Free Study Material from

All Lab Experiments

Classical Mechanics
for NET/Gate Physical Sciences
> Non-inertial frame of Reference <
> and Pseudo Forces, Part-1 <

06/Aug/2014

Non-Inertial Frames of Reference And Pseudo Forces

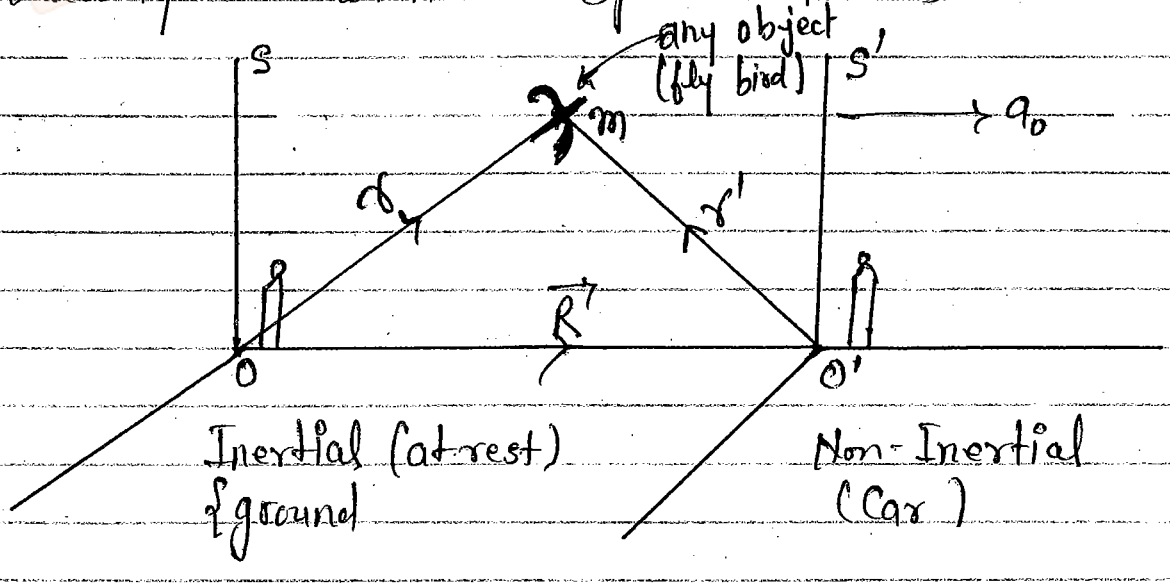
* Non-Inertial Frames :-

An accelarated frame is called Non-Inertial frames.
Accelarated frames are of two types -

1. Linearly Accelarated frame
2. Uniformly Rotating frame (ω = constt) OR Rotating frame.

1 Linearly Accelarated Frame :

Let us consider two frame S and S' where S is at rest and S' is moving with constant velocity 'v' with respect to S.



Let a_0 = accelaration of S'.

$$\vec{R} + \vec{r}' = \vec{r}$$

$$\vec{r}' = \vec{r} - \vec{R} \quad \text{—— (1)}$$

Differentiate twice with respect to time -

$$\frac{d^2 r'}{dt^2} = \frac{d^2 \vec{r}}{dt^2} - \frac{d^2 \vec{R}}{dt^2}$$

$$\vec{a}' = \vec{a} - \vec{a}_0$$

both side multiplying by m :-

$$m\vec{a}' = m\vec{a} - m\vec{a}_0$$

$$\boxed{\vec{F}' = \vec{F} - m\vec{a}_0}$$ Pseudo Force.

force on object at seen from S'

force on object at seen from S

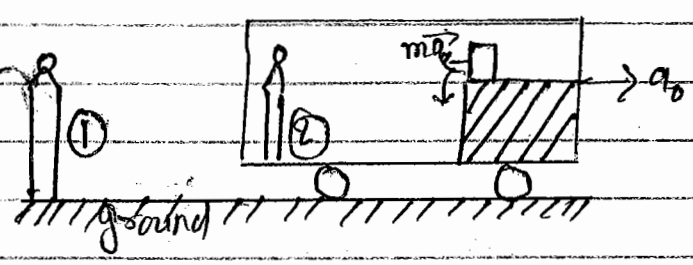
$$\boxed{\vec{F}' = \vec{F} + (-m\vec{a}_0)}$$

Note :-

Cause of fall of body in the transparent bus :-

Explanation of :-

first Person :- (Which is at ground) :-



Cause of fall of body in bus is ; There is no sufficient friction.

Second Person :-

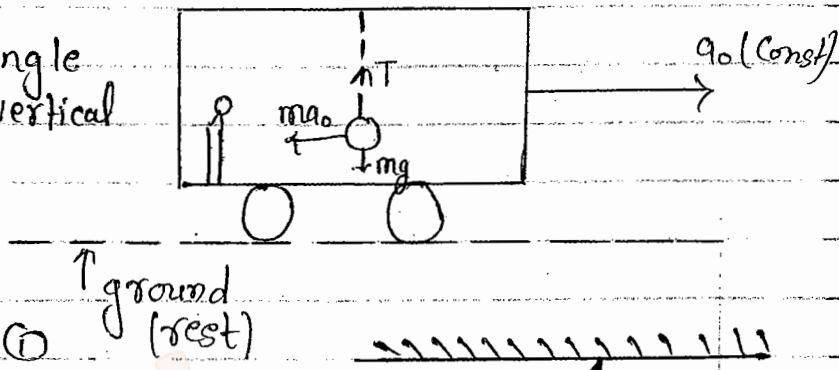
Due to pseudo force.

⇒ The origin of pseudo force is not known.

⇒ Concept of pseudo force is used when observation is made from non-inertial frame.

* Simple Pendulum in linearly accelerated frame :-

Let string makes angle θ with downward vertical in equilibrium position.



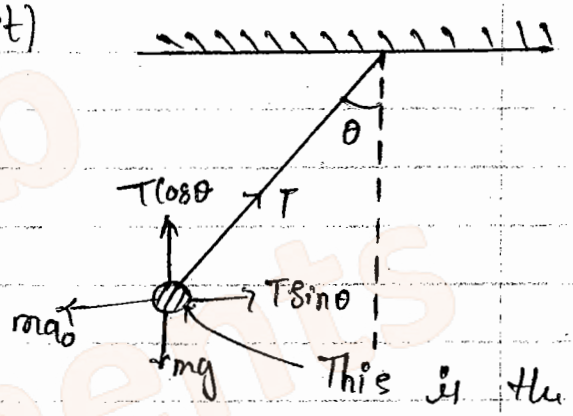
$T\sin\theta = ma_0$ — (I)

$T\cos\theta = mg$ — (II)

(I) / (II)

$$\tan\theta = \frac{a_0}{g}$$

$$\boxed{\theta = \tan^{-1}\frac{a_0}{g}}$$



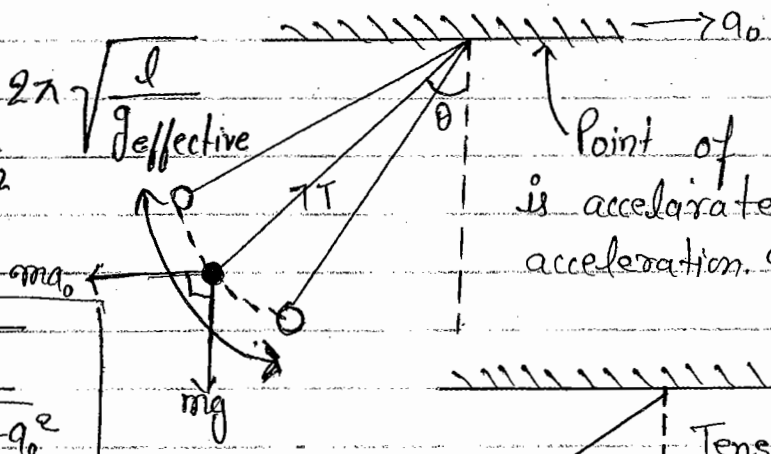
This is the Equilibrium position making angle θ with downward vertical.

* Time Period of oscillation in accelarated frame :-

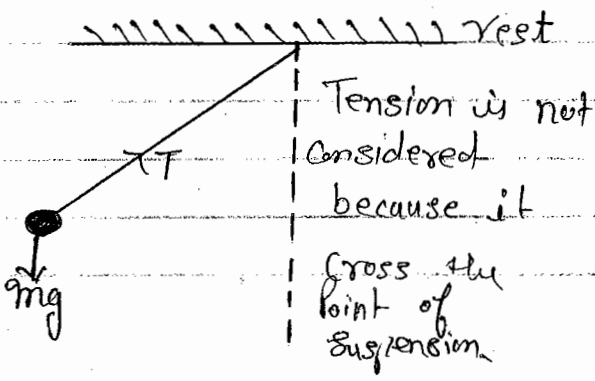
Time Period $(T) = 2\pi\sqrt{\frac{l}{g_{effective}}}$

$\because g_{eff} = \sqrt{g^2 + a_0^2}$

$$\therefore \boxed{T = 2\pi\sqrt{\frac{l}{\sqrt{g^2+a_0^2}}}}$$



Point of suspension is accelarated with acceleration a_0.

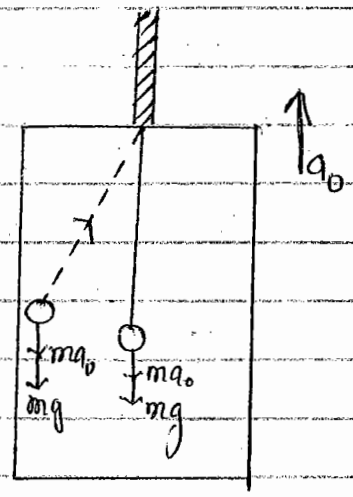


Tension is not considered because it cross the point of suspension.

* Pendulum in a Lift :-

In this case :-

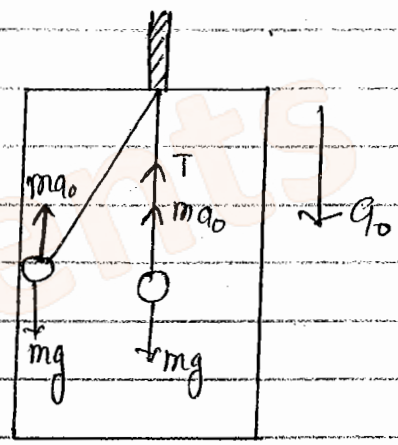
$g_{eff} = a_0 + g$



$\because \; T = 2\pi\sqrt{\dfrac{l}{g_{effective}}}$

So $\boxed{T = 2\pi\sqrt{\dfrac{l}{g+a_0}}}$

In this case :-

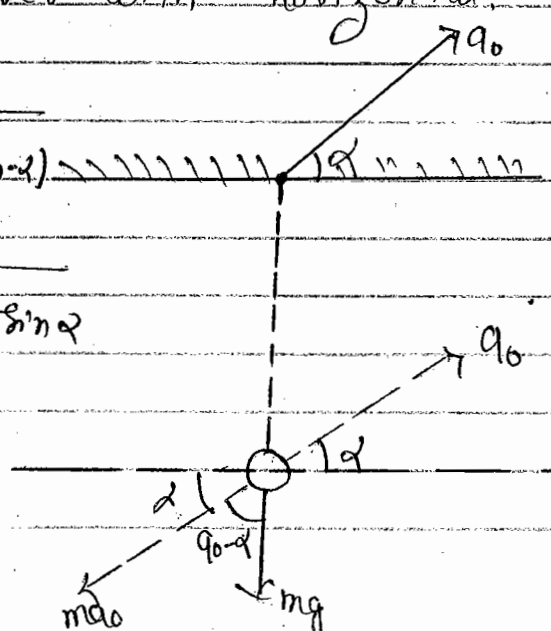


$g_{eff} = g - a_0$

$a_0 < g$

$\boxed{T = 2\pi\sqrt{\dfrac{l}{g-a_0}}}$

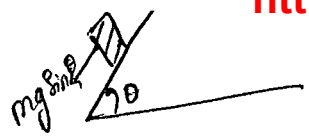
* When point of suspension is accelarated at an angle α ~~ver~~ with horizontal :-



