2 \times 2 - 2$$

$$\boxed{f = 2}$$

Q. Two particles are moving on surface of sphere find degree of freedom ?

Sol^n

$K = K_1 + K_2$



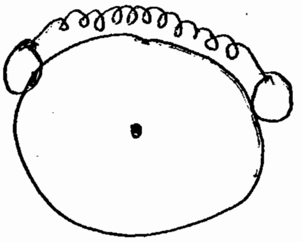
$$f = Nd - K$$

$N = 2,\ d = 3,\ K = 2$

$$f = 2 \times 3 - 2$$
$$= 6 - 2$$
$$\boxed{f = 4}$$

Constraints :- Particle moving on surface any where but distance from the center is always r.

* **Particle Connected by spring :-**

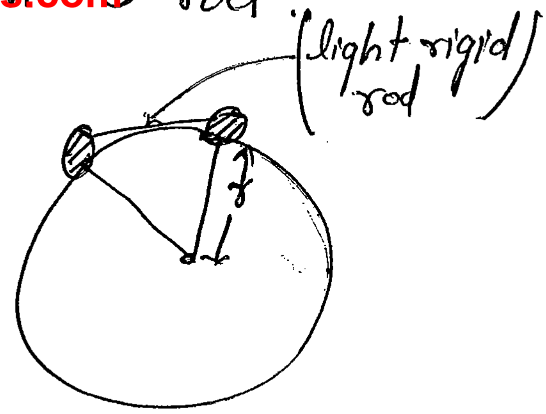


Spring does not put any condition (Constraints).

So. $\boxed{f = 4}$

* Particle Connected by light rod :-

Here there is extra condition (Constraints) arise that distance between particle is fixed.

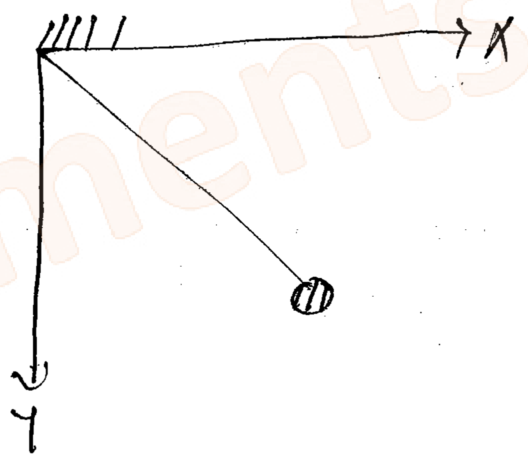


So $K = 3$

$$f = Nd - K$$
$$= 2 \times 3 - 3$$
$$= 6 - 3$$
$$\boxed{f = 3}$$

* Simple Pendulum :-

In simple pendulum, motion is confined in one plane.



$N = 1$
$d = 2$
$K = 1$

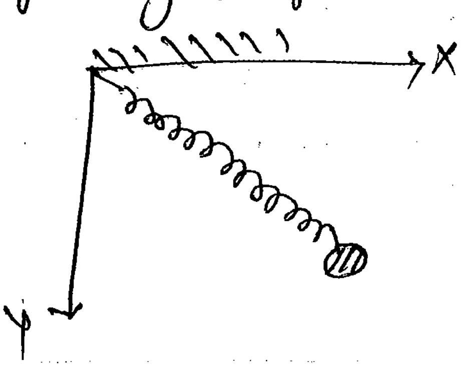
$f = Nd - K$
$= 2 \times 1 - 1$
$\boxed{f = 1}$

$N = 1$
$d = 3$
$K = 2$

$f = Nd - k$
$= 1 \times 3 - 2$
$= 3 - 2$
$\boxed{f = 1}$

① Confined in one plane
② Distance from point of suspension is l.

① If string is replaced by spring (or flexible string) :-

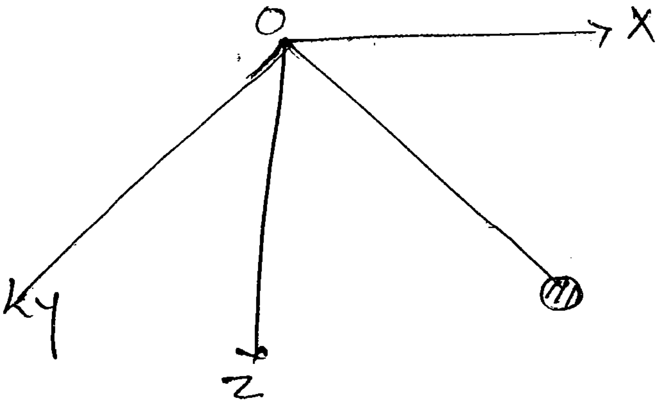


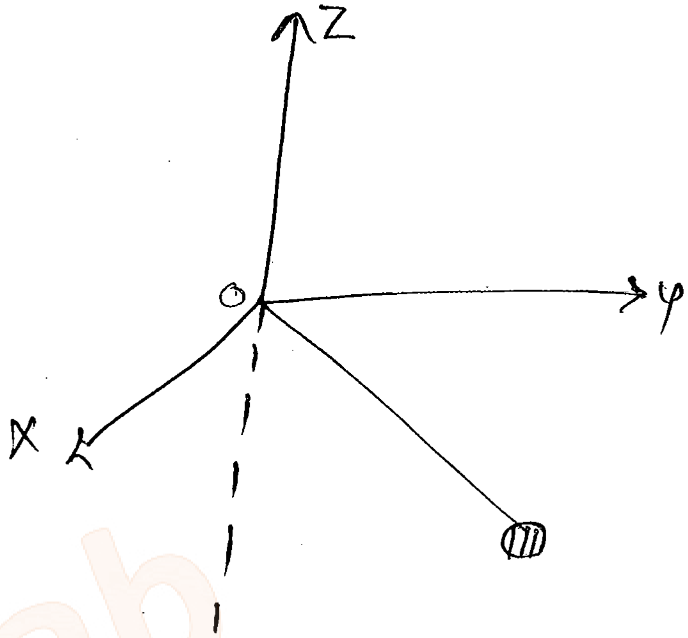
$d = 2$
$K = 0$
$N = 1$

$\boxed{\text{D.O.f.} = 2}$

* Spherical Pendulum :-

Motion is not confined in one plane.





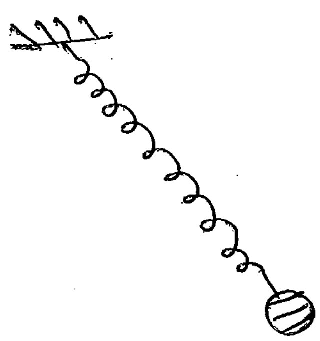
$d = 3$

$N = 1$

$K = 1$ (distance from O is fixed.)

$\boxed{D.O.F. = 2}$

If string is replaced by flexible string or Spring :-



$d = 3$

$K = 0$

$N = 1$

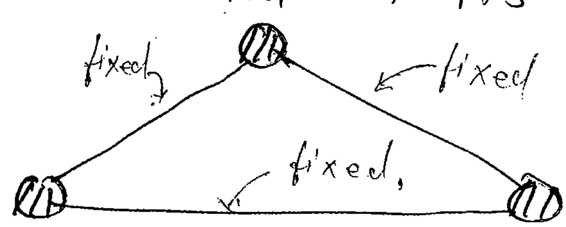
$\boxed{DOF = 3}$

Q. Three particle are connected by a light rod to each other as shown in figure. If system is moving in 3-Dimension then what is it's degree of freedom.

Sol^n $K = 3$, $N = 3$, $d = 3$

$f = Nd - K = 3\times3 - 3$

$\boxed{f = 6}$



If moving in 2-D.

$d = 2$

$N = 3$

$K = 3$

$DoF = 3 \times 2 - 3$

$= 6 - 3$

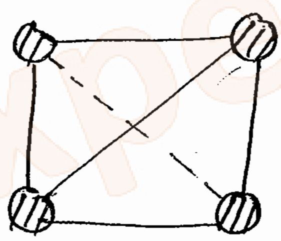
$\boxed{DoF = 3}$

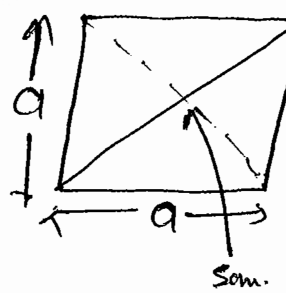
Q. 10 particles are connected to each other by light rod and system is moving in 3-D what is degree of freedom.

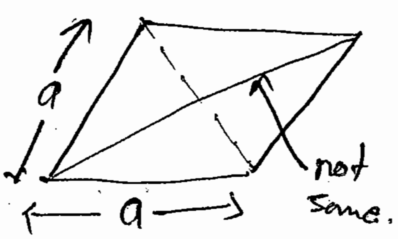
Sol^n If $N \geq 3$ and distance b/w the particle is fixed. And the system is moving in 3-D then degree of freedom is equal to 6.
If it is in 2-D it is equal to 3.
- If it is in 1-D it is equal to 1.

True for any rigid body

for $N = 4$:-







$DOF = Nd - K$

$= 4 \times 3 - 6$

$= 12 - 6$

$\boxed{DOF. = 6}$

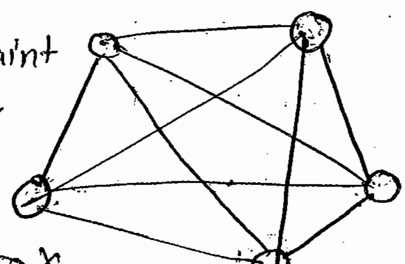
for $N = 5$:-

$N = 5$ K x

$K = 10$

$d = 3$

One constraint is redundant (not require).



$DOF = 5 \times 3 - 10 = 15 - 10 = 5$ x not true becoz by statement it is $= 6$

* Degree of freedom of rigid Body :-

$$\boxed{f = \frac{d(d+1)}{2}}$$

d = dimentionality of World.

3-D → $f = 6$ = 3 rotation + 3 translation
2-D → $f = 3$ = 2 translation + 1 rotation
1-D → $f = 1$ = 1 translation.

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* Constraints (Conditions)

or

Geometrical Condition on Coordinates

{There is a relation between Coordinates}

* Types of Constraints :-

Constraints → Holonomic (Complete), Non-Holonomic

Holonomic (Complete)

Relation b/w Coordinates is algebric equation

e.g. ⇒ $x^2 + y^2 = l^2$

Non-Holonomic

If relation b/w coordinates is differential equation

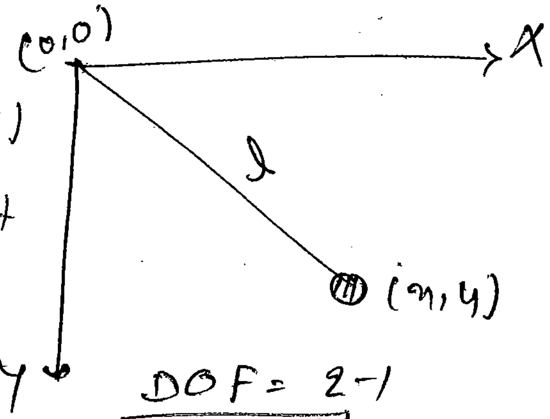
If relation b/w Coordinates is differential equation then it must be reducible to algebraic equation.

e.g. $x\,dy + y\,dx = 0$

$\int d(xy) = 0 \Rightarrow xy = \text{Const.}$ ← Algebraic eqn.

* Holonomic Constraints :-

1. Simple Pendulum :-



Variable Coordinates = 2 ⇒ (x,y)

$x^2 + y^2 = l^2$ ⟶ Constraint equation. (Holonomic)

$\boxed{y^2 = l^2 - x^2}$

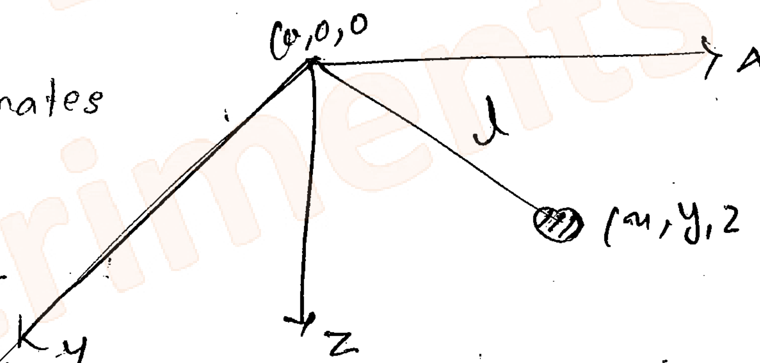
So if we know 'x' we can find 'y'. So it is dependent.

DOF = 2−1

$\boxed{\text{DOF} = 1}$

$\boxed{\text{DOF = No. of variable Coordinate − No. of Constraint equation (Holonomic)}}$

2. Spherical Pendulum :-



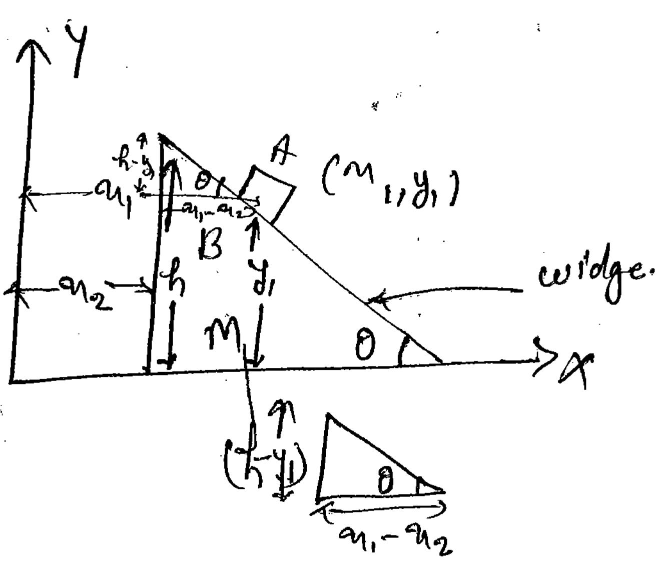
No. of variable Co-ordinates = 3 (x,y,z)

Constraints equation -

$x^2 + y^2 + z^2 = l^2$ (Holonomic)

$\boxed{\text{DOF} = 3-1 = 2}$

3. Block & Wedge :-



Here h and θ are fixed.

$\tan\theta = \dfrac{h - y_1}{x_1 - x_2}$

$h - y_1 = (x_1 - x_2)\tan\theta$

$\boxed{y_1 = h - (x_1 - x_2)\tan\theta}$ ← Constraint equation

$DOF = 3 - 1$

$\boxed{DOF = 2}$

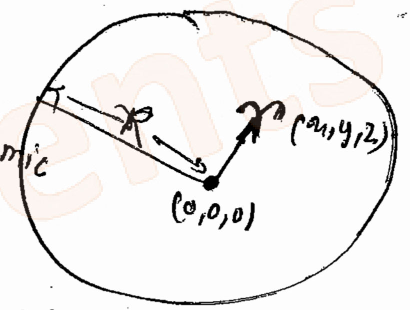
Non Holonomic Constraintes

Constraint equations are either inequality or non integrable differential equations.

* It does not reduce degree of freedom.

Ex - A particle (fly) moving ~~inside~~ inside a sphere.

$\boxed{\sqrt{x^2+y^2+z^2} \le R}$ ← Non Holonomic



$d = 3$
$N = 1$
$K = 0$

$DOF = Nd - K$
$= 1 \times 3 - 0$

$\boxed{DOF = 3}$

Here there is one constraint
⇒ the fly can not go out of the sphere. but it is non-Holonomic.

Ex - Disk Rolling on a plane (XY-Plane):-

Condition for Rolling:-

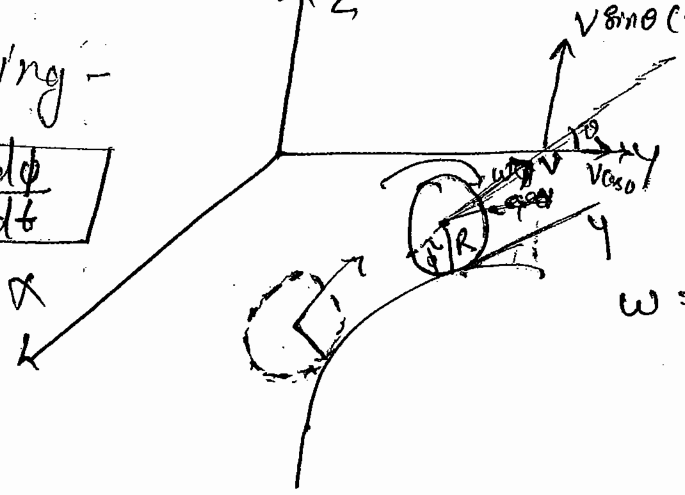
$\boxed{V = R\omega}$ $\boxed{V = R\frac{d\phi}{dt}}$

$V_y = V\cos\theta$

$\frac{dy}{dt} = R\frac{d\phi}{dt}\cos\theta$

$V\sin\theta(-\hat{x})+$

becoz disc Rolling in xy-plane so velocity in z-direcn = 0.



$\omega = \frac{d\phi}{dt}$

$$\boxed{dy = R\,d\phi \cos\theta} \text{ — (i)}$$

$$V_x = -V\sin\theta$$

$$\boxed{dx = -R\,d\phi \sin\theta} \text{ — (ii)}$$

Divide (i) by (ii)

$$\frac{dy}{dx} = -\cot\theta$$

$$\boxed{\int dy = -\int \cot\theta \, dx}$$

"Differential equation for x and y θ is variable. So it can not be integrated.

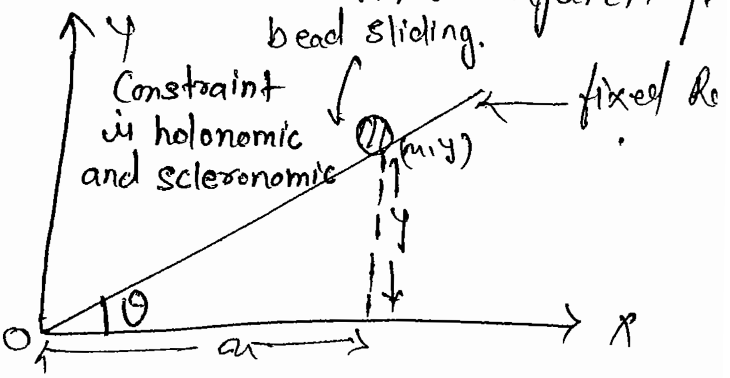
∴ Rolling is a non-holonomic constraint when motion is along a curved line. and it is holonomic constraint, if motion is along a straight line."

* Holonomic and Non-Holonomic Constraints have two types.

1. Scleronomic (rigid) : Time Independent :-

Constraint equation do not contain time explicitly. Therefore

Example :- Bead sliding on a fixed rod.



Constraint equation -

$$\tan\theta = \frac{y}{x}$$

$\boxed{y = x\tan\theta}$ — does not contain time.

Dof = No. of variable − (No of Constraint eqn)

= 2 − 1

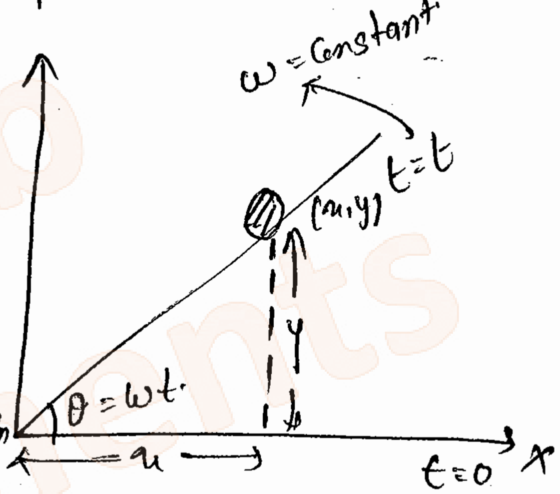
$\boxed{Dof = 1}$

2. Rheonomic Constraint: (Non Rigid)

{Time Dependent}

Constraint equation explicitly contain time.

Ex:- Bead sliding on a rotating rod.



Equation of Constraint -

$\tan\theta = \frac{y}{x}$

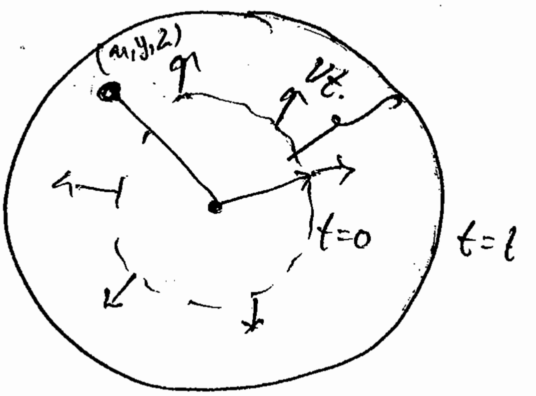
$\boxed{y = x \tan\omega t}$ ⇒ It contain time.

⇓

Holonomic and Rheonomic.

Ex:- Particle moving inside a expanding sphere.

$\boxed{\sqrt{x^2+y^2+z^2} \leq R_0 + vt}$



⇓

Non-Holonomic and Rhonomic.

* Generalised Coordinates : Convenient Coordinates

Note :- Do not use concepts about dynamics of system lernt from Newtonian Mechanics.

* "Convenient Coordinates chosen to simplify problem are called generalised coordinates."

"In most cases number of generalised coordinates is equal to Degree of freedom."

Q. A particle moving along Curve line (fixed) what is degree of freedom of particle?

Soln

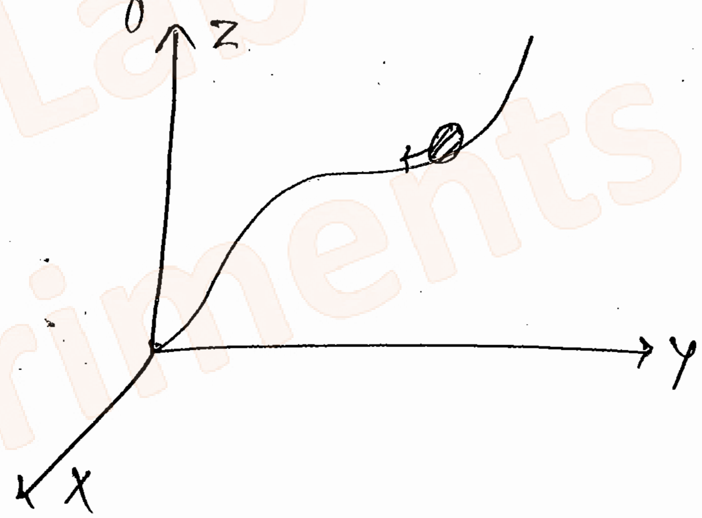
$N = 1$
$k = 1$
$d = 3$

$f = 3 \times 1 - 1 = 2$ ✗

but it is not true

Here degree of freedom

$\boxed{f = 1}$



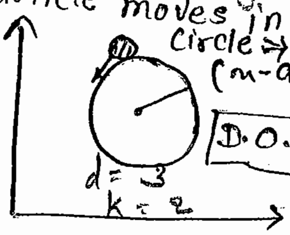
because A line is formed by intersection of two planes.

So ~~degree of freedom~~ of one plane = 1 and for another plane = 1

So total Constraints = 1+1 = 2

$f = 3 \times 1 - 2 = 1$ ✓

If Particle moves in Constraind Circle → $(x-a)^2 + (y-a)^2 = R$



$\boxed{D.O.F. = 1}$

$d = 3$
$k = 2$

Note :- If Particle moves along a fixed line (it may be a curved line) or straight line. then $\boxed{D.O.F = 1}$

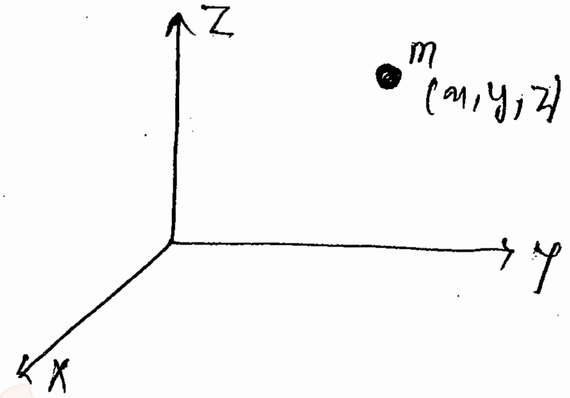
Because a line is formed by intersection of two planes.

* Kinetic Energy in Different Coordinate System :-

(1) Cartesian Co-ordinate :-

$$K.E. = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

(Most Useful)



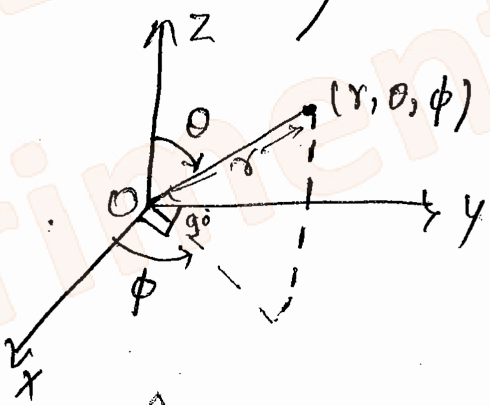
(2) Spherical Polar :-

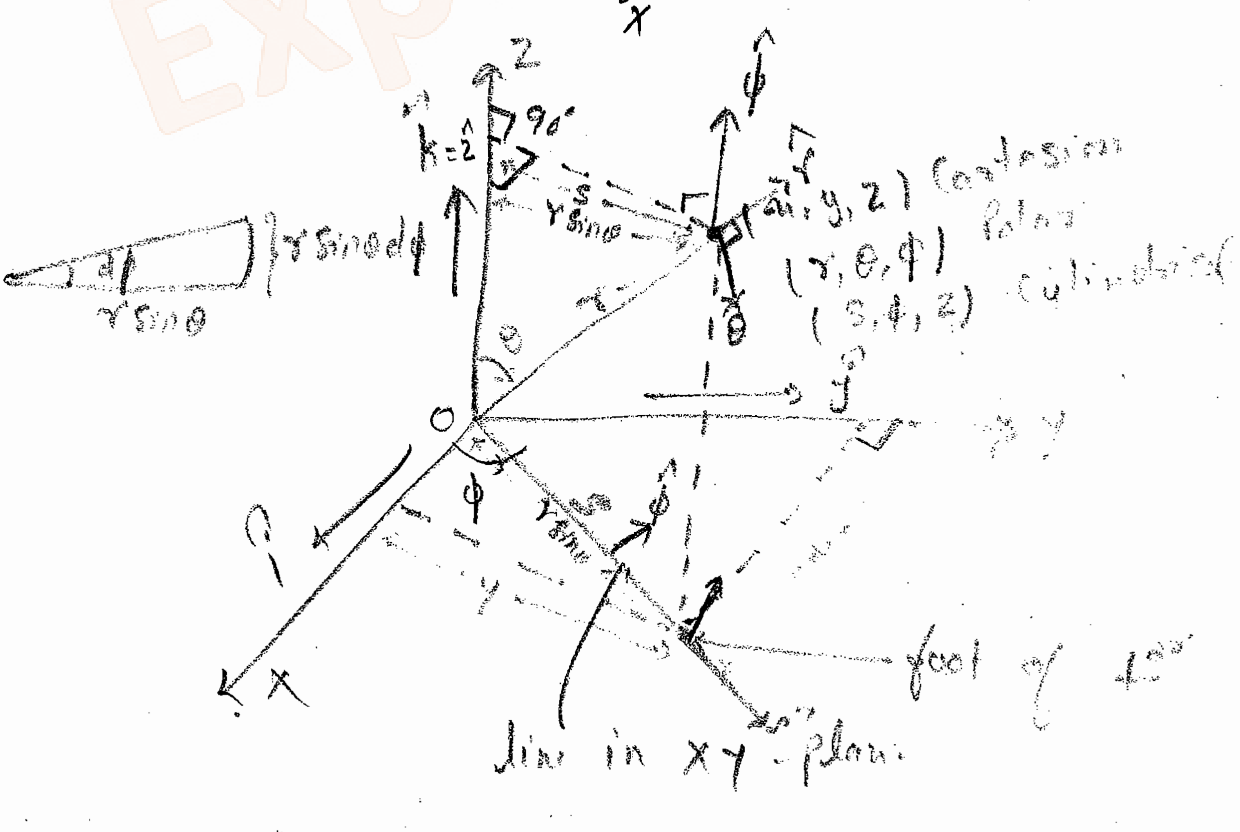
$$K.E. = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2)$$

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$





* How Write K.E. in Different Coordinate System:

Small displacement of particle in different coordinates.

$\Rightarrow$ $\boxed{d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k}}$ In Cartesian Coordinate.

$\Rightarrow$ In Spherical Polar :-

$$\boxed{d\vec{s} = d\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}}$$

$\Rightarrow$ In Cylindrical Coordinate :-

$$\boxed{d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}}$$

* Velocity :-

$$\vec{v} = \frac{d\vec{s}}{dt} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

$$\vec{v} = \frac{d\vec{s}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\hat{\phi}$$

$$v = \frac{d\vec{l}}{dt} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$

* Kinetic Energy :-

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$K = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin\theta\,\dot{\phi}^2\right)$$

$$K = \frac{1}{2}m\left(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2\right)$$

Note :- If a particle is moving in a plane/space and force on it is always directed towards a point or particle is attached to a point by a string/spring or rod then use of plane polar/spherical polar is convenient.

⇒ If case is not so in a plane then we can use Cartesian.