$g_{eff} = \sqrt{g^2 + a_0^2 + 2ga_0\cos(90-\alpha)}$

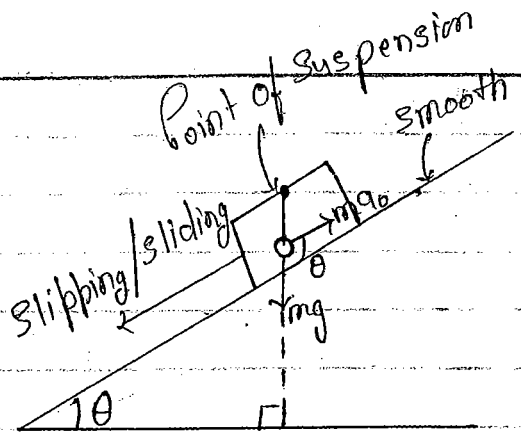
$g_{eff} = \sqrt{g^2 + a_0^2 + 2ga_0\sin\alpha}$

$\boxed{T = 2\pi\sqrt{\dfrac{l}{g_{eff}}}}$



$F = mg\sin\theta \quad a = \frac{mg\sin\theta}{m} \Rightarrow \boxed{a = g\sin\theta}$

Q. What is the time period of pendulum.

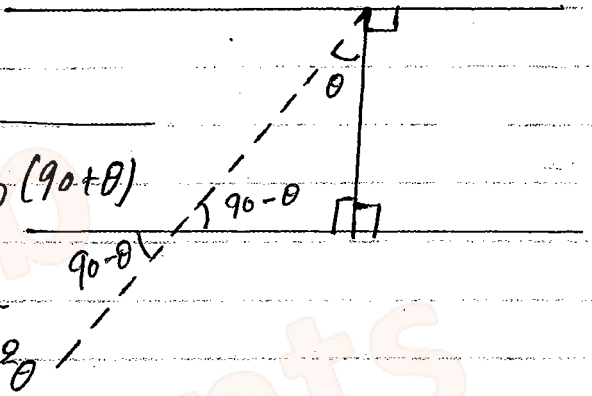


$a_0 = g\sin\theta$

$\therefore \quad \theta = 90 + \theta$

$g_{eff} = \sqrt{g^2 + a_0^2 + 2ga_0\sin(90+\theta)}$

$= \sqrt{g^2 + (g\sin\theta)^2 - 2g(g\sin\theta)\sin(90+\theta)}$



$= \sqrt{g^2 + g^2\sin^2\theta - 2g^2\sin^2\theta}$

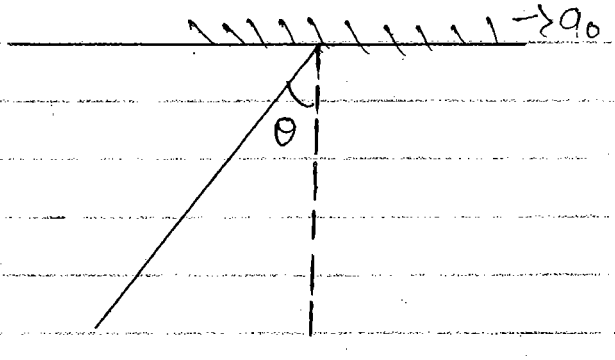
$= g\sqrt{1 + \sin^2\theta - 2\sin^2\theta} = g\sqrt{1 - \sin^2\theta}$

$g_{eff} = g\cos\theta$

So $\boxed{T = 2\pi\sqrt{\frac{l}{g\cos\theta}}}$

Q. Acceleration of point of suspension is $\sqrt{3}g$ in horizontal direction.

Sol^n



$\theta = \tan^{-1}\left(\frac{a_0}{g}\right)$

$= \tan^{-1}(\sqrt{3})$

$\boxed{\theta = 60^\circ}$ Ans

Given -

In stationary case time period = T

$$T = 2\pi\sqrt{\frac{l}{g}}$$

then

$$T' = 2\pi\sqrt{\frac{l}{g_{eff}}}$$

$$g_{eff} = \sqrt{a_0^2 + g^2}$$

$$= \sqrt{3g^2 + g^2} = \sqrt{4g^2}$$

$$g_{eff} = 2g$$

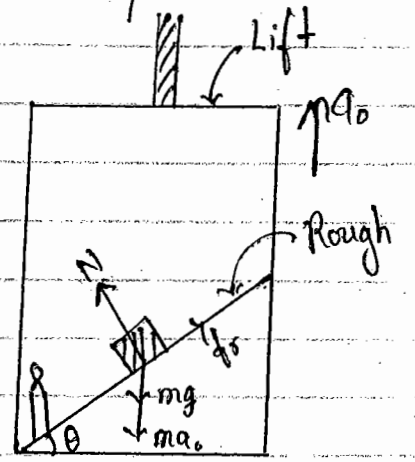
$$\boxed{T' = 2\pi\sqrt{\frac{l}{2g}}}$$

$$\boxed{T' = \frac{T}{\sqrt{2}}}$$ Ans

13/08/2014

Q. Block is not slidding on inclined plane what work done by the friction on Block. During the time lift move up by distance 'h'

Sol^n As seen from inside the lift, block is at rest.



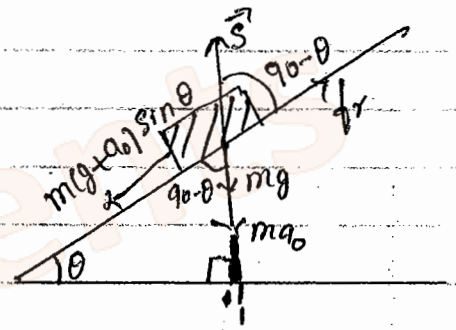
Therefore -

$$f_r = m(g+a_0)\sin\theta$$

Workdone = force . Displacement

$$= \vec{F}\cdot\vec{S}$$

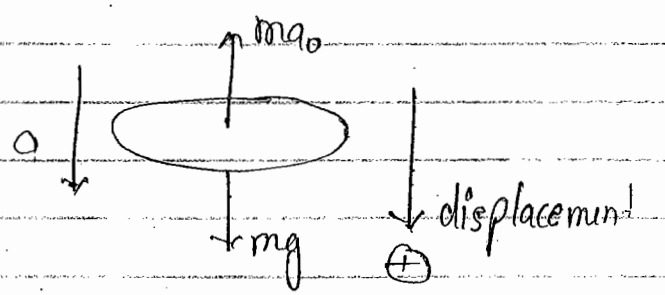
$$= f_r \cdot h \cos(90-\theta)$$

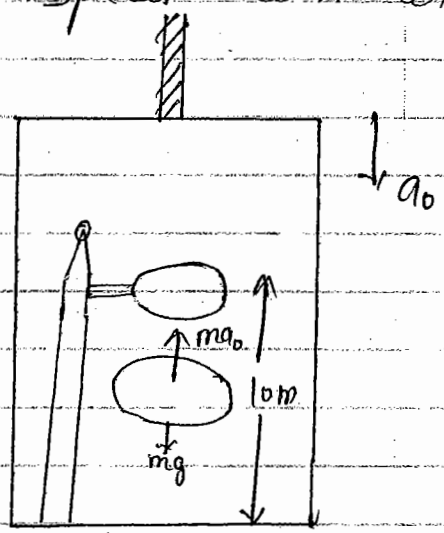


$$\boxed{W.D. = m(g+a_0)h\sin^2\theta}$$

Q. A lift is going downward direction with $a_0 = 4\,m/s^2$. A person drops a coin at 10 m distance from initial speed 0 with initial velocity. With what speed coin will hit the floor of lift.

Sol^n





Let a = acc^n of coin as seen by person -

$$mg - ma_0 = ma$$

$$a = (g - a_0)$$

$$V^2 = u^2 + 2as$$

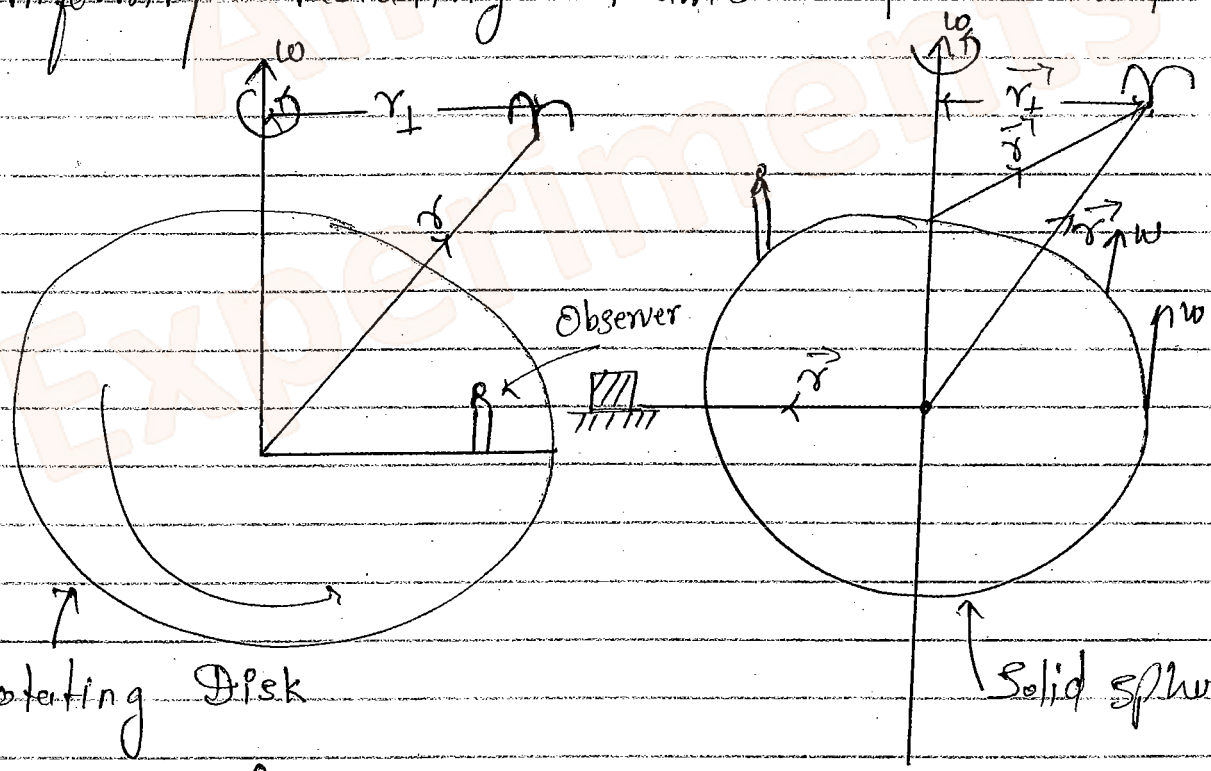
$$V^2 = 0 + 2(g - a_0) \times 10$$

$$= 2 \times 6 \times 10$$

$$V^2 = 120$$

$$\boxed{V = \sqrt{120}\ m/sec}$$ Ans

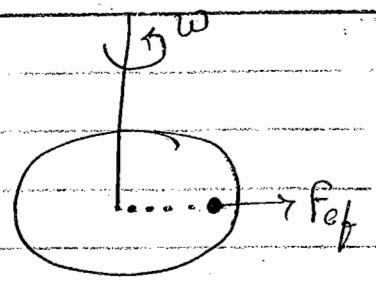
* Uniformly Rotating frame :- {ω = Constt} :-



Rotating frame is example of Non-Inertial frame.

When observation is made from rotating frame on all objects (irrespective of their location), following pseudo forces appears to act on them.

1. Centrifugal force :-



$$\boxed{F_{cf} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})}$$

m = mass of object.
$\vec{\omega}$ = angular velocity of rotating frame
$\vec{r}$ = position vector of object (with respect to a stationary point of frame).

Magnitude of Centrifugal Force :-

$$\boxed{\vec{F}_{cf} = m\omega^2 r_\perp}$$

$r_\perp$ = $\perp^{or}$ distance of object from axis of rotation.