⇒ Plane Polar :-
If we remove z from cylindrical coordinate system. then we get plane polar.

* How to write K.E. for more than one particle cases :-

1. Fist Method :- More General Method.

$$K.E. = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2)$$

2. Second Method :-
Use superposition method to calculate speed of second particle if speed of first particle is given. (less General).
[Use mostly for motion in one plane]

* Generalised Coordinate :-

Independent Coordinates (varia... chosen to simplify the problem.

<u>Notation :-</u> q_i

$$\frac{\partial q_i}{\partial q_j} = \delta_{ij}$$

$$\boxed{\frac{\partial q_1}{\partial q_2} = 0}$$

Generalised Velocity :-

$$\boxed{\dot{q}_i = \frac{dq_i}{dt}}$$

⇒ In lagrangean formulation q_i and $\dot{q}_i$ are taken to be independent while solving the problem.

$$\boxed{\begin{aligned} \frac{\partial \dot{q}_i}{\partial q_i} &= 0 \\ \frac{\partial q_i}{\partial \dot{q}_i} &= 0 \end{aligned}}$$

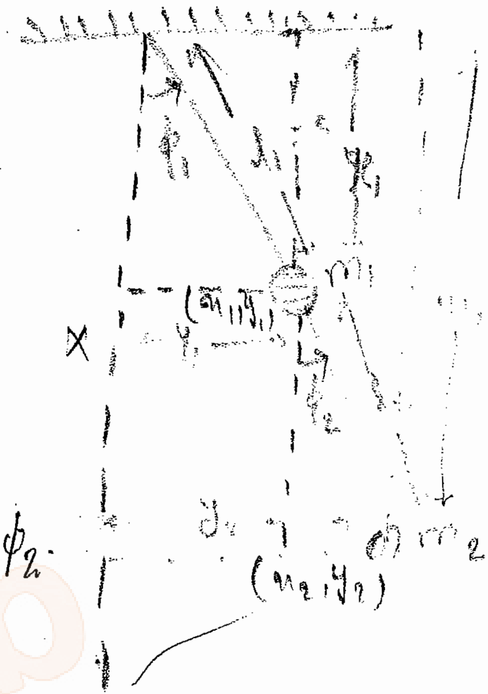
← Before problem has been solved

$$\boxed{\frac{\partial \dot{q}_i}{\partial \dot{q}_j} = \delta_{ij}}$$

After problem is solved q_i and $\dot{q}_i$ may turn out to be dependent (actually).

K.E. of Double Pendulum :- {both are moving in same plane}

Degree of freedom of system:-



$N = 2$, $d = 2$, $K = 2$

$f = Nd - K$

$f = 2 \times 2 - 2 = 4 - 2$

$\boxed{f = 2}$

Generalised Co-ordinates are ϕ_1 and ϕ_2.

$x_1 = l_1 \cos\phi_1$, $\dot{x}_1 = -l_1 \sin\phi_1 \dot{\phi}_1$

$y_1 = l_1 \sin\phi_1$, $\dot{y}_1 = l_1 \cos\phi_1 \dot{\phi}_1$

$x_2 = l_1 \cos\phi_1 + l_2 \cos\phi_2$, $\dot{x}_2 = -[l_1 \sin\phi_1 \dot{\phi}_1 + l_2 \sin\phi_2 \dot{\phi}_2]$

$y_2 = l_1 \sin\phi_1 + l_2 \sin\phi_2$, $\dot{y}_2 = [l_1 \cos\phi_1 \dot{\phi}_1 + l_2 \cos\phi_2 \dot{\phi}_2]$

We write K.E. of the system by first method -

$$K.E. = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

So $K.E. = \frac{1}{2} m_1 [l_1^2 \dot{\phi}_1^2] + \frac{1}{2} m_2 [l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)]$

So $\boxed{T = \frac{1}{2} \left[m_1 l_1^2 \dot{\phi}_1^2 + m_2 (l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)) \right]}$

* Second Method :

→ Velocity Superposition Method :-

$$\boxed{\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi}}$$

$$\vec{v}_1 = 0 + l_1\dot{\phi}_1\hat{\phi}_1$$

$$\boxed{\vec{V}_1 = l_1\dot{\phi}\hat{\phi}_1} \qquad \boxed{\vec{V}_2 = l_2\dot{\phi}_2\hat{\phi}_2}$$

Motion of m_2 depends upon motion of m_1.

So K.E. of $m_1 = \frac{1}{2} m_1 v_1^2$

$$= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2$$

Net velocity of $m_2 = \sqrt{v_1^2 + v_2^2 + 2v_1v_2 \cos(\phi_1 - \phi_2)}$

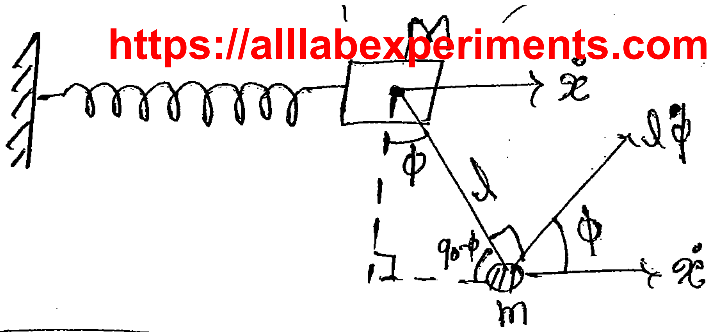
So K.E. of $m_2 = \frac{1}{2} m_2 (\text{net velocity})^2$

$$\text{K.E. of } m_2 = \frac{1}{2} m_2 \left[l_1^2\dot{\phi}_1^2 + l_2\dot{\phi}_2^2 + 2l_1l_2\dot{\phi}_1\dot{\phi}_2 \cos(\phi_1 - \phi_2)\right]$$

So Total K.E. of the system is -

$$\boxed{T = \frac{1}{2}\left[m_1l_1^2\dot{\phi}_1^2 + m_2\left(l_1^2\dot{\phi}_1^2 + l_2\dot{\phi}_2^2 + 2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)\right)\right]}$$

Q.



What is K.E. of the system?

Sol^n

$\boxed{DOF = 2}$

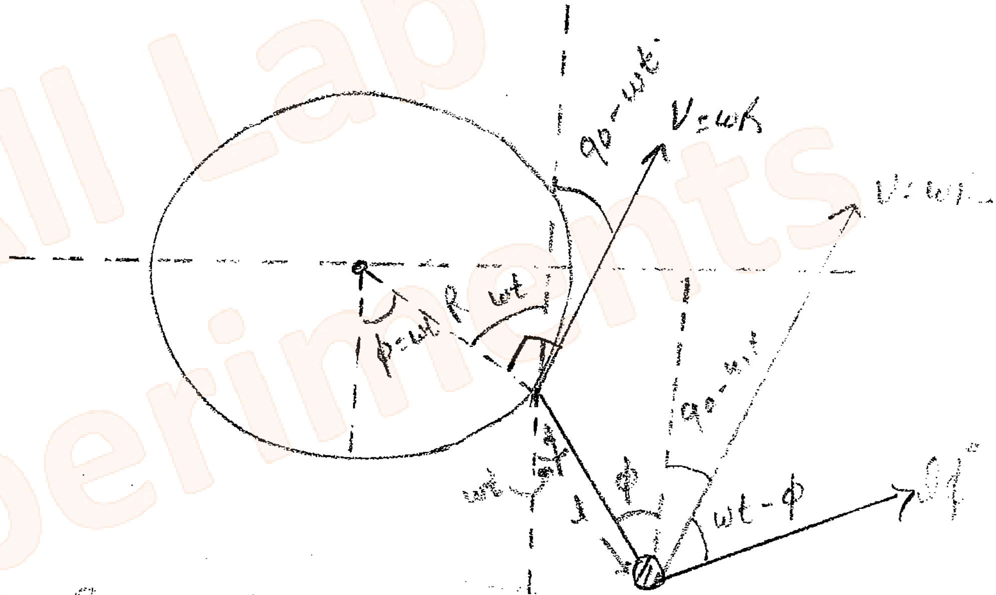
$V_1 = \dot{x}$

$V_2 = l\dot{\phi}$

$$\boxed{K.E. = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left(\dot{x}^2 + l^2 \dot{\phi}^2 + 2 l \dot{x} \dot{\phi} \cos\phi\right)}$$

A-10

Q.20



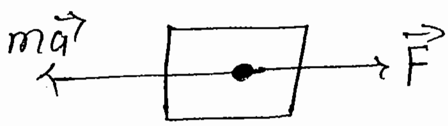
$$\boxed{K.E. = \frac{1}{2} m \left(l^2 \dot{\phi}^2 + \omega^2 R^2 + 2 l \dot{\phi} \omega R \cos(\omega t - \phi)\right)}$$

* Lagrangian formulation :

It is baised on following two approaches -

① D'Alembert's Principle. +
② Principle of virtual work. [Baised on Physics]

D'Alembert's principle converts Dynamic system into static system. by assuming a revese force.



$$\vec{F} - m\vec{a} = 0$$

Principle of Virtual Work:-

$$\sum_{i=1}^{3N} \vec{f_i} \cdot \delta\vec{r_i} = 0$$

② Variation Calculus (principle) Approach :- [Baised on Mathematics]
(Derivation of Schroedinger eq^n).

$$\text{Action } (S) = \int_{t_1}^{t_2} L\,dt$$

dynamics of system is such that action is extremum (only that dynamics is allowed in which action is extrimum).

$\delta S = 0$ [Condition for extremum]

$\delta \int L\,dt = 0$ [Condition for extremum]

$\downarrow$

This gives as Lagrangian's Equation :-

* Lagranges' Equation :-

First form :
$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i}$$

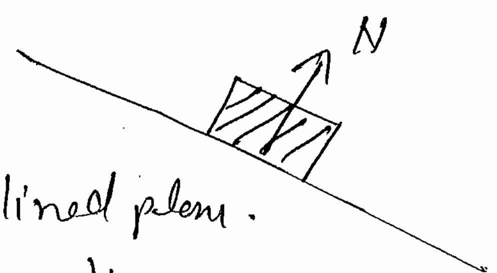
$T = K.E.$

$Q =$ Generalised force (Q includes all forces except constraint forces)

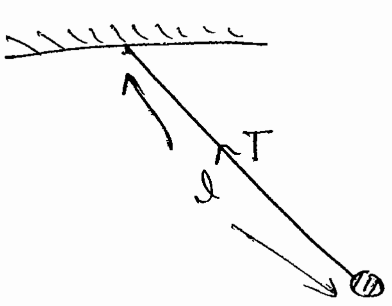
* Constraint forces :- forces arising due to some constraints.

Example :- Normal reation, tension etc.

Normal reaction is arised when object is started to slide down on a Inclined plane.



It is due to Contact between the inclined plane. surface and the object ~~between~~ is produced the normal reaction.



* Second form :-

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i}$$

Generalised force (It includes those forces which potential energy can not be written.).

$L = T - V$ = Lagrangian

V → Potential Energy

* For Constructive system (monogenic system) :-

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$

(Monogenic system is that system for which potential energy can not be written.)

* Relation between Q_i and f_i (actual force's component)

Q_i is may or may not be component of actual force.

$$\boxed{Q_i = \sum_i f_j \frac{\partial r_j}{\partial q_i}} \Rightarrow$$ Used to find Q_i if f_i is given.

$$\boxed{Q_\theta = \sum F_j \frac{\partial r_j}{\partial \theta}}$$

Imp.

* Relation b/w Generalised force and Generalised potential :- $\{U(q_i, \dot{q}_i)\}$:-

$$\boxed{Q_i = -\frac{\partial U}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_i}\right)}$$

Generalised force may or may not be unit of actual force.

Same as :-

$\vec{F} = -\vec{\nabla} V(x, y, z)$

$\vec{F}_x = -\frac{\partial V(x)}{\partial x}$

Correspondence b/w Q_i and q_i :-

"Product $Q_i q_i$ always has dimension of work. Although q_i and Q_i may not have dimension of length and force."

	q_i	Q_i	
(actual cordinate)	x	F_x	(actual component of force)
(Angle)	θ	F_θ	(torque)
(Thermodynamic co-ordinate variable)	Volume	P	(Pressure)
	temp.	entropy	

Generalised Momentum :-

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i}$$

Q_i → generalised force.

dimension

$$\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}}$$

$$\vec{F} = \frac{d(\vec{p})}{dt}$$

It may or may not depend on q_i and $\dot{q}_i$

$\frac{\partial p_i}{\partial q_i} = 0$ may or may not be equal to zero.

$\frac{\partial p_i}{\partial \dot{q}_i} = 0$ or may not be equal to zero.

p_i may or may not be component of actual momentum.

However product p_i and q_i always dimension of angular momentum.

	q_i	p_i
(actual Co-ordinate)	x	p_x (actual momentum)
(angle)	θ ↓ dimensionless	p_θ (angular momentum)

CSIR 2013 Dec

Q. $V = x^2 + y^2 + \frac{z^2}{2}$ which compound of angular momentum is conserved.?

Sol^n

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - \left[x^2 + y^2 + \frac{z^2}{2}\right]$$

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$$L = \frac{1}{2}m\{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\} - \left[r^2\sin^2\theta\cos^2\phi + r^2\sin^2\phi\sin^2\theta + \frac{r^2\cos^2\theta}{2}\right]$$

$$= \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2] - \left[r^2\sin^2\theta + r^2\frac{\cos^2\theta}{2}\right]$$

eq^n of r :- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r}$

$$m\ddot{r} - [mr\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2 - 2r\sin^2\theta + r\cos^2\theta]$$

eq^n of θ :-

ϕ = cyclic

p_ϕ = Conserved

$\boxed{L_z = \text{Constant}}$

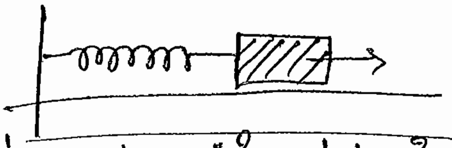
* Cyclic Coordinate / Ignorable / removable Coordinate :-

If Lagrangian does not depend on a coordinate then that coordinate is called cyclic coordinate.

Ex.- For a particle is moving in 3-D space (x, y, z).

$$\boxed{L = f(\dot{x}, \dot{y}, \dot{z}, x, y, t)}$$

z = is cyclic



$$\boxed{L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2}$$

No need to say that y and z are cyclic.

* Conservation theorem / Principle :-

If the system is conservative (monogenic).

Potential can be written for all forces.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

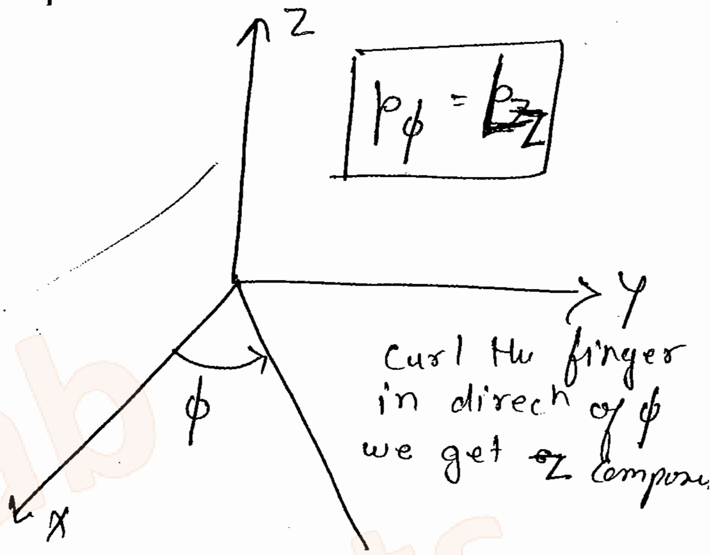
If L is no function of q_i (Cyclic)

$$\frac{\partial L}{\partial q_i} = 0$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$\rightarrow \boxed{\frac{\partial L}{\partial \dot{q}_i} = \text{Constant}} \Rightarrow \boxed{p_i = \text{Constant}}$$

Generalised momentum corresponding (conjugate) to cyclic generalised coordinate is conserved.
If x is cyclic then p_x is conserved or if ϕ (in spherical polar coordinate) is cyclic then p_ϕ = Constant. $\{P_\phi = L_z\}$.



* Problem Based On Lagrangian :-

$$L = T - V$$

Equation of motion = $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$.

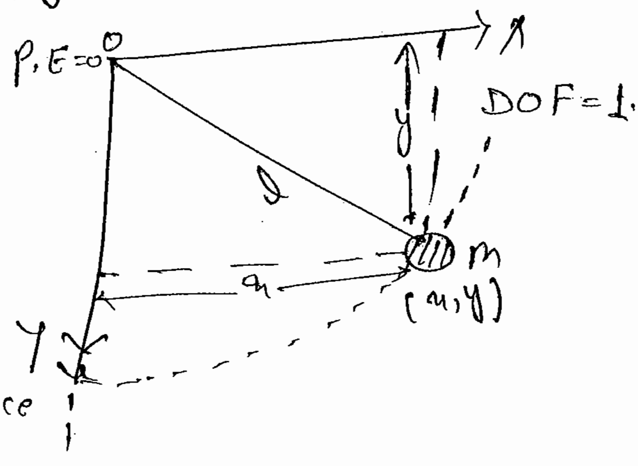
① Simple Pendulum :-

① Write Lagrangian in terms of x coordinate taken as shown in fig.

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$V = -mgy$$

Here -ve sign comes becoz of particle below the reference level.



$$L = T - V$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + mgy$$

$\because$ $x^2 + y^2 = l^2$

$\therefore$ $y^2 = l^2 - x^2$

$$\boxed{y = \sqrt{l^2 - x^2}}$$

$$\dot{y} = \frac{1}{2\sqrt{l^2 - x^2}} (-2x\dot{x}) \quad \left\{ \text{Diff. w.r.to time} \right)$$

$$\boxed{\dot{y} = \frac{-x\dot{x}}{\sqrt{l^2 - x^2}}}$$

So

$$L = \frac{1}{2} m \dot{x}^2 \left[1 + \frac{x^2}{l^2 - x^2} \right] + mg\sqrt{l^2 - x^2}$$

$$\boxed{L = \left[\frac{1}{2} m \dot{x}^2 + \frac{m \dot{x}^2 l^2}{l^2 - x^2} + mg\sqrt{l^2 - x^2} \right]}$$

Here $L = f(x)$ — x is not cyclic

$\Rightarrow$ p_x is not conserved.

To know whether p_y is conserved or not we must express L in terms of y. (becoz y is also changing when pendulum moves).

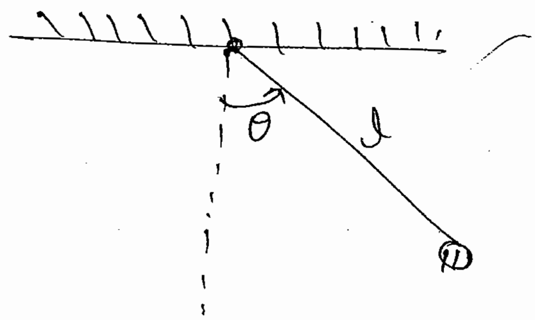
$\Rightarrow$ As pendulum oscillates in x-y plane only then z component of coordinate can not change. It can not be considered $\therefore$ z is cyclic

$\therefore$ p_z is conserved.

* Simple Pendulum in terms of θ Coordinate

We use here plane polar coordinate

$PE = 0$



DOF = 1

$$q = \theta$$

$$T = \frac{1}{2}\left(\dot{s}^2 + s^2\dot{\phi}^2\right)$$

$s = l =$ Constant $\therefore \dot{s} = 0$

$$\phi = \theta$$

$$\therefore T = \frac{1}{2}m\left(l^2 \dot{\theta}^2\right)$$

$$V = mgl\cos\theta$$

$$\therefore \boxed{L = \frac{1}{2}m\left(l^2\dot{\theta}^2\right) + mgl\cos\theta}$$

θ is not cyclic

$\therefore$ angular momentum is not Conserved.

Equation of motion –

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) - mgl\sin\theta = 0$$

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

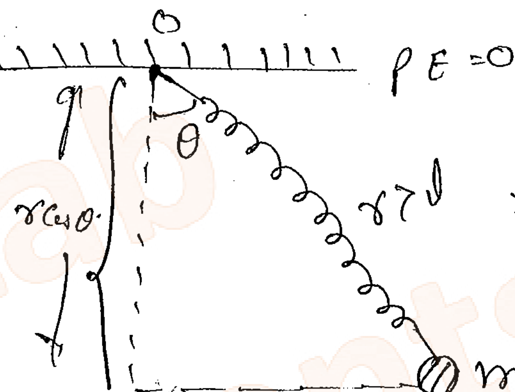
$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

Q. A simple pendulum consists of a small bob of mass 'm' suspended from light spring ~~natural length 'l'~~ with spring constant 'k'. Write Lagrangian of the system and eq^n of motion using plane polar coordinate (r,θ). Natural length of spring is l.

Sol^n

Case :- string $\longrightarrow$ spring

It is not a vertical oscillation it is a planar oscillation



DOF = 2

spring does not put any constraint.

$$T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = -mgr\cos\theta + \frac{1}{2} k (r-l)^2$$

$$L = T - V$$

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta - \frac{1}{2}k(r-l)^2$$

Equation of motion :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$q_1 = r, \quad q_2 = \theta$$

$$\frac{\partial q_i}{\partial q_j} = \delta_{ij} \quad , \quad \frac{\partial q_1}{\partial q_2} = 0$$

r – quatron →

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}(m\dot r) - mr\dot\theta^2 \cancel{- mg}\cos\theta + K(r-l) = 0$$

$$\boxed{\ddot r - r\dot\theta^2 - g\cos\theta + \frac{k}{m}(r-l) = 0} \longrightarrow (*)$$

θ – quatron →

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}(mr^2\dot\theta) + mgr\sin\theta = 0$$

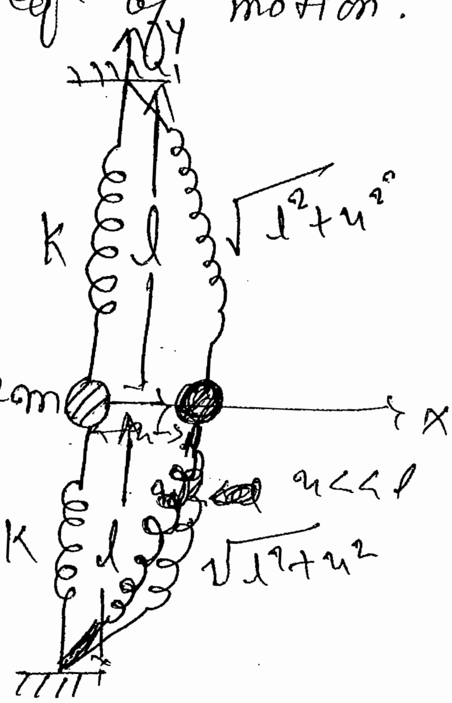
$$2r\dot r\dot\theta + r^2\ddot\theta + gr\sin\theta = 0$$

divide by r

$$\boxed{2\dot r\dot\theta + r\ddot\theta + g\sin\theta = 0} \longrightarrow (**)$$

NET-2013

Q. If the particle is slightly displaced along X-axis write its eqn of motion.



$\boxed{DOF = 1}$

Elongation $= \sqrt{l^2+x^2} - x$

$L = T - V$

$$= \frac{1}{2} m\dot x^2 \cancel{2} \frac{1}{\cancel{2}} K\left[\sqrt{l^2+x^2} - l\right]^2$$

$$\boxed{L = \frac{1}{2} m\dot x^2 - K\left[\sqrt{l^2+x^2} - l\right]^2}$$

Equation of motion: Here $\boxed{q = x}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt}[m\dot{x}] + 2k\left(\sqrt{l^2+x^2} - l\right)\cdot \frac{(\cancel{2}x)}{\cancel{2}\sqrt{l^2+x^2}} = 0$$

$$\Rightarrow m\ddot{x} + 2k\left(\sqrt{l^2+x^2} - l\right)\cdot\frac{x}{\sqrt{l^2+x^2}} = 0$$

$$\Rightarrow m\ddot{x} + 2k\left[1 - \frac{l}{\sqrt{l^2+x^2}}\right]x = 0$$

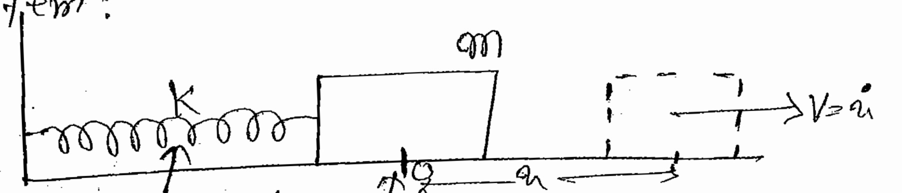
$$\Rightarrow \boxed{m\ddot{x} + 2kx\left[1 - \frac{l}{\sqrt{l^2+x^2}}\right] = 0}$$

$$\Rightarrow m\ddot{x} + 2kx\left[1 - \left(1 + \frac{x^2}{l^2}\right)^{-1/2}\right] = 0 \quad \left\{\because (1+x)^n = (1+nx)\right.$$ neglect higher term

$$\Rightarrow m\ddot{x} + 2kx\left[1 - \left(1 - \frac{x^2}{2l^2}\right)\right]$$

$$\Rightarrow \boxed{m\ddot{x} + \frac{kx^3}{l^2} = 0}$$ ← Eqn of motion for small x.

* Spring Mass System:-



Relaxed (no elongation or compression) mean position

Here elongation = x because the spring is oscillating along parallel to its length.

So

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}Kx^2$$

x = displacement from mean position.

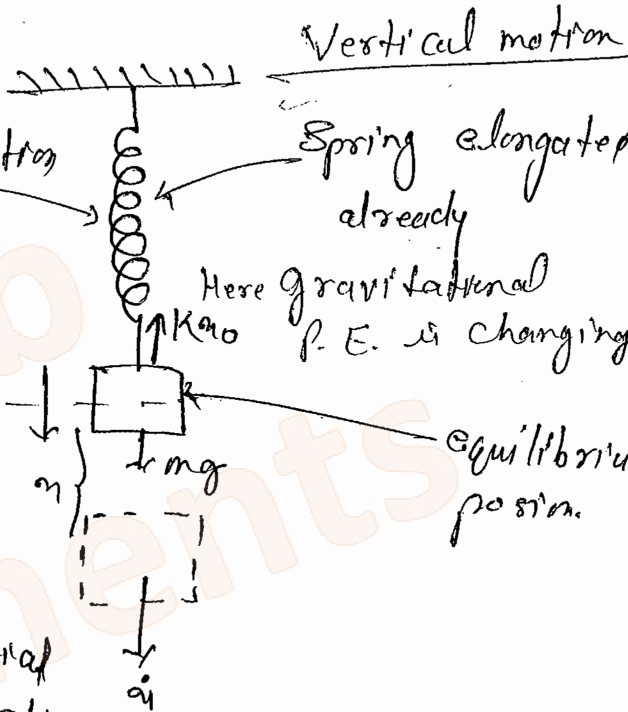
Equation of motion :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\boxed{m\ddot{x} + kx = 0}$$

Q.

Vertical motion



$L = T - V$

$T = \frac{1}{2} m \dot{x}^2$

$V = -mgx + \frac{1}{2} k (x_0 + x)^2$

$$L = \frac{1}{2} m \dot{x}^2 - \left[-mgx + \frac{1}{2} k (x_0 + x)^2\right]$$

$$\boxed{L = \frac{1}{2} m \dot{x}^2 + mgx - \frac{1}{2} k (x + x_0)^2}$$

Equation of motion :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$m\ddot{x} - mg + k(x + x_0) = 0$$

$$\boxed{m\ddot{x} + kx - mg + kx_0 = 0} \quad \text{——} (1)$$

At equilibrium position

$$mg = kx_0 \quad \text{--- (II)}$$

from (I) and (II)

$$\boxed{m\ddot{x} + kx = 0} \longrightarrow \text{Equation of motion at Equilibrium position.}$$

x is displacement from mean position.

Note :- "For spring mass system if we can neglect initial elongation and gravitational potential energy then also we get correct equation of motion."

NET -2013

Q. A particle moving in a potential

$$V(x, y, z) = x^2 + y^2 + \frac{z^2}{2}$$

If L_x, L_y and L_z be component of angular momentum then which is constant or conserved.

(a) L_x (b) L_y (c) L_z (d) None of these.

Sol^n Let us write Lagrangian in spherical polar coordinate.

$$L = T - V$$

$$\boxed{L = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2\right) - \left[r^2 \sin^2\theta + \frac{r^2 \cos^2\theta}{2}\right]}$$

$\because$ $x = r\sin\theta \cos\phi$, $y = r\sin\theta \sin\phi$, $z = r\cos\theta$.

$\because$ ϕ is not coming in L so ϕ is cyclic

So p_ϕ is conserved. || $\boxed{L_z \text{ is conserved}}$ Ans

* Lagrangian of a system is -

$$\boxed{L(x,\dot{x}) = \int_0^{\dot{x}} e^{-\alpha^2}\, d\alpha + \int_0^{x} e^{\alpha^2}\, d\alpha}$$

find eq^n of motion.