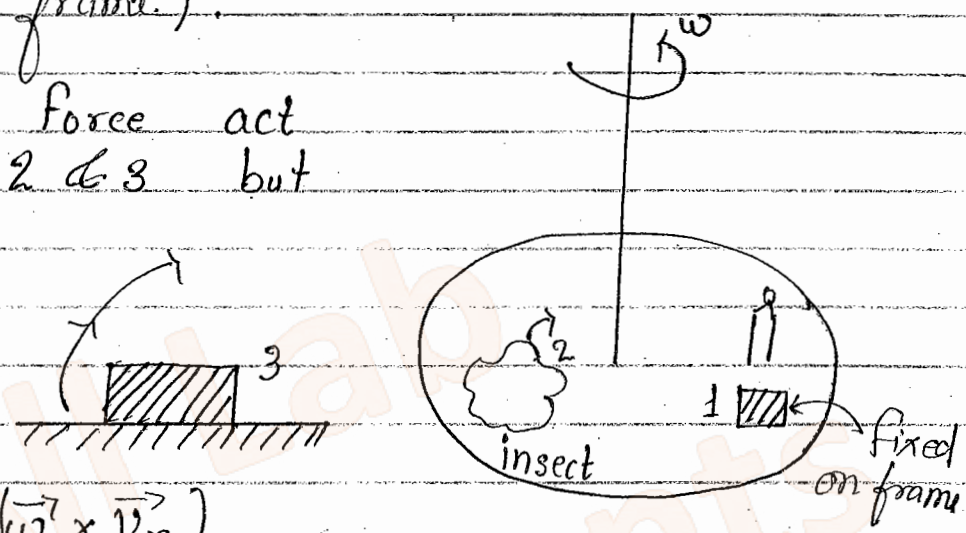
⇒ Centrifugal force acts on all object irrespective of their state of motion.

2. Coriolis Force :-

Coriolis :- A name of Mechanical Engeneer Coriolis was a mechanical engeneer. He is observing force on the liquid which is in rotating arm of the machine and unfortunately he discover a different type of force which is after his name Kla? Coriolis force.

"This force is appears to act on all object which appear to be moving when seen from the rotating frame (or object have speed with respect to rotating frame)."

Here Coriolis force act on object 2 & 3 but not on 1.



Expression of Coriolis force:-

$$\vec{F}_{cor.} = -2m(\vec{\omega} \times \vec{v}_r)$$

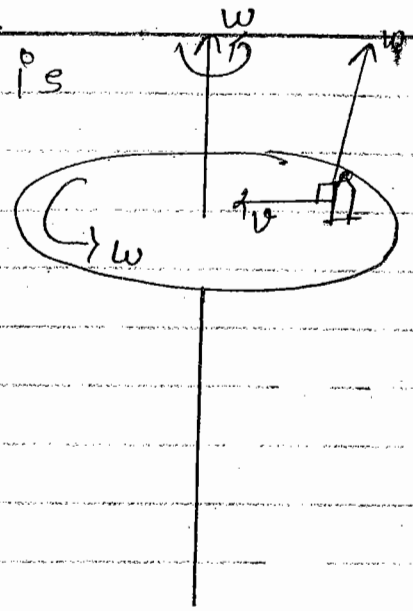
v_r = Velocity of object with respect to rotating frame

$$\boxed{\vec{F}_{cor} = 2m(\vec{v}_r \times \vec{\omega})}$$

* Some Basic Problems Based on Coriolis Force :-

5-22 A circular platform is rotating with a uniform angular speed ω counterclockwise about an axis passing through its center and perpendicular to its plane as shown in the figure. A person of mass m walks radially inwards with a uniform speed v on the plateform. The magnitude and the direction of the coriolis force (with respect to the direction along which the person walks is?

Curl the finger v to ω to get direction of



$v_r = v$

Axis is parallel to plane of paper.

$|\vec{F}_{cor}| = 2mv_r\omega \sin 90^\circ$

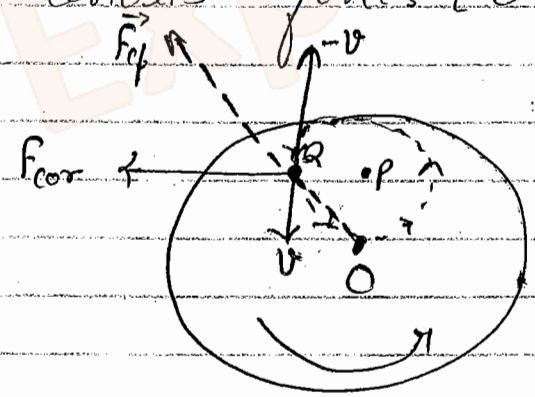
$= 2mv\omega \quad \{\sin 90^\circ = 1\}$

$\boxed{F_{cor} = 2mv\omega}$ towards his Right

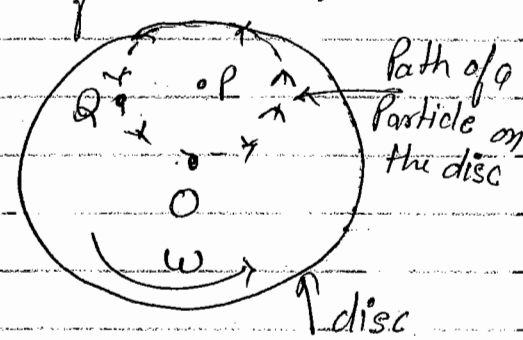
So option (c) is Correct.

A-5
Q.9 A circular disc is rotating in anticlockwise sence as shown in the figure. On the disc a particle moves in anticlockwise circle with center at P. At the instant particle at Q, which of the following options correctly represents directions of centrifugal and coriolis forces ['O' is center of the disc].

Sol^n



Axis is $\perp^r$ to the Plane of Paper.



When particle (insact) is at Q show direction of centrifugal and Coriolis force.

"Direction of centrifugal force ($\vec{F}_{cf}$) is always perpendicular to axis from object of rotation and away from it."

So option (d) is correct.

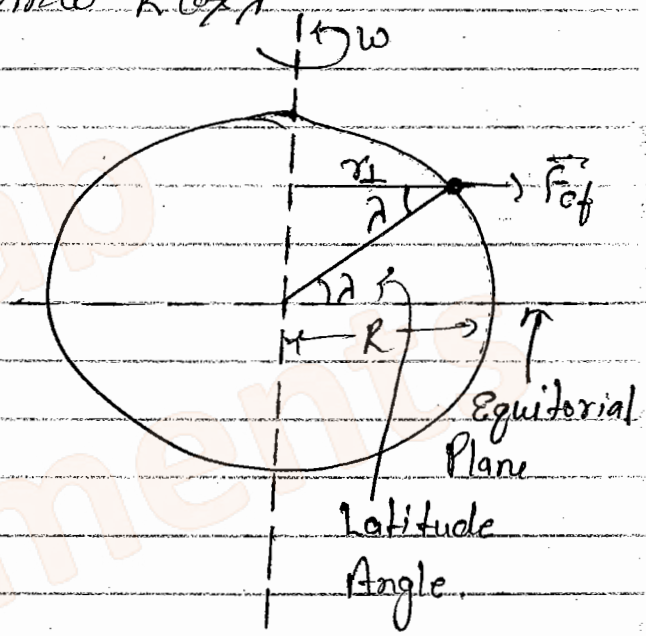
-5

5 A particle of mass m is lying on earth's surface at a location where latitude is λ. If ω be angular velocity of earth's spining motion and R be the radius of the earth then, centrifugal force on the particle is - ?

(a) $m\omega^2 R \sin\lambda$ (b) $m\omega^2 R \cos\lambda$ ✓

(c) $2m\omega^2 R \sin\lambda$ (d) $2m\omega^2 R \cos\lambda$

Sol:



$\vec{F}_{cf} = m\omega^2 r_\perp$

$\cos\lambda = \frac{r_\perp}{R}$

$r_\perp = R\cos\lambda$

So $\boxed{\vec{F}_{cf} = m\omega^2 R\cos\lambda}$

At equator (at $\lambda = 0$) :-

$\boxed{\vec{F}_{cf} = m\omega^2 R = max^m}$

At poles (At $\lambda = 90°$)

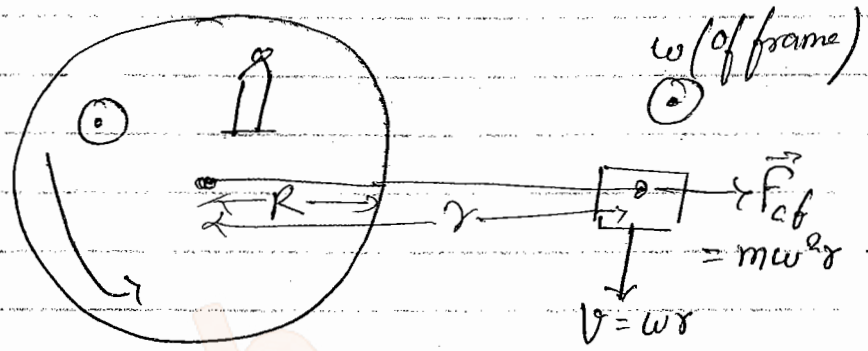
$\boxed{\vec{F}_{cf} = 0 = min^m}$

Imp. A disc is rotating in horizontal plane with uniform angular velocity ω an object on mass m is lying on the ground in the same plane at a distance r from the center what is centrifugal and

This velocity is due to rotation of rod.

Sol^n Coriolis force on the object as seen by a person, standing on the disc.

$v = \omega r$ → Velocity arising out of rotation is given by $\omega r_\perp$



$$\vec{F}_{cor} = 2m v_r \cdot \omega \sin 90^\circ$$

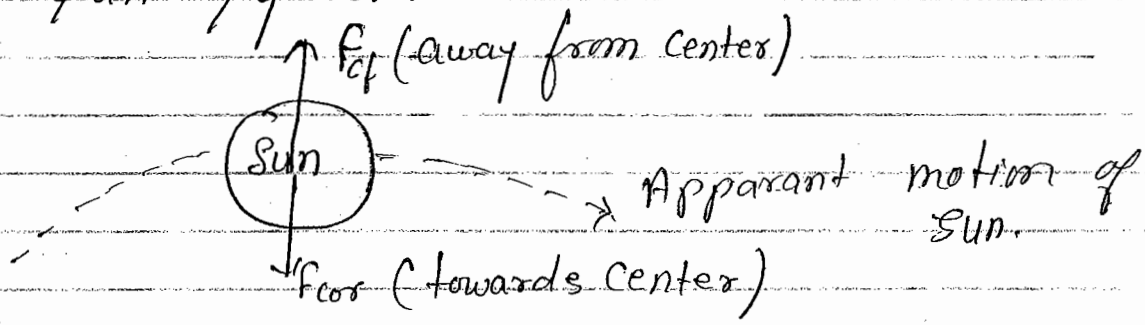
$$= 2m v_r \cdot \omega \cdot 1$$

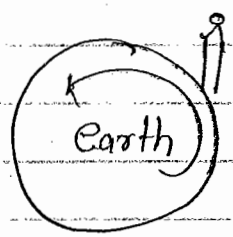
$$= 2m\omega r \cdot \omega$$

$$\boxed{\vec{F}_{cor} = 2m\omega^2 r}$$ Towards center.