Sol^n Eq^n of motion:-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(e^{-\dot{x}^2}\right) - e^{x^2} = 0$$

$$e^{-\dot{x}^2}\cdot\left[-2\dot{x}\ddot{x}\right] - e^{x^2} = 0$$

$$\boxed{2\dot{x}\ddot{x}e^{-\dot{x}^2} + e^{x^2} = 0}$$

TIFR 2014

Q. If $f(x) = \int_0^{\infty} f(t)\, dt$ plot $f(x)$

$$\frac{d f(x)}{dx} = f(x)$$

$$\int \frac{d f(x)}{f(x)} = \int dx$$

$$\log f(x) = x + C$$

$$\boxed{f(x) = Ke^{x}}$$

* Lagrangian Dynamics:-

Spherical Polar Co-ordinate :-

$$T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2)$$

In Cylindical Co-ordinate -

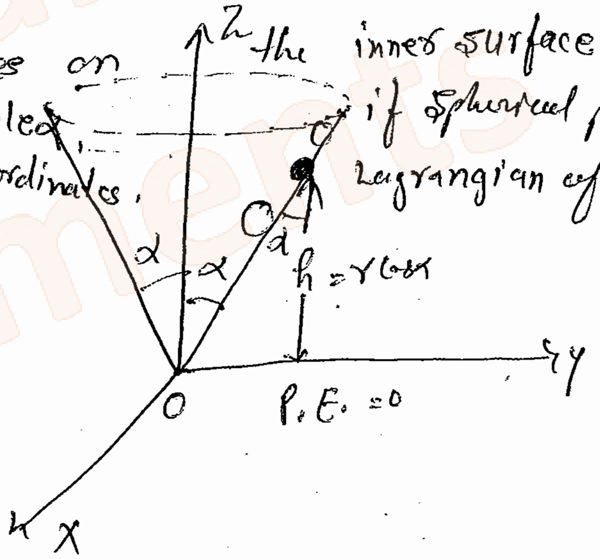
$$T = \frac{1}{2} m (\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2)$$

Plane Polar :-

$$T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2)$$

A-10
Q.9 A particle of mass 'm' moves on the inner surface of inverted cone of half vertex angle α, if spherical polar co-ordinate, are taken as generalised co-ordinates, Lagrangian of the System particle is -

Spherical polar (r, θ, ϕ)



$\theta = \alpha$, $\phi = 0$

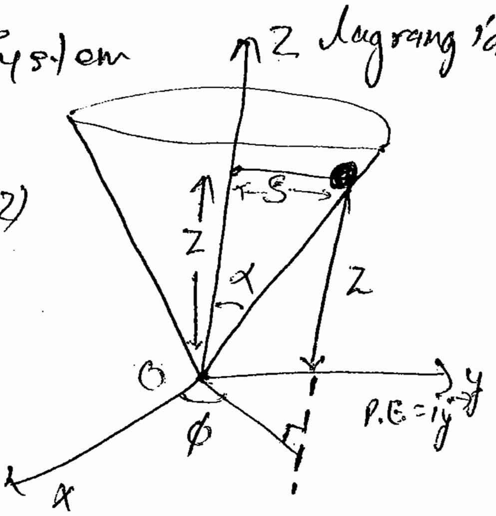
$\dot{\theta} = 0$

$L = T - V$

$$\boxed{L = \frac{1}{2} m (\dot{r}^2 + r^2 \sin^2\alpha \, \dot{\phi}^2) - mgr\cos\alpha}$$

A-10
Q.10 In cylinderical Co-ordinate System Lagrangian of bead in previous question is -

Cylindrical Coordinate (s, ϕ, z)



$$\tan\alpha = \frac{s}{z}$$

$$s = z\tan\alpha$$

$$\dot{s} = \dot{z}\tan\alpha$$

$$L = T - V$$

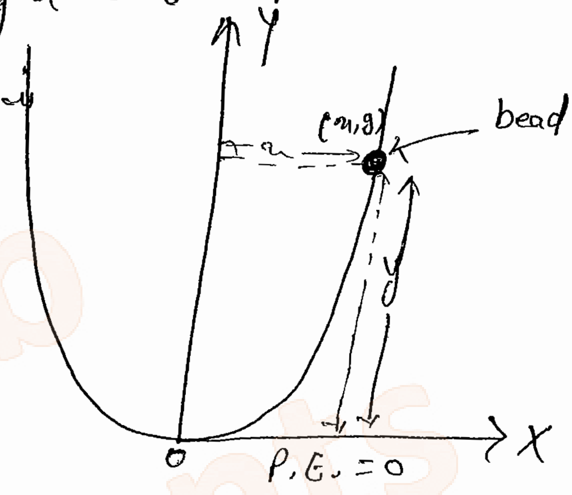
$$L = \frac{1}{2}m\left(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2\right) - mgz$$

$$\boxed{L = \frac{1}{2}m\left(\dot{z}^2\tan^2\alpha + z^2\tan^2\alpha\,\dot{\phi}^2 + \dot{z}^2\right) - mgz}$$

A-10

Q.1 A bead of mass 'm' slides along a wire kept in vertical pl as shown in fig. The eqn of wire is $y = \alpha x^2$. Lagrangian of the bead is - ?

$y = \alpha x^2$



Lagrangian of bead

$$\therefore \dot{y} = 2\alpha x\dot{x}$$

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$= \frac{1}{2}m\left(\dot{x}^2 + 4\alpha^2x^2\dot{x}^2\right) - mg\cdot\alpha x^2$$

$$\boxed{L = \frac{1}{2}m\dot{x}^2\left(1 + 4\alpha^2x^2\right) - mg\alpha x^2}$$

Write equation of motion :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left[m\dot{x}\left(1 + 4\alpha^2x^2\right)\right] - 4\alpha^2m\dot{x}^2x + 2mg\alpha x = 0$$

$$\ddot{x}\left(1 + 4\alpha^2x^2\right) + 8\alpha^2x\dot{x}^2 - 4\alpha^2\dot{x}^2x + 2mg\alpha x = 0$$

$$\boxed{\ddot{x}\left(1 + 4\alpha^2x^2\right) + 4\alpha^2x\dot{x}^2 + 2mg\alpha x = 0}$$

Note :- "Lagrangian is always constant at any instant of time."
In Lagrangian initial velocity does not consider.

-10
.2

$$L = T - V$$

$$\boxed{L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy}$$

Write eq^n of motion :-

x - equation :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$m\ddot{x} - 0 = 0$$

$$\ddot{x} = 0$$

$$\boxed{\dot{x} = \text{Constant}}$$

y - equation :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

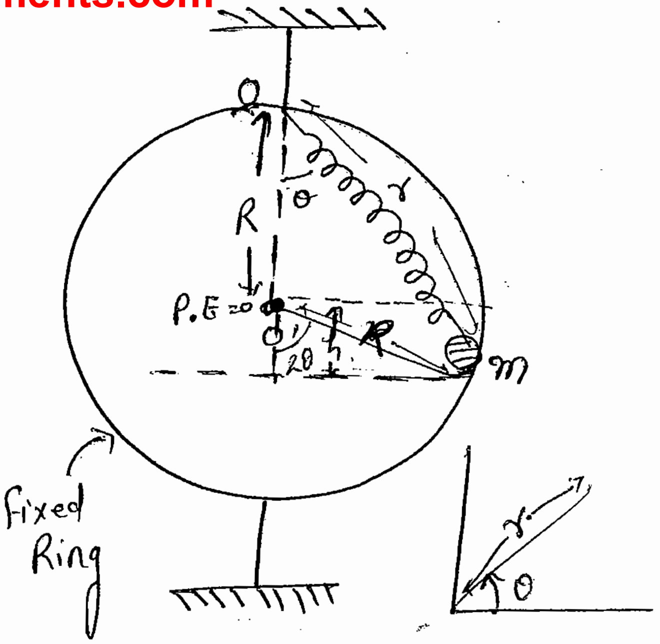
$$m\ddot{y} + mg = 0$$

$$m(\ddot{y} + g) = 0$$

$$\ddot{y} + g = 0$$

$$\boxed{\ddot{y} = -g}$$

A-10
Q.4

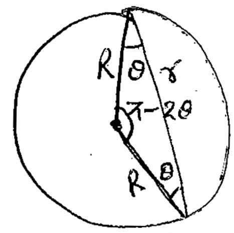


$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = -mgh + \frac{1}{2} k(r - 2R)^2$$

Use Sine rule

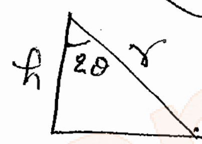
$$\frac{R}{\sin\theta} = \frac{r}{\sin(\pi - 2\theta)}$$



$$\cancel{\frac{R}{\sin\theta}} = \frac{r}{\sin 2\theta} = \cancel{\frac{r}{2\sin\theta\cos\theta}}$$

$$\therefore \quad r = 2R\cos\theta$$

$$\dot{r} = 2R\sin\theta.\dot{\theta}$$



$$h = R\cos 2\theta$$

$$L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + mgh - \frac{1}{2} k(r - 2R)^2$$

$$= \frac{1}{2} m\left(4R^2\sin^2\theta.\dot{\theta}^2 + 4R^2\cos^2\theta.\dot{\theta}^2\right) + (mg.R\cos 2\theta) - \frac{1}{2} k(2R\cos\theta - 2R)^2$$

$$= \frac{1}{2} m\left(4R^2\dot{\theta}^2\right) + (mgR\cos 2\theta) - \frac{1}{2} k\left(4R^2\cos^2\theta + 4R^2 - 8R^2\cos\theta\right)$$

$$= \frac{1}{2} m\left(4R^2\dot{\theta}^2\right) + mgR\cos 2\theta - 2kR^2\cos^2\theta + 2R^2k + 4R^2k\cos\theta$$

$$= 2mR^2\dot{\theta}^2 + mgR\cos 2\theta - 2kR^2\cos^2\theta + 2R^2k + 4R^2k\cos\theta$$

$$\boxed{L = 2mR^2\dot{\theta}^2 - 2kR^2(1 - \cos\theta)^2 + mgR\cos 2\theta}$$

Ans

Q.5 In previous question, equation of motion of bead for small value of θ is ?

(a) $\ddot{\theta} + \frac{g}{R}\theta = 0$

(b) $\ddot{\theta} + \frac{2g}{R}\theta$

(c) $\ddot{\theta} + \frac{g}{2R}\theta = 0$

(d) $\ddot{\theta} + \frac{4g}{R}\theta = 0$

Sol^n

for small values of θ

$\sin\theta = \theta$

$\cos\theta = 1$

$$L = 2mR^2\dot{\theta}^2 - 2kR^2(1-\cos\theta)^2 + mgR\cos 2\theta$$

$$L = 2mR^2\dot{\theta}^2 + mgR\cos 2\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) + \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow 4mR^2\ddot{\theta} + mgR \cdot 2\sin 2\theta = 0$$

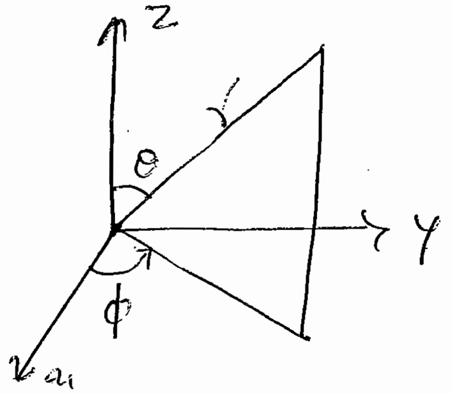
$$4mR^2\ddot{\theta} + 4mgR\theta = 0$$

$$4mR^2\left(\ddot{\theta} + \frac{g}{R}\theta\right) = 0$$

$$\boxed{\ddot{\theta} + \frac{g}{R}\theta = 0}$$

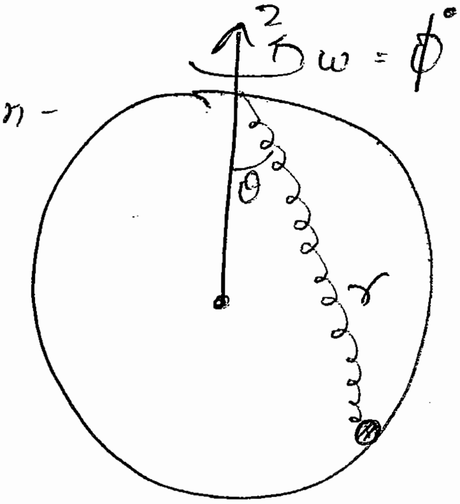
Ans

* If the ring is rotated with angular speed ω as shown :-



Write Lagrangian -

$\sin\theta \simeq \theta$
$\cos\theta \simeq 1$



$p_\phi = L_z$

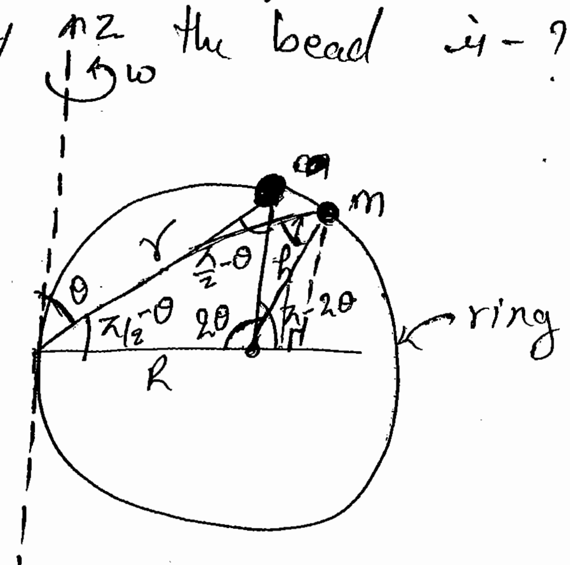
ϕ represents rotation about z axis (on spherical & cylindrical coordinate)

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2\right)$$

$$\boxed{T = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta \, \omega^2\right)}$$

A-10

Q.11 A bead of mass 'm' is sliding on a vertical circular loop of radius R. The loop is rotated with constant angular velocity ω about a tangential axis shown in figure. Lagrangian of the bead is - ?



Sol^n

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

$$V = mgh = mgR\sin(\pi - 2\theta)$$

$$V = mgR\sin 2\theta$$

Use sine Rule

$$\frac{R}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{r}{\sin 2\theta}$$

$$\boxed{r = 2R\sin\theta}$$

$$\dot{r} = 2R\cos\theta\,\dot{\theta}$$

$$\dot{\phi} = \omega,$$

$$T = \frac{1}{2} m\left(4R^2\cos^2\theta\,\dot{\theta}^2 + 4R^2\sin^2\theta\,\dot{\theta}^2 + 4R^2\sin^2\theta\,\omega^2\right)$$

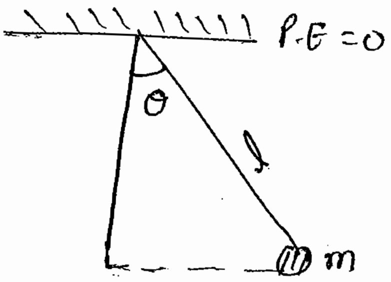
$$T = \frac{1}{2} m\left(4R^2\dot{\theta}^2 + 4R^2\sin^2\theta\,\omega^2\right)$$

$$L = T - V$$

* Simple Pendulum :-

Oscillations is in plane. $r = l$, $\dot{r} = 0$

$$\boxed{L = \frac{1}{2} m l^2\dot{\theta}^2 + mgl\cos\theta}$$



Spherical Pendulum :-

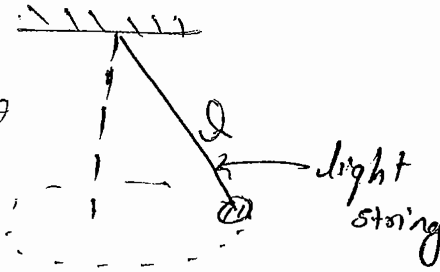
Plane of oscillation is not fixed (there is oscillation about z-axis)

Use spherical polar.

$r = l$, θ $\dot{r} = 0$

$$L = \frac{1}{2} m (\dot\theta^2 l^2 + l^2 \sin^2\theta\, \dot\phi^2) + mgl\cos\theta$$

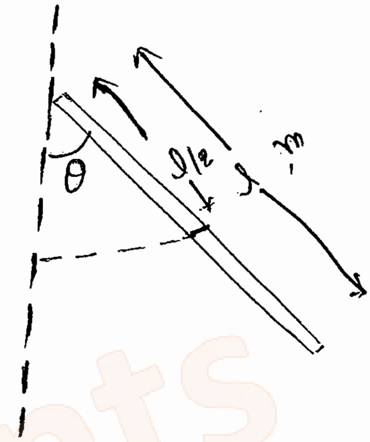
$$\boxed{L = \frac{1}{2} m l^2 (\dot\theta^2 + \sin^2\theta\, \dot\phi^2) + mgl\cos\theta}$$



* A thin rod suspended from one end :-

$$L = \frac{1}{2}\left(\frac{ml^2}{3}\right)(\dot\theta^2 + \sin^2\theta\, \dot\phi^2) + mg\frac{l}{2}\cos\theta$$

$$\boxed{L = \frac{-ml^2}{6}(\dot\theta^2 + \sin^2\theta\, \dot\phi^2) + \frac{mgl\cos\theta}{2}}$$



* A rod is suspended from middle (free to rotate in any orientation) :-

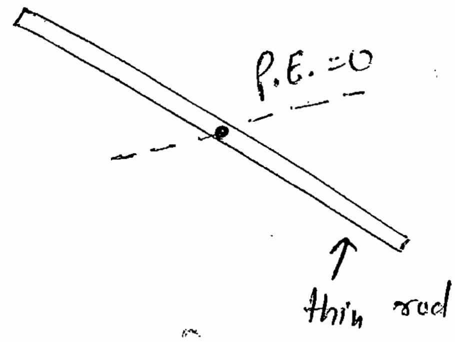
Centre of is not moving.

$K = \frac{1}{2} I\omega^2$

$K = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega}$

$$L = \frac{1}{2}\left(\frac{ml^2}{12}\right)\left(\dot\theta^2 + \sin^2\theta\, \dot\phi^2\right)$$

$$\boxed{L = \frac{ml^2}{24}(\dot\theta^2 + \sin^2\theta\, \dot\phi^2)}$$



A-10
Q. 12

A-10 Q.25 Lagrangian of a system is $L = a\dot{x}^2 - bx^2$ then

(a) $x = c_1 t + c_2 t^2 + c_3$ (b) $x = c_1 e^{-c_2 t} + c_3$

(c) $x = c_1 \sin(c_2 t + c_3)$ (d) $x = c_1 e^{-c_2 t} \sin(c_2 t + c_3)$

Soln

Equation of motion -

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$2a\ddot{x} + 2bx = 0$$

$$\ddot{x} + \frac{b}{a}x = 0$$

$$\boxed{\text{roots} = \pm i\sqrt{\frac{b}{a}}}$$ purely imaginary.

$$\boxed{x = c_1 \sin(c_2 t + c_3)}$$

B-A-2
Q.47 NET 2012 Dec.

Soln

$$L = \frac{1}{2}m\dot{x}^2 - bx$$

Equation of motion:-

$$m\ddot{x} + b = 0$$

$$\ddot{x} = -\frac{b}{m}$$

$$\frac{d^2x}{dt^2} = -\frac{b}{m}$$

Integrate -

$$\frac{dx}{dt} = -\frac{b}{m}t + c_1$$

$$\boxed{x = -\frac{b}{m}t^2 + c_1 t + c_2}$$

Ans

Imp. A-10

Q.24 A planet of mass 'm' revolves around the Sun of mass M in an elliptical orbit. If motion of planet is confined in one plane, Lagrangian of the system is -?

(a) $\frac{1}{2}(\dot{r}^2+r^2\dot{\theta}^2)+\frac{GMm}{r}$

(b)

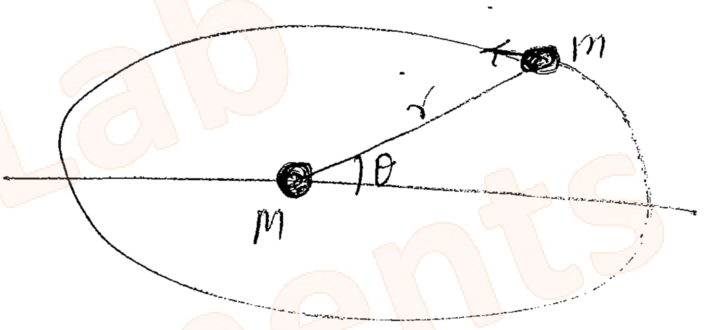
(c)

(d)

Sol^n

$T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)$

$V=-\frac{GMm}{r}$



$L = T-V$

$$\boxed{L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+\frac{GMm}{r}}$$

Q.22

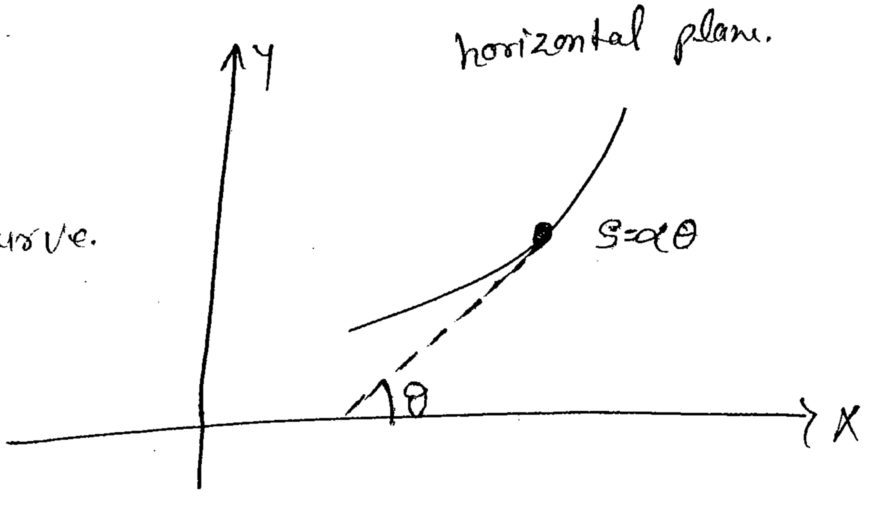
Sol^n

Equation of curve

s = dist along curve.

$L=\frac{1}{2}mv^2$ —0

$=\frac{1}{2}m\left(\frac{ds}{dt}\right)^2$



$\therefore s = \alpha\theta$

$$\boxed{L = \frac{1}{2} m\alpha^2\dot{\theta}^2}$$

$$\frac{ds}{dt} = \alpha\dot{\theta}$$

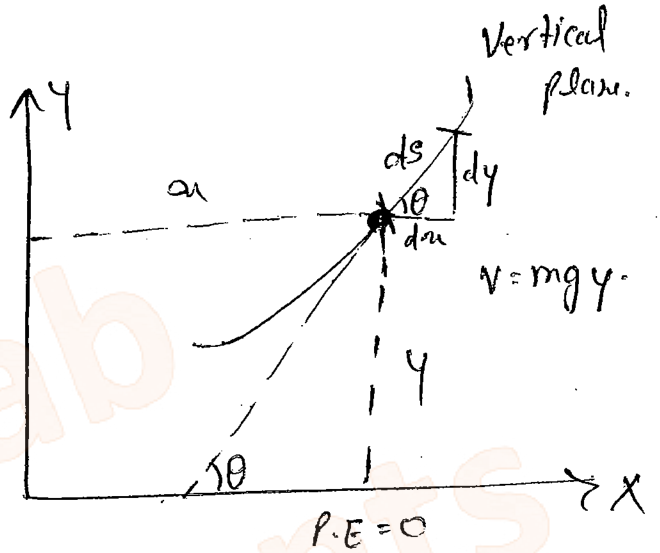
Q. In previous question plane is vertical write lagrangian in θ co-ordinate.

Sol^n

$L = T - V$

$V = mgy$

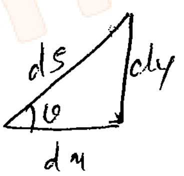
$L = \frac{1}{2} m\alpha^2\dot{\theta}^2 + mg\alpha\cos\theta$



$\sin\theta = \frac{dy}{ds}$

$s = \alpha\theta$

$ds = \alpha d\theta$



$\sin\theta = \frac{dy}{\alpha d\theta}$

$\int dy = \int \alpha \sin\theta \, d\theta$

$$\boxed{y = -\alpha\cos\theta}$$

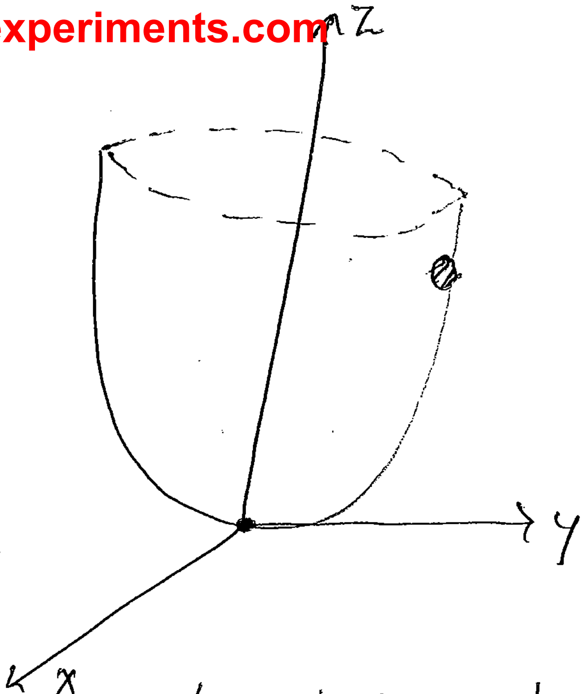
NET-2011

Q.38 A particle of mass 'm' moves inside a bowl. If the surface of the bowl is given by the equation $z = \frac{1}{2}a(x^2+y^2)$, where 'a' is a constant, the Lagrangian of the particle is.

Sol^n

$$\boxed{z = \frac{1}{2} a(x^2 + y^2)}$$

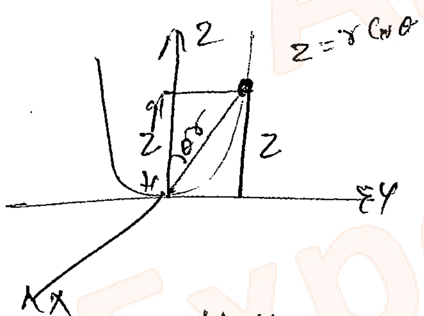
a = Constant.



He co-ordinate is not given so we check which co-ordinate system we use –

When we ~~See~~ option we confused whether it is spherical ~~pto~~ polar of cylindrical ~~po~~ co-ordinate.

So we check potential energy in both region so identify the co-ordinate system.



$V = mgz$

$\boxed{V = mgr\cos\theta}$ not match

Whether we check $z = \frac{1}{2} a(x^2+y^2)$ given

So $z = \frac{1}{2} a r^2 \sin^2\theta$ where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$

which is also mismatch.

So we check in cylindrical co-ordinate:–

$V = mgz$

$x = r\cos\phi$

$y = r\sin\phi$

$z = z$

So $\boxed{V = mg\frac{1}{2}ar^2}$ match in 3 options.

Here options are independent of z so we have to remove z.

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right)$$

$$z = \frac{1}{2} a r^2$$

$$\dot{z} = a r \dot{r}$$

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2 + a^2 r^2 \dot{r}^2\right)$$

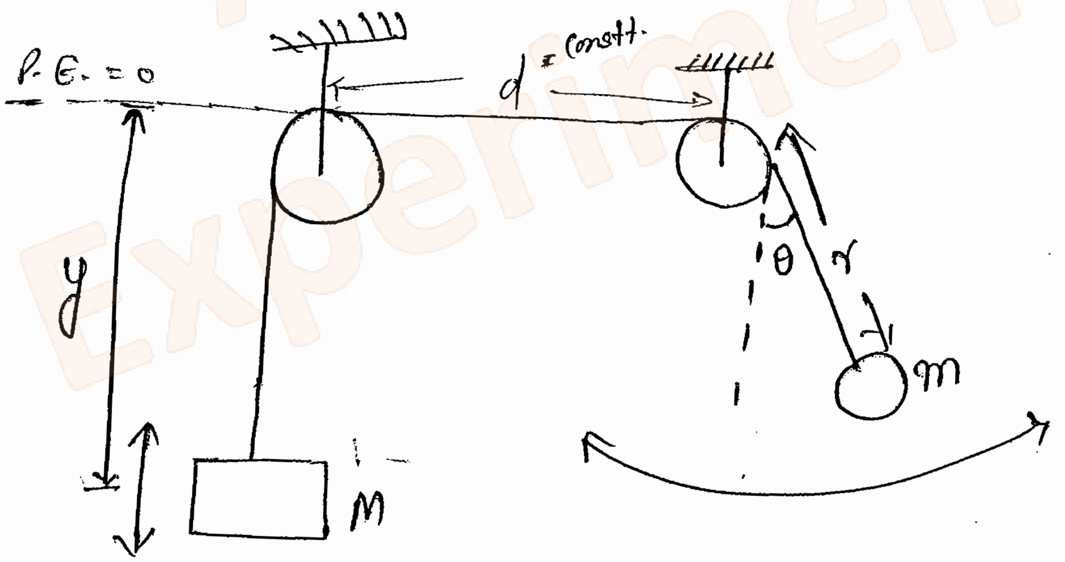
$$\boxed{T = \frac{1}{2} m\left(\dot{r}^2(1 + a^2 r^2) + r^2\dot{\phi}^2\right)}$$

$$L = T - V$$

$$L = \frac{1}{2} m\left(\dot{r}^2(1 + a^2 r^2) + r^2\dot{\phi}^2\right) - \frac{1}{2} m g a r^2$$

$$\boxed{L = \frac{1}{2} m\left(\dot{r}^2(1 + a^2 r^2) + r^2\dot{\phi}^2 - g a r^2\right)}$$ Ans

Q.