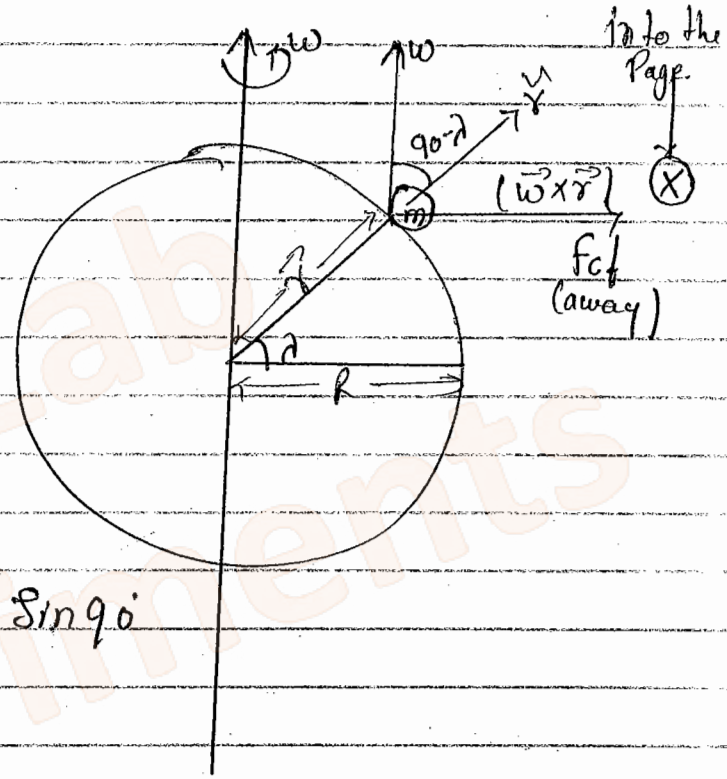
$$\boxed{\vec{F}_{cf} = m\omega^2 r}$$ (away from center)

* Earth Sun System :-





Sol^n Second Method :-



$$\vec{f}_{cf} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\vec{f}_{cf} = m(\vec{\omega}\times\vec{r})\times\vec{\omega}$$

$$|\vec{F}_{cf}| = |m(\underline{\vec{\omega}\times\vec{r}})\,\underline{\times\vec{\omega}}|$$

$$\vec{f}_{cf} = m|\vec{\omega}\times\vec{r}|\,|\vec{\omega}|\sin 90^\circ$$

$$= m|\vec{\omega}\times\vec{r}|\cdot\omega\cdot 1$$

$$= m\omega|r|\sin(90-\lambda)\cdot\omega\cdot 1$$

$$\boxed{|\vec{F}_{cf}| = m\omega^2 R\cos\lambda}$$ Ans

* Earth: A non-Inertial (rotating) frame:

Earth spins about its axis, therefore, therefore it is a non-inertial frame.
Angular velocity of earth's spining motion is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{24h} = 7.27 \times 10^{-5} \text{ rad/sec.}$$

Due to rotation of earth following effects arise-

(a) All objects on earth's surface experience centrifugal force which is directed away from the axis of rotation.

(b) All objects moving on earth's surface experience coriolis force in addition to centrifugal force.

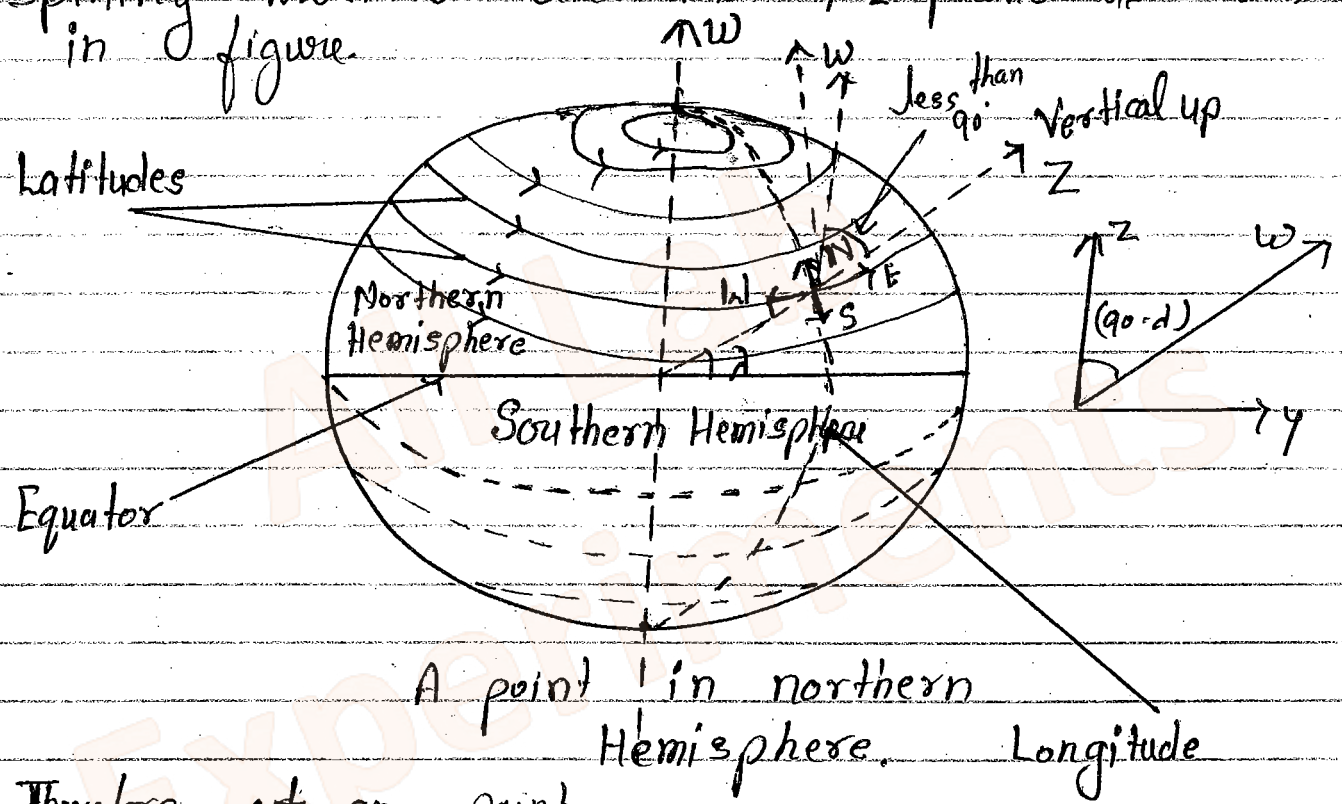
(c) An observer standing on earth's surface notices that on all objects a centrifugal and a coriolis force (if $v' \neq 0$) act irrespective of the location of the object. ~~This~~ That is the object which is being observed by the observer may or may not be on the earth's surface. For example we on the earth, see that the ~~Sun~~ Sun rises and sets. If we analyse the dynamics of the sun we will find that Sun experiences a centrifugal force away from the earth and a coriolis force towards the earth.

Coordinate System on Earth's Surface:-

To analyse dynamics of objects from earth's surface we define local cartesian co-ordinate system as follows-

(i) +X-axis : Along local east, -X axis : Along local west
(ii) +Y-axis : Along local north, -Y axis : Along local south
(iii) +Z-axis : Along local vertical upward
-Z-axis : Along local vertical downward.

Therefore angular velocity vector of earth's spining motion lies in Y-Z plane as shown in figure.



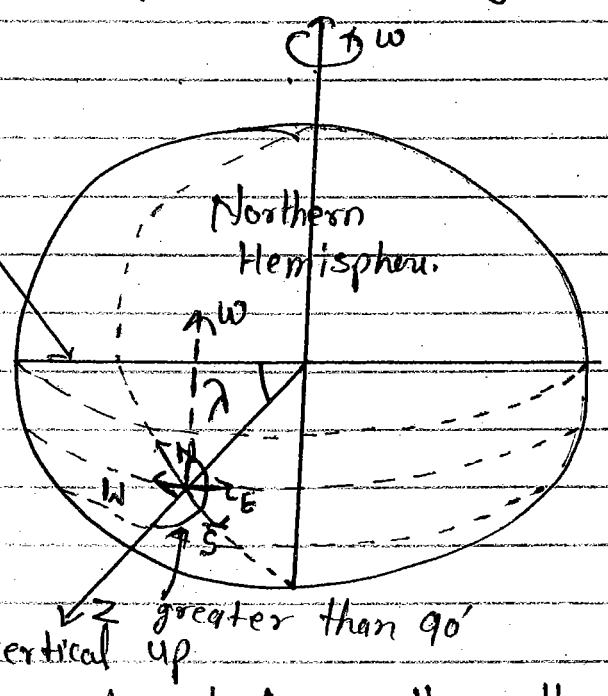
Therefore at any point in northern hemisphere,

$\vec{\omega} = \omega(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$

Where λ is latitude of the point.

And at any point in southern hemisphere.

$\vec{\omega} = \omega(\cos\lambda\,\hat{j} - \sin\lambda\,\hat{k})$



A-5

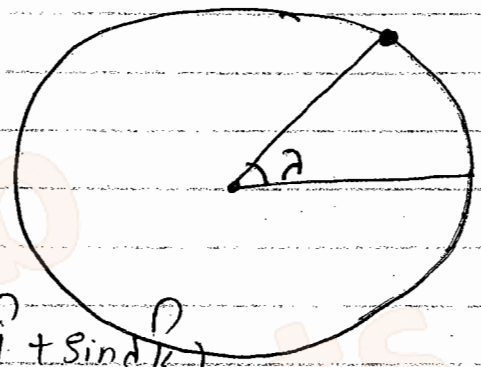
Q.6 In the previous question if the particle starts moving along the latitude from east to west with constant speed V_0, coriolis force on the particle will be (magnitude).

(a) $2mV_0\omega$ (b) $2mV_0\omega\cos\lambda$

(c) $2mV_0\omega\sin\lambda$ (d) $2mV_0\omega\sin\lambda\cos\lambda$

Soln.

$$\vec{\omega} = \omega(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$$



$$\vec{v} = -V_0\,\hat{i}$$

$$\vec{F}_{cor} = 2m\vec{v}\times\vec{\omega}$$
$$= -2mV_0\omega\,\hat{i}\times(\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k})$$

$$\vec{F}_{cor} = -2mV_0\omega(\cos\lambda\,\hat{k} - \sin\lambda\,\hat{j})$$

$$|\vec{F}_{cor}| = 2mV_0\omega\sqrt{\cos^2\lambda + \sin^2\lambda}$$

$$\boxed{|\vec{F}_{cor}| = 2mV_0\omega}$$

* Eastward deviation of a freely falling object in northern hemisphere:-

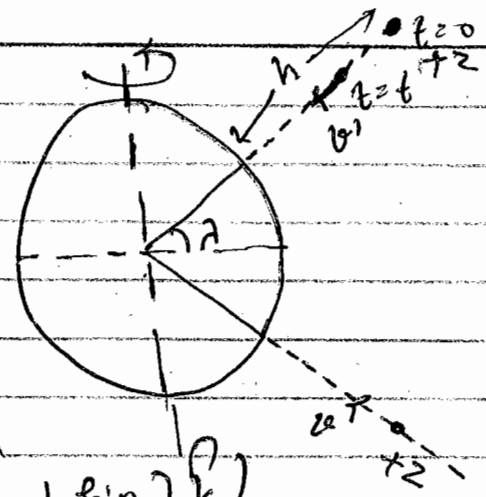
Suppose a particle falls from a height h at a place where latitude is λ.

Here we assume that $h << R_e$

Velocity of particle at time t is -

$$\vec{v}' = -gt\,\hat{k}$$

Coriolis force on the particle -



$$\vec{\omega} = \omega(\cos\lambda\hat{j} + \sin\lambda\hat{k})$$

So $\vec{F}_{cor} = 2m\vec{v}\times\vec{\omega}$

$$= -2m\omega g t\hat{k}\times(\cos\lambda\hat{j} + \sin\lambda\hat{k})$$

$$\vec{F}_{cor} = 2m\omega g\cos\lambda t\,\hat{i}$$

Force is towards east so particle will be deviate towards east, in both hemisphere.