Write the eq^n of motion of the two blogs.

Sol^n DOF = 2.

Length of string is constant. if it is inextensible

$$y + d + r = \text{Constant} = \text{length of string} = l.$$

$\dot{y} + 0 + \dot{r} = 0$

$$\boxed{\dot{y} = -\dot{r}}$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m \dot{y}^2$$

$$\boxed{T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m \dot{r}^2}$$

$$V = -Mgy - mgr\cos\theta$$

$$\boxed{V = -Mg(l - d - r) - mgr\cos\theta.}$$

$$\boxed{L = \frac{1}{2}\left[(m + M)\dot{r}^2 + mr^2\dot{\theta}^2\right] + Mg(l - d - r) + mgr\cos\theta.}$$

Equ^n of motion -

r - eq^n :-

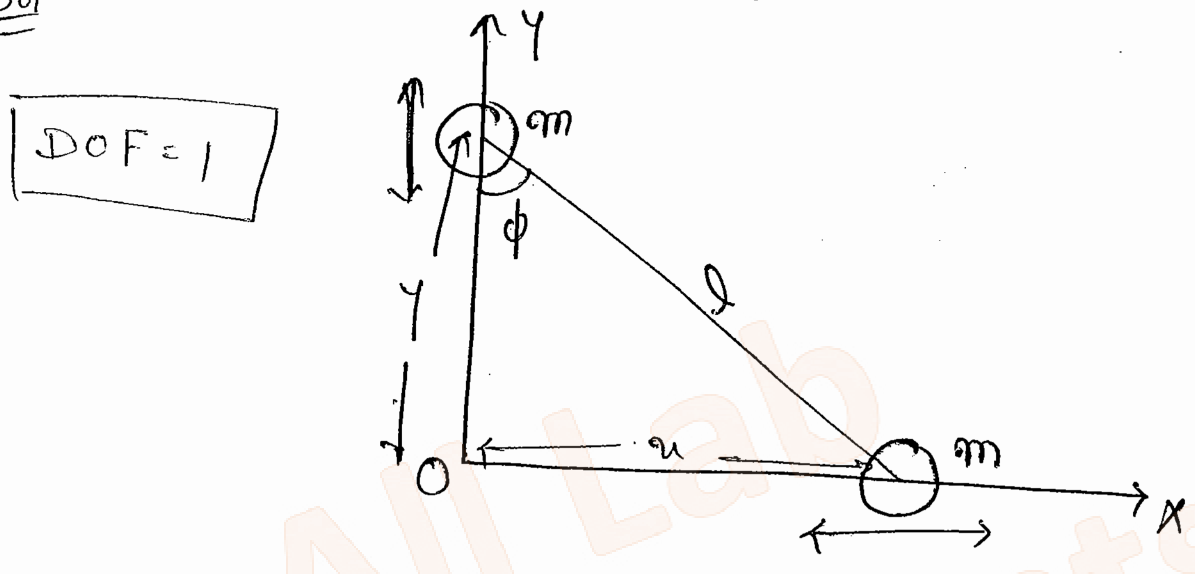
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\boxed{(m+M)\ddot{r} + Mg - mg\cos\theta - mr\dot{\theta}^2 = 0}$$ eq^n of motion.

$\ddot{r} = g(m\cos\theta - M)$

Q. Two particle connected by a light rod. Particles are constrained to move along two $\perp^{or}$ lines. If ϕ is generalied co-ordinate What is K.E. of the system.

Sol^n



$$K.E. = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2$$

$$K.E. = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

from $\triangle$

$$x = l \sin\phi \Rightarrow \dot{x} = l \cos\phi \, \dot{\phi}$$

$$y = l \cos\phi \Rightarrow \dot{y} = -l \sin\phi \, \dot{\phi}$$

So $\boxed{K.E. = \frac{1}{2} m l^2 \dot{\phi}^2}$

* Small Oscillation :-

Normal Modes :

It is a special type of oscillation in which all particles of system oscillates with same frequency.

Finding frequency of Normal modes :-

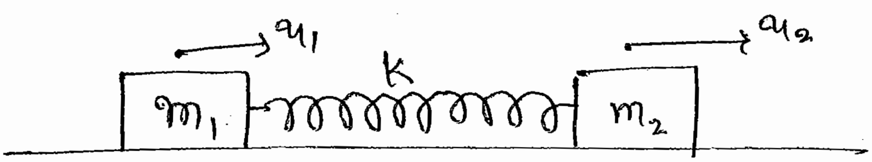
It is obtained by solving following matrix equation or determinant.

$$\left| \hat{V} - \omega^2 \hat{T} \right| = 0$$

Where $\hat{V}$ is a matrix obtained from ~~kinetic~~ potential energy.
$\hat{T}$ " " " " Kinetic "

To obtain $\hat{V}$ we take cofficients of q_i, q_j in P.E.
" $\hat{T}$ " " " K.E.

Q. Two masses connected by a light spring. masses are free to oscillate.



find frequency (angular) of Normal modes?

Sol^n DOF = 2

Let x_1 and x_2 be displacement of m_1 and m_2 from their mean position

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2}\left(m_1 \dot{x}_1 \dot{x}_1 + m_2 \dot{x}_2 \dot{x}_2\right)$$

$$\therefore \quad \hat{T} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

{ Here $\frac{1}{2}$ is not consider becoz it is canceled out finally.

$$V = \frac{1}{2} k (x_2 - x_1)^2 = \frac{1}{2} k \left(x_1^2 + x_2^2 - 2 x_1 x_2\right)$$

$$V = \frac{1}{2}\left(k x_1^2 + k x_2^2 - k x_1 x_2 - k x_2 x_1\right)$$

$$\therefore \quad \hat{V} = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$$

Secular Equation :-

$$\left| \hat{V} - \omega^2 \hat{T} \right|$$

$$\begin{vmatrix} k - m_1 \omega^2 & -k \\ -k & k - m_2 \omega^2 \end{vmatrix} = 0$$

$$(k - m_1 \omega^2)(k - m_2 \omega^2) - k^2 = 0$$

$$\cancel{K^2} - K(m_1+m_2)\omega^2 + m_1 m_2 \omega^4 - \cancel{K^2} = 0$$

$$\omega^2 \left[-K(m_1+m_2) + m_1 m_2 \omega^2 \right] = 0$$

$$\omega^2 = 0 \longrightarrow \omega = 0$$

$$\omega^2 = \frac{K(m_1+m_2)}{m_1 m_2}$$

$$\omega = \sqrt{\frac{K}{\left(\frac{m_1 m_2}{(m_1+m_2)}\right)}}$$

Here −ve term is not considered becoz freq. is never negative

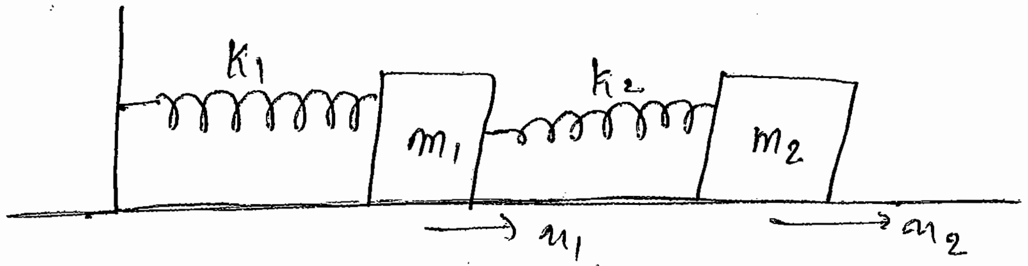
$$\boxed{\omega = \sqrt{\frac{K}{\mu}}}$$

Where

$$\boxed{\mu = \frac{m_1 m_2}{m_1 + m_2}}$$

Note :- If system is not rigidly fixed then one of freq. comes out to be zero.

Q.



If x_1 and x_2 are displacement from mean positions.

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = \frac{1}{2} K_1 x_1^2 + \frac{1}{2} K_2 (x_2 - x_1)^2$$

$$\hat{T} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

$$\hat{V} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}$$

<u>Note</u> :- If x and θ are involved in a question then to calculate ω we will consider $(l\theta)$ and x as two coordinates [so that the two co-ordinates so that the two co-ordinates have same dimension.

* Approximation :-

$$mgl \cos\theta \simeq mgl\left(1 - \frac{\theta^2}{2}\right)$$

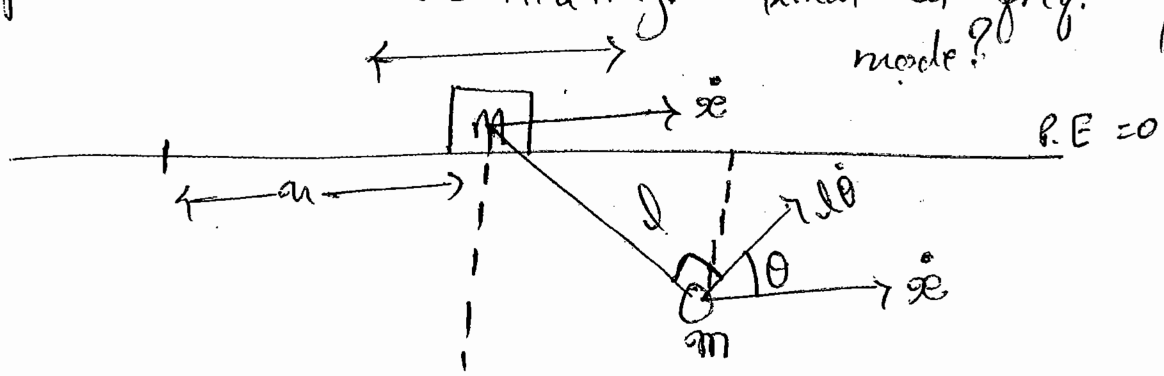
$$\dot{x}\dot{\theta} \cos\theta \simeq \dot{x}\dot{\theta}\left(1 - \frac{\theta^2}{2}\right)$$

$$\simeq \dot{x}\dot{\theta} - \dot{x}\dot{\theta}\frac{\theta^2}{2}$$

for small oscillation → very small ≈ 0

$$\simeq \dot{x}\dot{\theta}$$

Q. Block can move forward and backward and pendulum is oscillating. What is freq. of normal mode?



$$T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m\left[\dot{x}^2 + (l\dot{\theta})^2 + 2\dot{x}(l\dot{\theta})\cos\theta\right]$$

for small oscillations

$$T \simeq \frac{1}{2}\left[M\dot{x}^2 + m\left[\dot{x}^2 + (l\dot{\theta})^2 + 2\dot{x}(l\dot{\theta})\right]\right]$$

$x \rightarrow$ first coordinate

$l\theta \rightarrow$ Second Co-ordinate.

$$\hat{T} = \begin{pmatrix} M+m & m \\ m & m \end{pmatrix}$$

$$V = -mgl\cos\theta$$

$$= -mgl\left(1 - \frac{\theta^2}{2}\right) = -mgl + \frac{1}{2} mgl\,\theta^2$$

$$= -mgl + \frac{1}{2}\underline{\frac{mg}{l}}\,\underline{(l\theta)^2}$$

So $\hat{V} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{mg}{l} \end{pmatrix}$

So Secular equation -

$$|\hat{V} - \omega^2\hat{T}| = 0$$

$$|\omega^2\hat{T} - \hat{V}| = 0$$

$$\begin{vmatrix} \omega^2(M+m) & m\omega^2 \\ m\omega^2 & m\omega^2 - \frac{mg}{l} \end{vmatrix} = 0$$

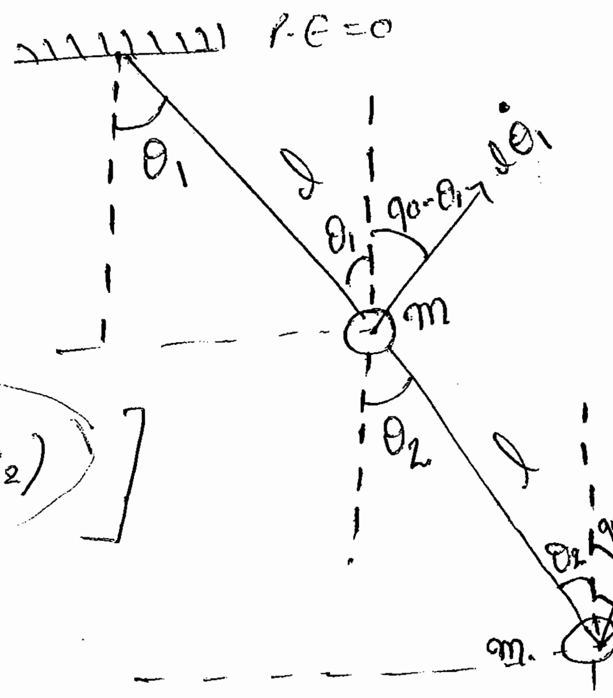
$$\omega^2(M+m)\cancel{m}\left(\omega^2 - \frac{g}{l}\right) - m^{\cancel{2}}\omega^4 = 0$$

$$\omega^2\left[(M+m)\left(\omega^2 - \frac{g}{l}\right) - m\omega^2\right] = 0$$

$$\Rightarrow \boxed{\begin{aligned} \omega &= 0 \\ \omega &= \sqrt{\frac{g}{l}\left(1 + \frac{m}{M}\right)} \end{aligned}}$$

If $M \to \infty$ then we get result of simple pendulum.

Q. Write $\hat{T}$ and $\hat{V}$.



$$T = \frac{1}{2} m (l\dot{\theta}_1)^2 + \frac{1}{2} m\left[(l\dot{\theta}_1)^2 + (l\dot{\theta}_2)^2 + 2(l\dot{\theta}_1)(l\dot{\theta}_2)\underbrace{\cos(\theta_1 - \theta_2)}_{\approx 1}\right]$$

$$\hat{T} = \begin{pmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{pmatrix}$$

θ_1 = first co-ordinate

θ_2 = second co-ordinate.

$$V = -mgl\cos\theta_1 - mgl(\cos\theta_1 + \cos\theta_2)$$

$$= -mgl\,[2\cos\theta_1 + \cos\theta_2]$$

$$= -mgl\left[2\left(1 - \frac{\theta_1^2}{2}\right) + \left(1 - \frac{\theta_2^2}{2}\right)\right]$$

$$V = -3mgl + \frac{1}{2}\left[2mgl\,\theta_1^2 + mgl\,\theta_2^2\right]$$

$$\hat{V} = \begin{pmatrix} 2ml & 0 \\ 0 & mgl \end{pmatrix}$$

Hamiltonian Dynamics

A transition from Lagrangian function to Hamiltonian function.

$$L \xrightarrow[\text{Transformation}]{\text{Legendre}} H$$

$$\boxed{H(q_i, p_i) = p_i \dot{q}_i - L(q_i, \dot{q}_i)}$$

Hamiltonian.