Equation of motion of the particle in eastward direction is -

$$\vec{F}_{cor} = m\frac{d^2x}{dt^3} \quad \text{or} \quad \frac{d^2x}{dt^2} = 2\omega g\cos\lambda t$$

We have following initial condition which we use to solve above equation -

at, $t=0$, $x=0, y=0, z=h$ —— (a)

at, $t=0$, $\frac{dx}{dt}=0$, $\frac{dy}{dt}=0$, $\frac{dz}{dt}=0$ —— (b)

Integrating the above differential equation we get, $\frac{dx}{dt} = \omega g\cos\lambda t^2 + c_1$

Using (b) we get $c_1 = 0$. Therefore $\frac{dx}{dt} = \omega g\cos\lambda t^2$

Integrating again we get, $x = \frac{1}{3}\omega g\cos\lambda t^3 + c_2$

Using (a) we get $c_2 = 0$

$$\therefore \boxed{x = \frac{1}{3}\omega g \cos\lambda t^3}$$

Where t is total time taken to reach earth surface.

$$\therefore t = \sqrt{\frac{2h}{g}}$$

$$\therefore \boxed{x = \frac{1}{3}\omega g \cos\lambda \left(\frac{2h}{g}\right)^{3/2}}$$

If a particle falls from a height of 100 metre at equator ($\lambda = 0$) then using above equation we get.

$$x = \frac{1}{3} \times 7.27 \times 10^{-5} \times 9.8 \times \left(\frac{2 \times 100}{9.8}\right)^{3/2}$$

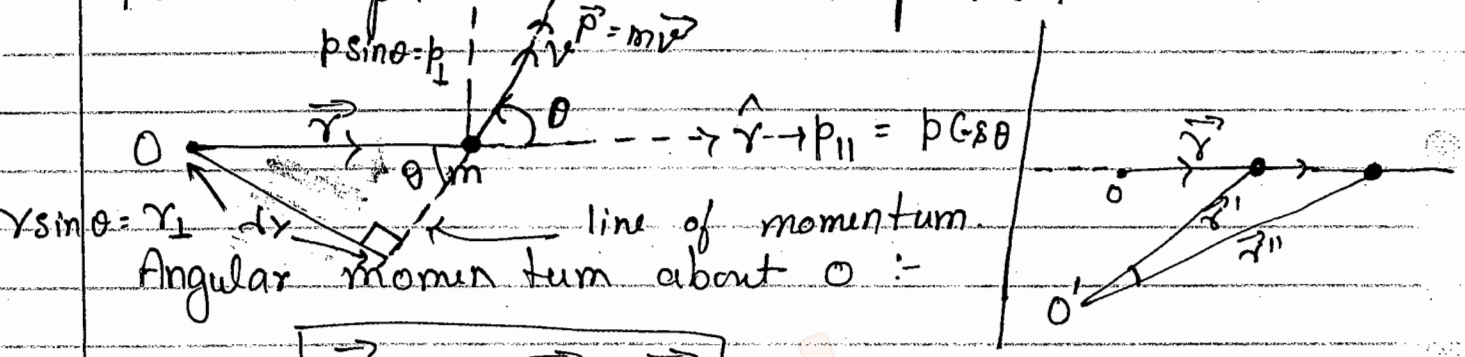
$$= 2.19 \approx 2.2 \text{ c.m.}$$

Note :-

The expression derived for eastward deviation is an approximate one. In the derivation we have assumed that particle is uniformly accelerated in downward direction which is not true, when particle attains a velocity in X (each direction) there arises a component of coriolis force in vertical direction, due to which downward acceleration becomes a function of V_x'. Because of V_x' component of velocity, particle also experience a force in southward direction. Therefore a freely falling object in northern hemisphere actually deviates in south-east direction.

* Angular Momentum And Torque :-

{ For a particle, about a point } :-



Angular momentum about O :-

$$\boxed{\vec{L} = \vec{r} \times \vec{p}} \Rightarrow \text{Vector form}$$

$\vec{r}$ = position of particle from point O.

Use right hand thumb rule to find the direction of $\vec{L}$.

<u>Magnitude form :-</u>

$$\boxed{|\vec{L}| = rp\sin\theta = rp_\perp}$$

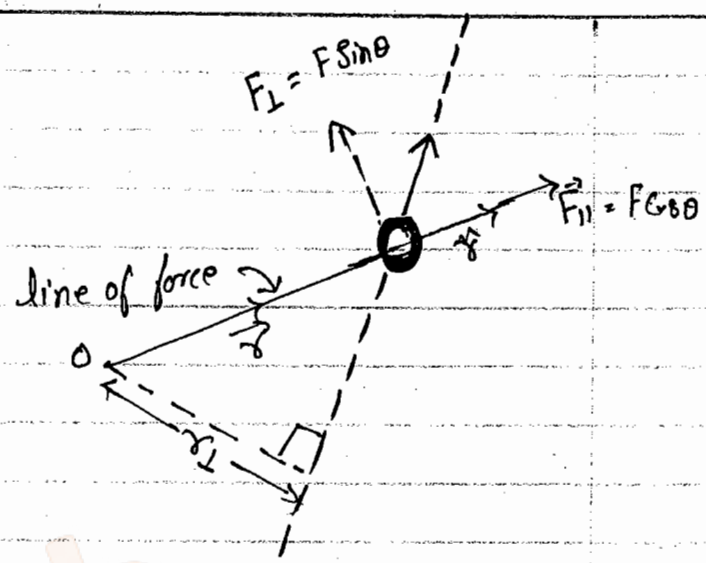
$$(\because p_\perp = p\sin\theta)$$

$$\therefore \boxed{|\vec{L}| = r_\perp p}$$

$$(\because r_\perp = r\sin\theta)$$

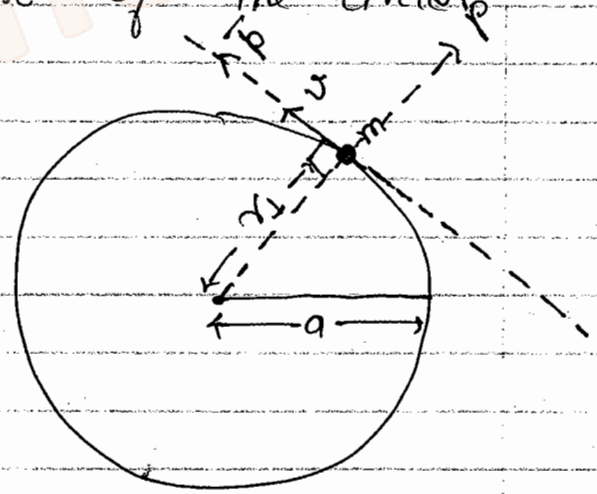
$$\therefore \boxed{L = rp\sin\theta = rp_\perp = pr_\perp}$$

* Torque :-



Qu A particle is moving in a circle of radius 'a' with constant speed v calculate its angular momentum about its centre of the circle.

Sol^n



$L = p\, r_\perp$

$$\boxed{L = m v a}$$

$(\because p = mv)$ $r_\perp = a$

for Direction -

$\vec{L} = \vec{r} \times \vec{p}$ {finger curl from r to p}

So direction is up the plane of paper. Up the plane of paper.

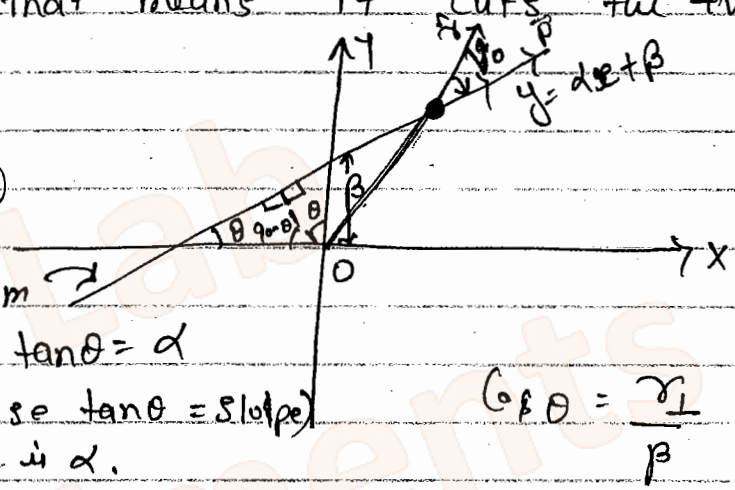
Qu A particle of mass m is moving with speed v_0 along a straight line $y = (\alpha x + \beta)$ $\alpha > 0$ $\beta > 0$ in x-y plane. Calculate angular momentum about origin.

Sol^n

$$y = \underset{\text{slope}}{\alpha}\, x + \underset{\text{intersept}}{\beta}$$

Slope is +ve that means angle from +ve x direction is less than 90°.

Intersept is +ve that means it cuts the +ve y-axis.



$$L = p\, r_\perp \quad \text{—} (1)$$

→ line of momentum

(Particle moving about this line)

$\tan\theta = \alpha$ (because $\tan\theta$ = slope which is α.)

$$\cos\theta = \frac{r_\perp}{\beta}$$

$$\because \tan\theta = \frac{\alpha}{1} = \frac{\text{Perpendicular}}{\text{Base}} \quad \therefore r_\perp = \beta \cos\theta$$

$$\cos\theta = \frac{r_\perp}{\beta} = \frac{\text{Base}}{\text{Hypoten.}}$$

$$\text{So } r_\perp = \beta \cdot \frac{\text{Base}}{\text{Hypoten.}} = \beta \cdot \frac{1}{\sqrt{\alpha^2+1}}$$

$$\text{So } \boxed{r_\perp = \frac{\beta}{\sqrt{\alpha^2+1}}}$$

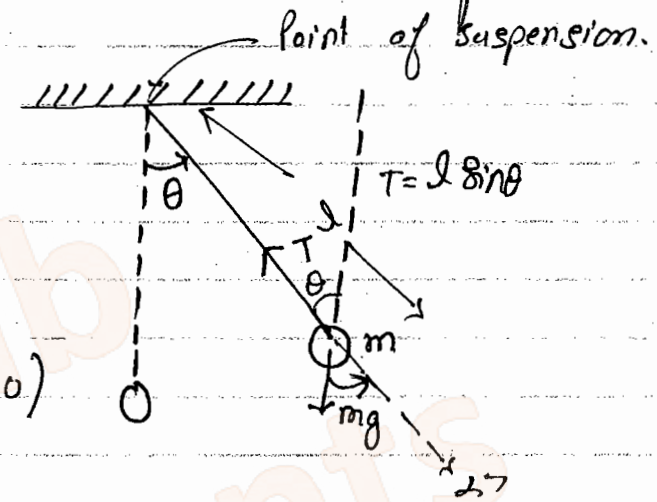
$$\text{So } \boxed{L = m v_0 \frac{\beta}{\sqrt{\alpha^2+1}}} \quad \{\text{from } (1)\}$$

direction of angular momentum is into the page or (−Z) axis.

$r \times F$

Ques A simple pendulum consisting of a small bob of mass m and light string of length l is deflected by an angle θ from lowest position. Calculate torque about point of suspension in this position.

Sol^n



$$\tau = F\, r_\perp$$

① Due to tension :-

$$\tau = T \cdot 0 = 0 \quad (\because r_\perp = 0)$$

$$\boxed{\tau = 0}$$

If a force passes through a point so that force will not produce any torque about point.

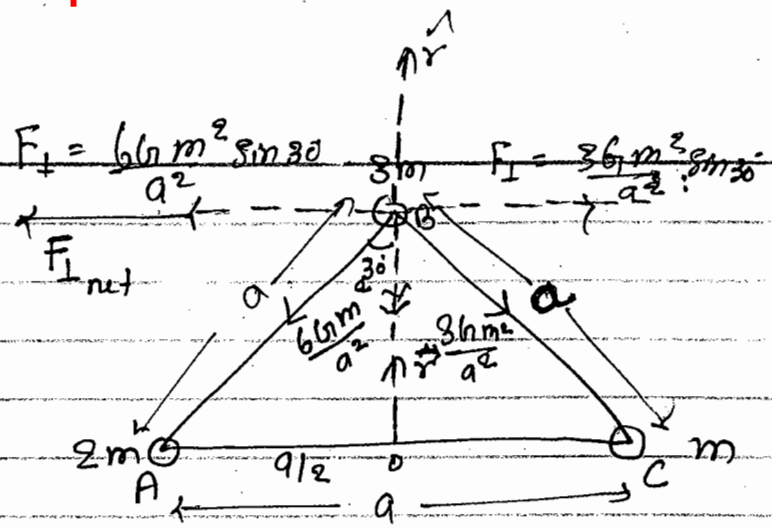
② Due to mg :-

$$\tau = mg \cdot l \sin\theta$$

$$\boxed{\tau = mgl\sin\theta} \quad (r_\perp = l\sin\theta)$$

Direction of τ is into the plane of paper.

Ques What is torque on mass $3m$ due to forces of other masses about point O ?



1n

(1) $\tau = F_\perp r$

$r = \frac{\sqrt{3}}{2} a$

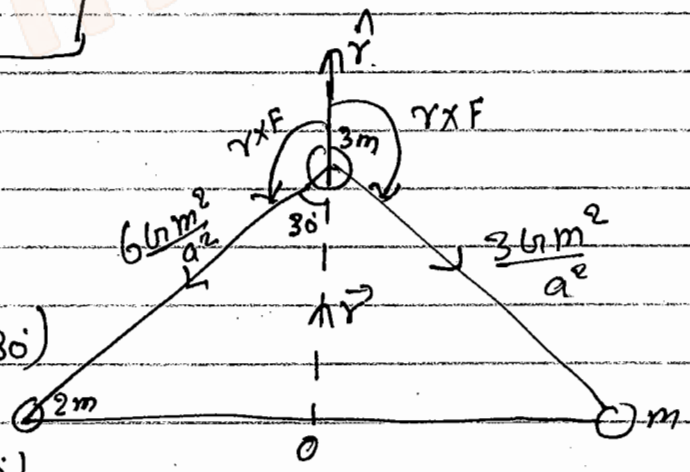
$F_{\perp net} = \frac{6m^2G}{a^2}\sin 30^\circ - \frac{3Gm^2}{a^2}\sin 30^\circ$

$F_{\perp net} = \frac{3Gm^2}{a^2}\sin 30^\circ$

So $\tau = \frac{3Gm^2}{a^2} \cdot \frac{\sqrt{3}}{2} a \cdot \frac{1}{2}$

$$\boxed{\tau = \frac{3\sqrt{3}Gm^2}{4a}}$$

Second Method :-



$\tau = rF\sin\theta$

$= \frac{\sqrt{3}}{2} a \frac{6Gm^2}{a^2} \sin(\pi - 30^\circ) - \frac{\sqrt{3}}{2} a \frac{3Gm^2}{a^2} \sin(\pi - 30^\circ)$

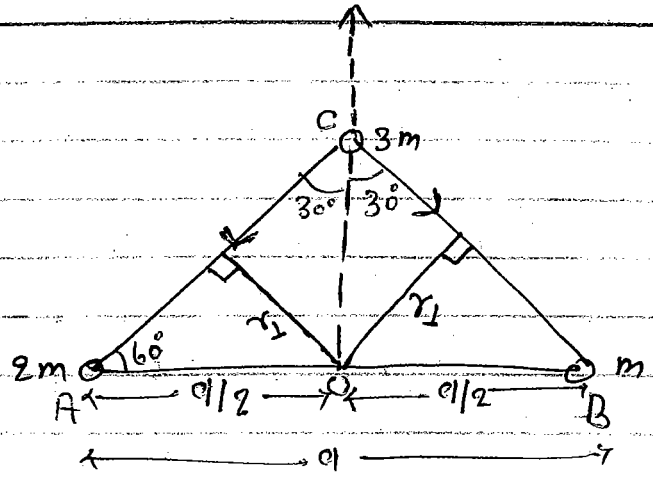
$\tau = \frac{\sqrt{3}}{2} a \frac{3Gm^2}{a^2} \{2 - 1\} \sin(\pi - 30^\circ)$

$= \frac{3\sqrt{3}}{2} \frac{Gm^2}{a} \sin 30^\circ = \frac{3\sqrt{3}}{2} \frac{Gm^2}{a} \cdot \frac{1}{2}$

$$\boxed{\tau = \frac{3\sqrt{3}}{4} \frac{Gm^2}{a}}$$ up the plane of paper.

A

Third Method :-



$$\tau = \vec{F} r_\perp$$

$$\sin 60^\circ = \frac{r_\perp}{a/2}$$

$$\frac{\sqrt{3}}{2} = \frac{2 r_\perp}{a}$$

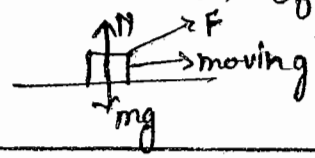
$$\boxed{r_\perp = \frac{\sqrt{3}a}{4}}$$, $\vec{F} =$

$$\vec{F} = \frac{6 G m^2}{a^2} \sin 30^\circ$$

$$\tau = \vec{F} r_\perp$$

$$\tau =$$

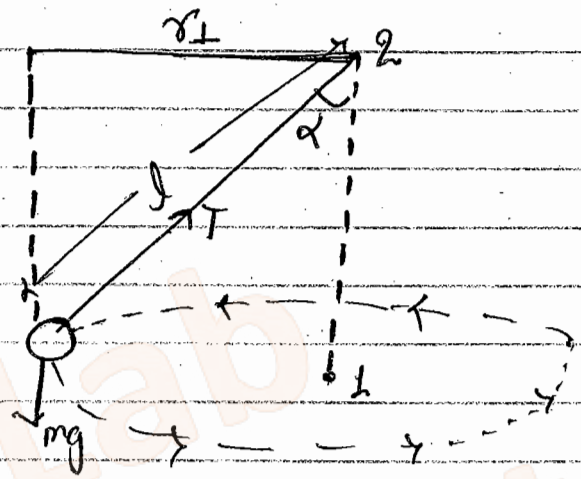
force is resolve in the direction of motion.



28/Aug/2014

A conical pendulum consist of a bob of mass m and light string of length l. If string of pendulum makes an angle α with downward vertical.

Sol^n Torque about point 2:-



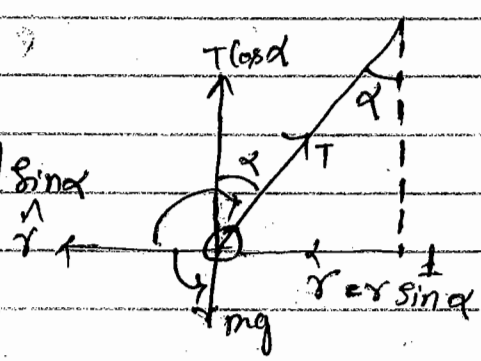
Here tension is not produces Torque because it passes through the point of suspension.

So $\tau_2 = mg\, r_\perp$

$$\boxed{\tau_2 = mg\, l \sin\alpha}$$

About point 1:-

$\tau = \vec{F}_\perp r$, $\tau = F_\perp r_\perp$



$\tau_1 = mgl \sin\alpha - T\cos\alpha \cdot l \sin\alpha$

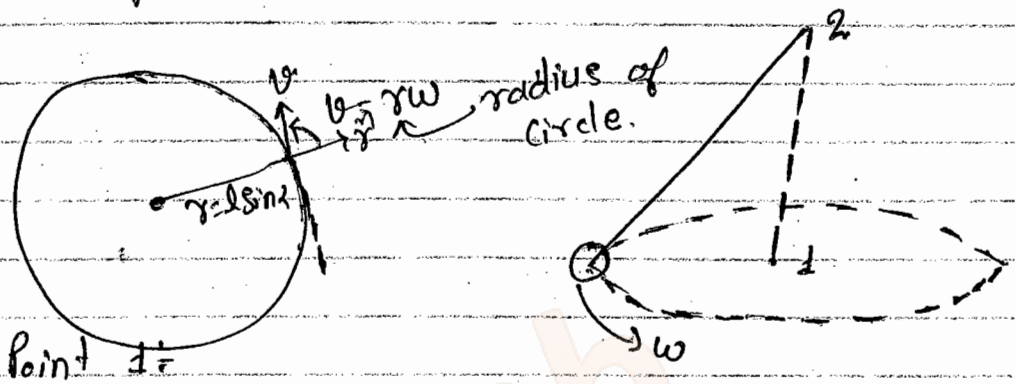
$$\boxed{\tau_1 = l \sin\alpha\,(mg - T\cos\alpha)}$$

∵ Here there is vertical equilibrium.

So ∴ $T\cos\alpha = mg$.

$\vec{L} = \vec{r} \times \vec{p}$

Ques In previous question if angular speed of conical pendium is ω Calculate angular momentum about points 1 and 2.



About Point 1:

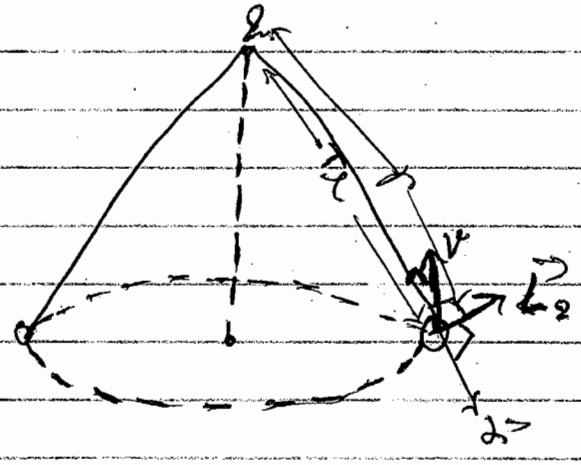
$L = p \cdot r_\perp$

$= mv \cdot r$

$= m\omega r^2$

$\boxed{L_1 = m\omega l^2 \sin^2\alpha}$ (And direction of angular momentum along the vertical line.)

About point 2:



$L_2 = p r_\perp$

$= mv r_\perp$

$= m\omega l \sin\alpha \cdot r_\perp$

$= m\omega l \sin\alpha \cdot l$

$\boxed{L_2 = m\omega l^2 \sin\alpha}$

It is not along the vertical line. It is perpendicular to string.

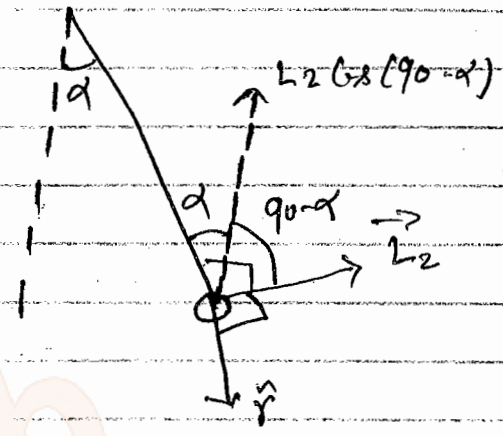
Let us take component of L_2 along vertical line,

L_2 (along vertical line) =

$= L_2 \sin\alpha$

$= m\omega l^2 \sin\alpha . \sin\alpha$

$$\boxed{L_2 = m\omega l^2 \sin^2\alpha = L_1}$$



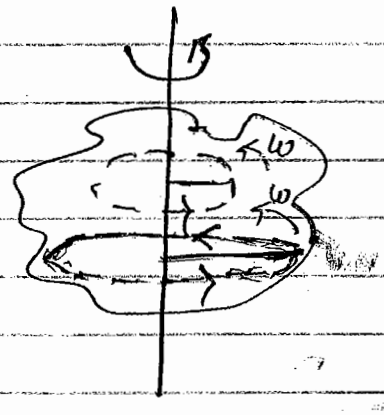
Conclusion :-

If we calculate angular momentum about different points of axis of rotation then magnitude as well as direction will be different. However value of angular momentum about or along the axis [i.e. component along the axis is always same].
And its value is given as -

$$\boxed{L_{axis} = m\omega r_\perp^2}$$ {for single particle}

Where $r_\perp$ = perpendicular distance from axis of Rotation

* If there is a rigid body :-



So every particle moves in a circle of different radius, but same angular speed.

For a rigid body :-

$$L_{axis} = \sum_i m_i \omega r_{\perp i}^2$$

$$\boxed{L_{axis} = \omega \sum_{i=1}^{n} m_i r_{\perp i}^2}$$ for system of particles.