In hamiltonian dynamics q_i & p_i are taken as independent variable.

$$\frac{\partial p_i}{\partial q_i} = 0 \quad , \quad \frac{\partial q_i}{\partial p_i} = 0$$

$\frac{\partial \dot{q}_i}{\partial q_i}$ may not be zero

$\frac{\partial \dot{q}_i}{\partial p_i}$ may not be zero.

$$\frac{\partial p_i}{\partial p_j} = \delta_{ij} \quad , \quad \frac{\partial q_i}{\partial q_j} = \delta_{ij}$$

* Hamilton's Equation of motion:-

It is observed from -

① Legendre Transform

② Variational principle.

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}}$$

$$\boxed{\dot{p}_i = \frac{-\partial H}{\partial q_i}}$$

* **Dynamical Variable :-** A function of any quantity which is function of q_i, p_i and t
If A is dynamical variable, Then -

$$A = A(q_i, p_i, t)$$

* **Possion's Equations of Motion :-** It gives the rate of change of a dynamical variable.

$$\frac{dA}{dt} = \frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i + \frac{\partial A}{\partial t}$$

$$= \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial A}{\partial p_i}\left(\frac{-\partial H}{\partial q_i}\right) + \frac{\partial A}{\partial t}$$

$$= \underbrace{\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i}}_{\text{Possion Bracket of } A \text{ with } H.} + \frac{\partial A}{\partial t}$$

$$\boxed{\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t}}$$

This is possion's eqn of motion.

If A does not explicitly depends on time t.

$$\frac{\partial A}{\partial t} = 0$$

$$\frac{dA}{dt} = [A, H]$$

$$\boxed{\dot{A} = [A, H]}$$

If A is constant

$$\dot{A} = 0$$

$$\boxed{[A, H] = 0}$$

If a quantity is constant then its possion's Bracket with H is zero.

* Hamilton's Equation in terms of possion's bracket :-

$$\boxed{\begin{aligned} \dot{q}_i &= [q_i, H] \\ \dot{p}_i &= [p_i, H] \end{aligned}}$$ → Gate 2014

or

$$\boxed{\begin{aligned} [q_i, H] &= \frac{\partial H}{\partial p_i} \\ [p_i, H] &= -\frac{\partial H}{\partial q_i} \end{aligned}}$$

* Conversion of Lagrangian (L) into Hamiltonian (H):-

Steps:-

(I) Use $p_i = \frac{\partial L(q_i, \dot{q}_i)}{\partial \dot{q}_i}$ and find $\dot{q}_i$

(II) Use $H = p_i \dot{q}_i - L$ and put the value of $\dot{q}_i$

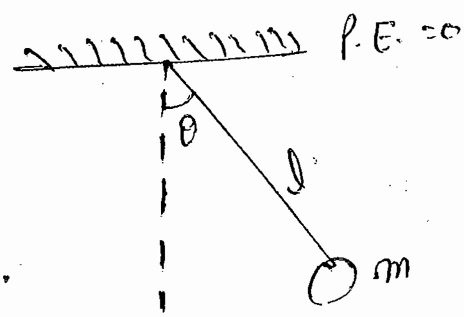
* Conversion of Hamiltonian (H) into Lagrangian (L):-

Steps:-

(1) Use $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and find p_i

(2) Use $L = p_i \dot{q}_i - H$ and put value of p_i

Simple Pendulum



$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

Suppose it is given find H.

Sol^n

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\dot{\theta} = \frac{p_\theta}{ml^2}$$

$$H = p_\theta \dot{\theta} - L$$

So $H = p_\theta \dot{\theta} - \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta$

put $\dot{\theta}$

$$H = \frac{p_\theta^2}{ml^2} - \frac{1}{2}\frac{p_\theta^2}{ml^2} - mgl\cos\theta$$

$$\boxed{H = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta}$$

Net 2014 June

Q. $L = \frac{1}{2} m\dot{q}^2 - \frac{a}{2} q\dot{q}^2$ find H ?

Solⁿ

$$p = \frac{\partial L}{\partial \dot{q}}$$

$$p = m\dot{q} - a q\dot{q} = \dot{q}(m - aq)$$

$$\boxed{\dot{q} = \frac{p}{m - aq}}$$

$$H = p\dot{q} - L = p\dot{q} - \frac{1}{2} m\dot{q}^2 + \frac{a}{2} q\dot{q}^2$$

put $\dot{q}$

$$H = \frac{p^2}{m - aq} - \frac{1}{2}\dot{q}^2 (m - aq)$$

$$= \frac{p^2}{m - aq} - \frac{1}{2}\frac{p^2}{(m - aq)^{\cancel{2}}} \cancel{(m - aq)}$$

$$\boxed{H = \frac{p^2}{2(m - aq)}}$$

Q. $L = \frac{1}{2} m v^2 + \vec{a}\cdot\vec{v}$ where $\vec{v}$ = velocity, $\vec{a}$ = Constant vector.

find $H = ?$

Soln Some mathematical methods -

$$\vec{a} \longleftrightarrow a_i$$

$$\vec{a}.\vec{b} \longleftrightarrow a_i b_i$$

$$\vec{a}^2 \longleftrightarrow a_i^2$$

{when repetation of index then we assume Sumation}

Summation over repeated index (i) is assumed.

$$(\vec{a}\times\vec{b})_i = \varepsilon_{ijk}\, a_j b_k$$

ε_{ijk} → Levicivita Tensor

$$\varepsilon_{ijk} = \hat{i}.(\hat{j}\times\hat{k})$$

if two index are equal then $\varepsilon = 0$

$$\varepsilon_{iik} = \hat{i}\cdot(\hat{i}\times\hat{k}) = 0$$

$$\varepsilon_{ijk} = -\varepsilon_{ikj}$$

$$\varepsilon_{ijj} = 0$$

$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$$

Now come in question -

$$L = \frac{1}{2} m v_i^2 + a_i v_i$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial v_i}$$

generalised velocity

$$p_i = m v_i + a_i$$

$$v_i = \frac{p_i - a_i}{m}$$

$$H = p_i \dot{q}_i - L$$

$$= p_i v_i - \frac{1}{2} m v_i^2 - a_i v_i$$

$$H = p_i \left(\frac{p_i - a_i}{m} \right) - \frac{1}{2} \frac{(p_i - a_i)^2}{m} - \frac{a_i (p_i - a_i)}{m}$$

$$= \frac{(p_i - a_i)}{m} (p_i - a_i) - \frac{(p_i - a_i)^2}{2m}$$

$$= \frac{(p_i - a_i)^2}{2m}$$

$$\boxed{H = \frac{(\vec{p} - \vec{a})^2}{2m}}$$

* Some Standard Lagrangian and Corresponding Hamiltonian :-

(1) Free Particle (V = 0)

(i) Non - Relativistic :-

$$L = \frac{1}{2} m v^2$$

$$H = \frac{p^2}{2m}$$

(ii) Relativistic Case :-

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}}$$ ← It gives correct expression for momentum.

$$H = \sqrt{p^2 c^2 + m_0^2 c^4}$$

(2) Particle moving in electromagnetic field (ϕ, $\vec{A}$)

ϕ ↑ Electric Potential, $\vec{A}$ ↑ mag. Vector poten.

(i) Non - Relativistic :-

$$L = \frac{1}{2} m v^2 - q\phi + q\vec{A}\cdot\vec{v}$$

$$H = \frac{(\vec{P} - q\vec{A})^2}{2m} + q\phi$$

$P.E(V) = \underbrace{q\phi - q\vec{A}\cdot\vec{v}}$ { q = charge. }

It gives correct Lorentz force.

(ii) Relativistic Case :-

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} - q\phi + q\vec{A}\cdot\vec{v}$$

$$H = \sqrt{(\vec{P} - q\vec{A})^2 c^2 + m_0^2 c^4} + q\phi$$

$$L = T - V$$

$$L = mc^2 - m_0c^2$$

× $$L = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2$$ → We can not write Lagrangian because it can not give correct expression for momentum.

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial v} = \frac{m_0 v^2}{(1-v^2/c^2)^{3/2}}$$

but this is not correct.

Correct expression for momentum.

$$p = mv = \frac{m_0 v}{\sqrt{1-v^2/c^2}}$$

but $$L = -m_0c^2\sqrt{1-v^2/c^2}$$

$$\frac{\partial L}{\partial v} = p = \frac{m_0 v}{\sqrt{1-v^2/c^2}}$$ This is the correct expression of Lagrangian.

* Conversion of Lagrangian into Hamiltonian for relativistic case :-

$$L = -m_0c^2\sqrt{1-\frac{\vec{v}^2}{c^2}}$$

$$L = -m_0c^2\sqrt{1-\frac{v_i^2}{c^2}} \qquad \{\because \vec{a} \rightarrow a_i\}$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial v_i}$$

$$p_i = \frac{-m_0\cancel{c^2}}{\cancel{2}\sqrt{1-\frac{v_i^2}{c^2}}}\left(-\frac{\cancel{2}v_i}{\cancel{c^2}}\right)$$

$$p_i = \frac{m_0 v_i}{\sqrt{1-\frac{v_i^2}{c^2}}}$$

Now squaring both side

$$p_i^2\left(1-\frac{v_i^2}{c^2}\right)=m_0^2 v_i^2$$

$$p_i^2 = v_i^2\left[m_0^2+\frac{p_i^2}{c^2}\right]$$

$$v_i^2 = \frac{p_i^2 c^2}{m_0^2c^2+p_i^2}$$

$$H = p_i v_i - L$$

$$= p_i v_i + m_0 c^2\sqrt{1-\frac{v_i^2}{c^2}}$$

$$= \frac{p_i p_i c}{\sqrt{m_0^2c^2+p_i^2}} + m_0c^2\sqrt{1-\frac{p_i^2}{m_0^2c^2+p_i^2}}$$

$$= \frac{p_i^2 c}{\sqrt{m_0^2c^2+p_i^2}} + \frac{m_0c^2\, m_0 c}{\sqrt{m_0^2c^2+p_i^2}}$$

$$= \frac{c\,(p_i^2+m_0^2c^2)}{\sqrt{p_i^2+m_0^2c^2}} = c\sqrt{p_i^2+m_0^2c^2}$$

$$= \sqrt{p_i^2c^2+m_0^2c^4}$$

So $\boxed{H = \sqrt{p^2c^2+m_0^2c^4}}$

Net - 2015 ...

Q. If $L = -\sqrt{1-\dot{q}^2} - V(q)$ find $H = ?$

Sol^n

$$\boxed{H = \sqrt{p_q^2 + 1} + V(q)}$$

Standard form. { here $m_0 = 1$, $c = 1$

* Question on $H \longrightarrow L$:-

Q. $H = \frac{p^2}{2m} + \vec{a}\cdot\vec{p}$ find Lagrangian ?

Sol^n

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \Rightarrow \because H = \frac{p_i^2}{2m} + a_i p_i$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = v_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m} + a_i$$

$$p_i = (v_i - a_i)m$$

$$L = p_i\dot{q}_i - H = p_i\dot{q}_i - \frac{p_i^2}{2m} - a_i p_i$$

put p_i

$$L = (v_i - a_i)m v_i - \frac{m}{2}(v_i - a_i)^2 - a_i m(v_i - a_i)$$

$$= m(v_i - a_i)(v_i - a_i) - \frac{m}{2}(v_i - a_i)^2$$

$$= m(v_i - a_i)^2 - \frac{m}{2}(v_i - a_i)^2$$

$$= \frac{1}{2}m(v_i - a_i)^2$$

So $$\boxed{L = \frac{1}{2}m(\vec{v} - \vec{a})^2}$$ Ans

Q. Hamiltonian of the system is $H = ap_r^2 + \frac{bp_\theta^2}{r^2} + c\cos\theta$
Where a, b, c are constants find lagrangian?

Soln Here two co-ordinates are used which is r and θ. So velocity corresponding to r is $\dot{r}$ and corresponding to θ is $\dot{\theta}$.

$$q_i = r, \theta$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = 2ap_r$$

$$\boxed{p_r = \frac{\dot{r}}{2a}}$$

Now

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{2bp_\theta}{r^2} \Rightarrow \boxed{p_\theta = \frac{r^2\dot{\theta}}{2b}}$$

$$L = p_i\dot{q}_i - H$$

$$= p_r\dot{r} + p_\theta\dot{\theta} - ap_r^2 - \frac{bp_\theta^2}{r^2} - c\cos\theta$$

Put p_r and p_θ

$$L = \frac{\dot{r}^2}{2a} + \frac{r^2\dot{\theta}^2}{2b} - \frac{\dot{r}^2}{4a} - \frac{r^2\dot{\theta}^2}{4b} - c\cos\theta$$

$$\boxed{L = \frac{\dot{r}^2}{4a} + \frac{r^2\dot{\theta}^2}{4b} - c\cos\theta}$$

* Lagrangian and Hamiltonian of a particle in different co-ordinate system:-

① Plane Polar (r, θ) :-

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \text{P.E. term}$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \text{P.E. term.}$$

(ii) Spherical Polar (r, θ, ϕ) :-

$$L = \frac{1}{2} m (1 \cdot \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2) - \text{P.E. term}$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2\theta} + \text{P.E.}$$

(iii) Cylindrical Coordinate (s, ϕ, z) or (r, ϕ, z) :-

$$L = \frac{1}{2} m (\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2) - \text{P.E. term.}$$

$$H = \frac{p_s^2}{2m} + \frac{p_\phi^2}{2ms^2} + \frac{p_z^2}{2m}$$

Q. If $H = a p_\theta^2 + b \cos\theta$ write eqn of motion of system (2nd order differential eqn).

Soln Write Hamilton's Equation.

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = 2a p_\theta$$

$$\dot{p}_\theta = \frac{-\partial H}{\partial \theta} = b \sin\theta$$

$$\boxed{\begin{aligned} \dot{\theta} &= 2a p_\theta \\ \dot{p}_\theta &= b \sin\theta \end{aligned}} \longrightarrow \text{first order differential Equation.}$$

To get second order diff. eq^n. in θ put $\dot{p}_\theta$ in I^{st} Equation.

diff. w.r.t. time.

first eq^n — $\ddot{\theta} = 2a \dot{p}_\theta$

$$\ddot{\theta} = 2ab \sin\theta$$

$$\boxed{\ddot{\theta} - 2ab \sin\theta = 0} \longleftarrow \text{second order differential eq}^n \text{ in } \theta.$$

Q. If $H = \dfrac{p_\theta^2}{2ml^2} + mgl(1 - \cos\theta)$ find second order diff. eq^n in θ.

Sol^n

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{2 p_\theta}{2ml^2}$$

$$\dot{p}_\theta = \frac{-\partial H}{\partial \theta} = -mgl \sin\theta$$

$$\ddot{\theta} = \frac{\dot{p}_\theta}{ml^2} = -\frac{\cancel{m}g\cancel{l}\sin\theta}{\cancel{m}l^{\cancel{2}}}$$

$$\boxed{\ddot{\theta} + \frac{g}{l} \sin\theta = 0}$$