This may or may not be total angular momentum.

$$\boxed{L_{axis} = \omega I}$$

$$I = \sum_{i=1}^{N} m_i r_{\perp i}^2 = \text{Moment of Inertia}$$

* Moment of Inertia about an axis:-

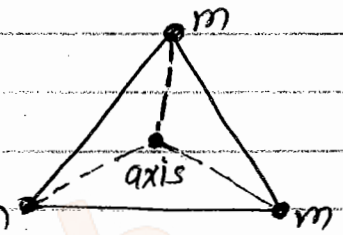
$$\boxed{I = \sum_{i=1}^{N} m r_{\perp i}^2} \rightarrow \text{for descreate case}$$

$$\boxed{I = \int dm\, r_{\perp}^2} \rightarrow \text{Continuous case}$$

$r_{\perp}$ = $\perp^r$ distance of elementary part dm from axis.

Three particles of mass m are placed at the corners of an equilateral triangle of side 'a'. Calculate M.I. about axis passes through centroide and ⊥ to its plane.

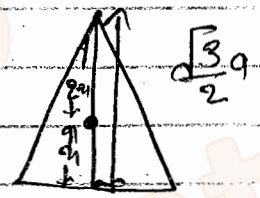
$$I = \sum_{i=1}^{3} m_i r_{\perp i}^2$$



$$I = m\frac{a^2}{3} + m\cdot\frac{a^2}{3} + m\cdot\frac{a^2}{3}$$

$$= \frac{\cancel{3}ma^2}{\cancel{3}}$$

$$\boxed{I = ma^2}$$

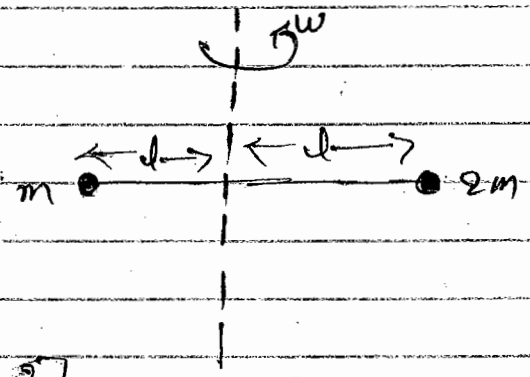


$$3x = \frac{\sqrt{3}}{2}a$$

$$x = \frac{a}{2\sqrt{3}}$$

$$2x = \frac{a}{\sqrt{3}}$$

What is angular momentum about axis of rotation?



$$L_{axis} = I\omega$$

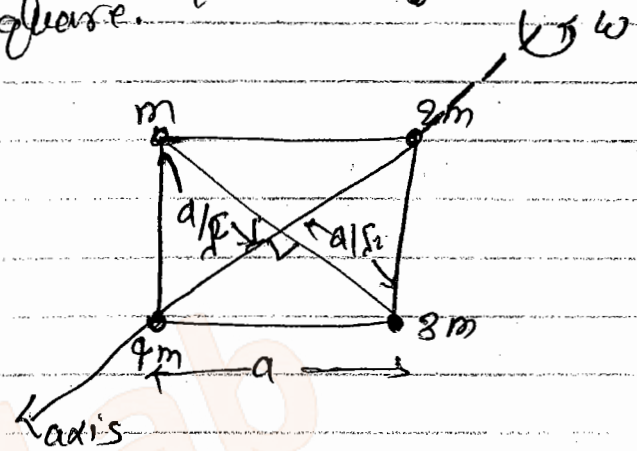
$$= \omega\left[\sum_{i=1}^{2} m_i r_{\perp i}^2\right]$$

$$= \omega\left[m \times l^2 + 2ml^2\right]$$

$$= 3ml^2\omega$$

Q. Four particles are placed at corners of square as shown in fig. Calculate M.I. of system about diagonal of square.

Sol^n



$$I = \sum_{i=1}^{4} m_i r_{\perp i}^2$$

$$= m \times \frac{a^2}{2} + 2m \times 0 + 3m \times \frac{a^2}{2} + 4m \times 0$$

$$= 4m \times \frac{a^2}{2} = 2ma^2$$

$$\boxed{I = 2ma^2}$$

Note :-

Mass density :-

Linear $(\lambda) = \frac{\text{mass}}{\text{length}} = \frac{m}{l} = \frac{dm}{dl}$ → non-Uniform.

Surface $(\sigma) = \frac{\text{mass}}{\text{area}} = \frac{dm}{dA} = \frac{m}{A}$ → Uniform

Volume $(\rho) = \frac{\text{mass}}{\text{Volume}} = \frac{dm}{dV} = \frac{m}{V}$

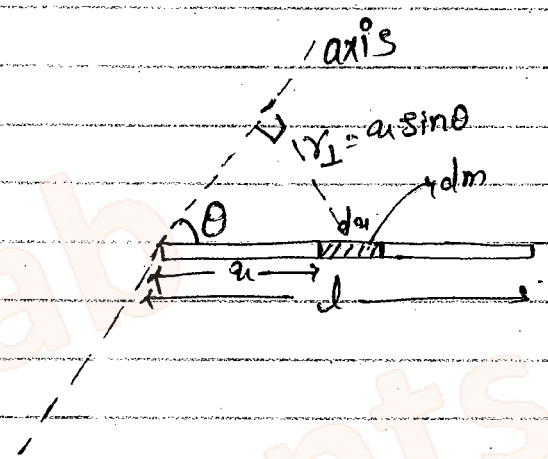
* M.I. of a Continuous object:-

A thin rod {Uniform}:-

What is moment of Inertia about axis shown in figure.

Mass of rod = m

length of rod = L



$$I = \int dm\, r_\perp^2$$

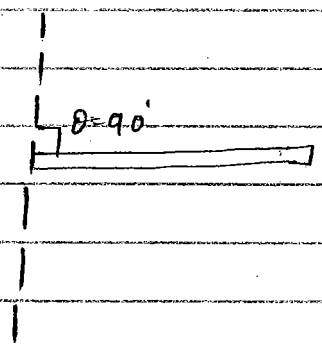
$$= \int_0^L \frac{M}{L} dx \cdot x^2 \sin^2\theta$$

$$= \frac{M}{L}\sin^2\theta \int_0^L x^2\, dx = \frac{ML^2}{3}\sin^2\theta$$

Case I :-

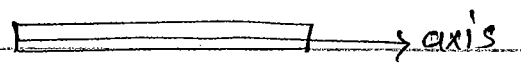
If $\theta = 90^\circ$

then $\boxed{I = \frac{ML^2}{3}}$



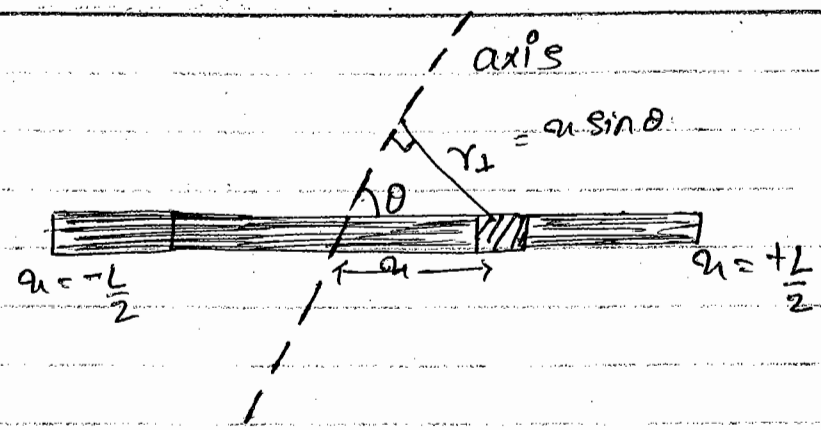
Case II :-

If $\theta = 0^\circ$



then $\boxed{I = 0}$

Case III :-



$$I = \int_{-\frac{L}{2}}^{+\frac{L}{2}} dm \, r_\perp^2$$

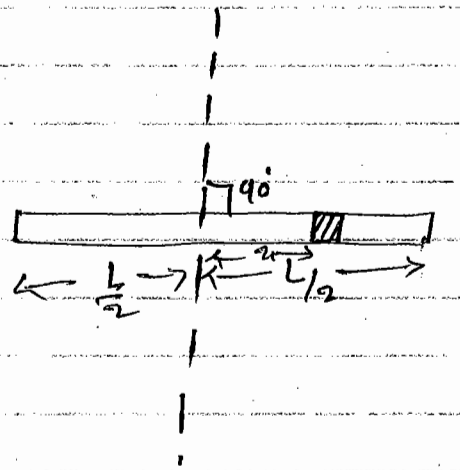
$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{M}{L} dx \cdot x^2 \sin^2\theta$$

$$= \frac{M}{L} \sin^2\theta \int_{-\frac{L}{2}}^{+\frac{L}{2}} x^2 dx = \frac{M}{L} \sin^2\theta \left[\frac{x^3}{3}\right]_{-\frac{L}{2}}^{+\frac{L}{2}}$$

$$= \frac{1}{3}\frac{M}{L} \sin^2\theta \left[\frac{L^3}{8} + \frac{L^3}{8}\right] = \frac{1}{3}\frac{M}{\cancel{L}} \sin^2\theta \left[\frac{\cancel{2}L^{\cancel{3}}}{\cancel{8}\,4}\right]$$

$$\boxed{I = \frac{ML^2 \sin^2\theta}{12}}$$

Case IV :-



2. Four thin rods adjoit to get to form a square loop. If mass of thin rod is M and length is L. What is moment of inertia of square plan about it ~~square~~ diagonal plan.

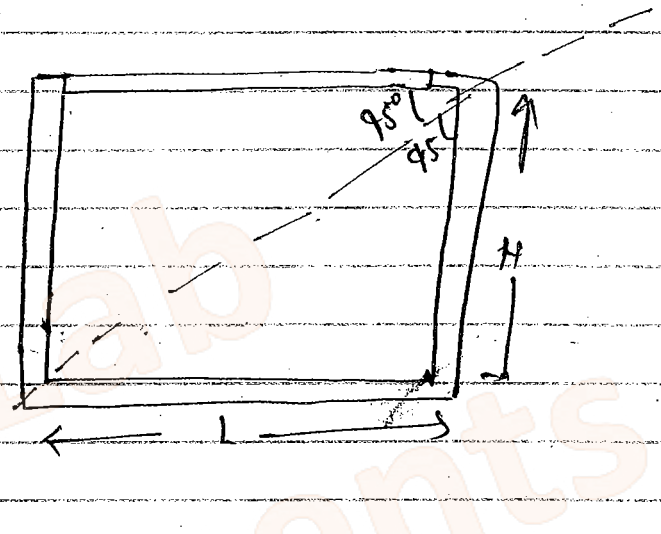
Sol

$$I = \frac{ML^2}{3}\sin^2\theta$$

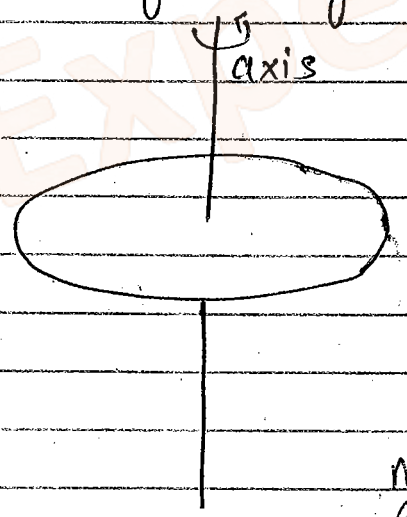
$$= 4 \times \frac{ML^2}{3}\sin^2 45^\circ$$

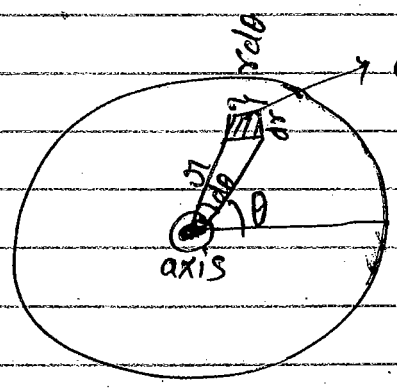
$$= 4 \times \frac{ML^2}{3} \times \left(\frac{1}{\sqrt{2}}\right)^2$$

$$\boxed{I = \frac{2ML^2}{3}}$$



2. M.I. of Uniform disc :-





$dA = r\,d\theta\,dr$
$= r\,dr\,d\theta$

Mass = M
Radius = R

$$\sigma = \frac{M}{\pi R^2}$$

$$\sigma = \frac{dm}{dA}$$

$$\frac{M}{R^2} = \frac{dM}{dA}$$

$$\boxed{dm = \frac{M}{R^2} \cdot dA}$$

$$I = \int dm\, r_\perp^2$$

$$= \int \frac{M}{\pi R^2} dA \cdot r^2$$

$$= \frac{M}{\pi R^2} \iint r\, dr\, d\theta \cdot r^2$$

$$= \frac{M}{\pi R^2} \int_0^R r^3 dr \int_0^{2\pi} d\theta$$

$$= \frac{\cancel{2}\cancel{\pi} M}{\cancel{\pi} \cancel{R^2}} \cdot \frac{R^{\cancel{4}}}{\cancel{4}}$$

$$\boxed{I = \frac{MR^2}{2}}$$

{ Memorize Read list of booklet. }

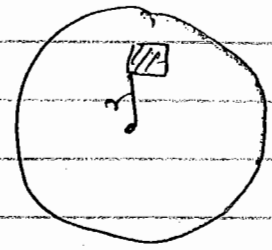
Ques A circular disc of mass M and Radius R have non-uniform density varying a distance from center as $\sigma = \sigma_0\left(1 - \frac{r}{R}\right)$ Calculate moment of inertia of disc in terms of A and R about $\perp^{or}$ axis through it centre.

Sol^n

$$\because \sigma = \frac{dM}{dA}$$

$$\sigma_0\left(1 - \frac{r}{R}\right) = \frac{dM}{dA}$$

$$dM = \sigma_0\left(1-\frac{r}{R}\right)dA$$



$$I = \int dM\, r_\perp^2$$

$$= \int \sigma_0\left(1-\frac{r}{R}\right)\cdot r^2 \cdot dA$$

$$= \iint \sigma_0\left(1-\frac{r}{R}\right) r\,dr\,d\theta \cdot r^2$$

$$= \sigma_0 \int_0^R \left(1-\frac{r}{R}\right) r^3\,dr \int_0^{2\pi} d\theta$$

$$= 2\pi\sigma_0\left[\int_0^R r^3\,dr - \int_0^R \frac{r^4}{R}\,dr\right]$$

$$= 2\pi\sigma_0\left[\frac{R^4}{4} - \frac{R^5}{5R}\right]$$

$$= 2\pi\sigma_0\left[\frac{R^4}{4} - \frac{R^4}{5}\right]$$

$$= 2\pi\sigma_0\left[\frac{R^4}{20}\right]$$

$$\boxed{I = \frac{\pi\sigma_0 R^4}{10}} \quad \text{—— (A)}$$

To eliminate σ_0 integrate dM :-

$$\int dM = \sigma_0 \int_0^R \left(1-\frac{r}{R}\right) r\,dr \int_0^{2\pi} d\theta$$

$$= 2\pi\sigma_0\left[\frac{R^2}{2} - \frac{R^3}{3R}\right]$$

$$= \sigma_0 2\pi \cdot \frac{R^2}{6}$$

$$M = \frac{\sigma_0 \pi R^2}{3}$$

$$\boxed{\sigma_0 = \frac{3M}{\pi R^2}}$$

Put in (A)

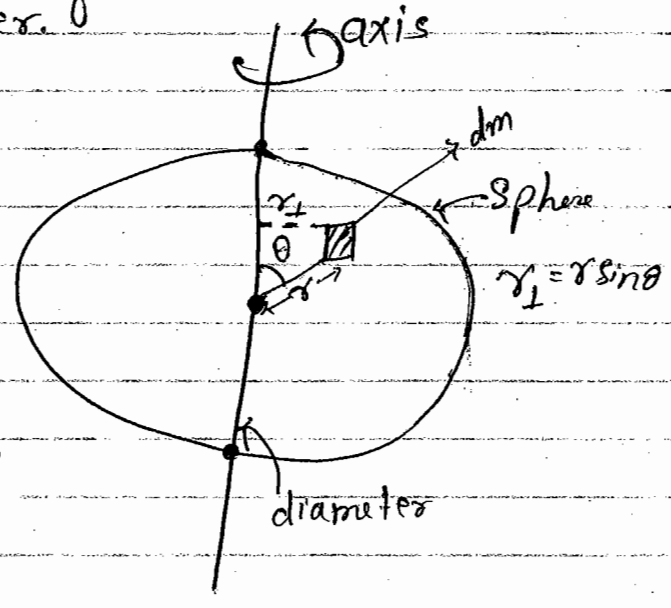
$$I = \frac{3M}{\cancel{\pi} R^{\cancel{2}}} \cdot \frac{\cancel{\pi} R^{\cancel{4}}}{10}$$

$$\boxed{I = \frac{3MR^2}{10}}$$

Ans

Q. A solid sphere of mass M and radius R has volume mass density $\rho = kr^2$ where k is constant. r is distance from centre calculate M.I. about its diameter.

Sol^n



$$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\rho = \frac{dm}{dv}$$

$$kr^2 = \frac{dm}{dv}$$

$$dm = kr^2 \cdot r^2 dr \cdot \sin\theta \, d\theta \cdot d\phi$$

$$= kr^4 dr \sin\theta \, d\theta \cdot d\phi$$

$$I = \int dm \, r_\perp^2$$

$$= \int dm \, r^2 \sin^2\theta$$

$$I = K\int_0^R r^6\,dr \int_0^{\pi} \sin^3\theta\,d\theta \int_0^{2\pi} d\phi$$

$$= \frac{KR^7}{7}\cdot 2\pi \cdot \int_0^{\pi} \sin^2\theta \cdot \sin\theta\,d\theta$$

$$\int \cancel{\sin^3\theta\cdot d\theta}$$
$$\cancel{\sin^2\theta\,dt}$$
$$\cancel{2\sin\theta}\int_0^{\pi} \sin^3\theta$$
$$\int_0^{\pi}(1-\cos^2\theta)\sin\theta\,d\theta$$
$$-\int_{-1}^{+1}(1-t^2)\,dt$$

$$= \frac{KR^7}{7}\cdot 2\pi \int_{-1}^{+1}(1-t^2)\,dt$$

$$= \frac{KR^7}{7}\cdot 2\pi\left[\left[t\right]_{-1}^{+1} - \left[\frac{t^3}{3}\right]_{-1}^{+1}\right]$$

$$= \frac{KR^7}{7}\cdot 2\pi\left[1+1\right] - \left[\frac{1}{3}+\frac{1}{3}\right]$$

$$= \frac{KR^7}{7}\,2\pi\cdot\left(2-\frac{2}{3}\right) = \frac{KR^7}{7}\,2\pi\cdot\left(\frac{6-2}{3}\right)$$

$$= \frac{8\pi KR^7}{7\times 3} = \frac{8\pi KR^7}{21} \quad \underline{\text{Ans}} \quad \text{—— (4)}$$

Now to remove K

$$\int dM = K\int_0^R r^4\,dr \int_0^{\pi} \sin\theta\,d\theta \int_0^{2\pi} d\phi$$

$$M = \frac{KR^5}{5}\cdot\left[-\cos\theta\right]_0^{\pi}\cdot 2\pi$$

$$= \frac{2\pi KR^5}{5}\cdot 2$$

$$M = \frac{4\pi KR^5}{5}$$

$$\boxed{K = \frac{5M}{4\pi R^5}}$$

Put the value of k in eq^n (6)

$$I = \frac{\cancel{8}^{2}\pi R^7}{21} \times \frac{5M}{\cancel{2}\pi R^5}$$

$$= \frac{10MR^{\cancel{7}}}{21R^{\cancel{5}}} = \frac{10MR^2}{21}$$

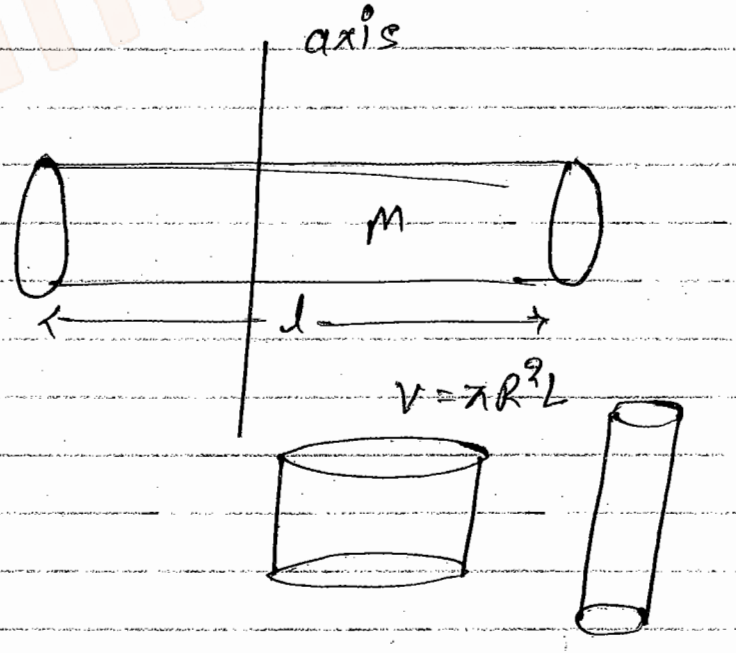
$$\boxed{I = \frac{10MR^2}{21}}$$ Ans

Q.4 What should be ratio of radius and length of a solid cylinder of uniform mass density so that its moment of inertia through the axis and $\perp^{or}$ to its length is minimum for given volume.

Sol^n

$$I = \frac{MR^2}{4} + \frac{ML^2}{12}$$

given $V = \text{Constt.}$
$M = \text{Constt.}$



$$V = \pi R^2 l$$

$$R^2 = \frac{V}{\pi l}$$

$$I = M\left[\frac{R^2}{4} + \frac{l^2}{12}\right]$$

$$I = M\left[\frac{V}{4\pi l} + \frac{l^2}{12}\right]$$

$$I = f(l)$$

for I to be minimum -

$$\frac{dI}{dl} = 0$$

$$0 = m\left[\frac{-V}{9\pi l^2} + \frac{l}{6}\right]$$

$$\frac{l}{6} = \frac{V}{9\pi l^2}$$

$$\frac{\pi R^2 l}{9\pi l^2} = \frac{l}{6}$$

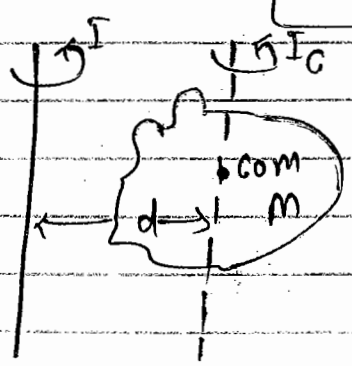
$$\left(\frac{R}{L}\right)^2 = \frac{9}{6}$$

$$\boxed{\frac{R}{L} = \frac{2}{\sqrt{6}}}$$

* Parallel Axis Theorem :-

Applicable for 1d, 2d and 3d object. We take two parallel axes one of them must pass through centre of mass.

$$\boxed{I = I_c + Md^2}$$

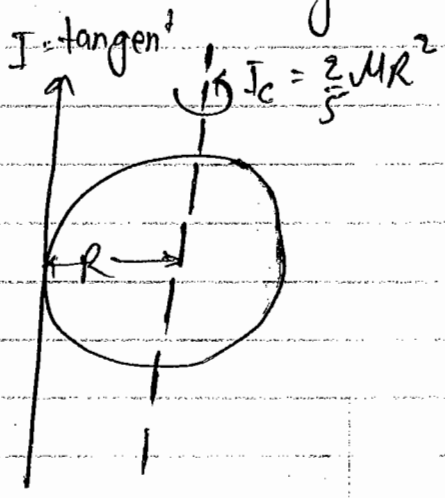


m = mass of object

d = $\perp^{or}$ distance b/w the axis.

Application :-

1 M.I of a uniform sphere about its tangent



M = mass of sphere

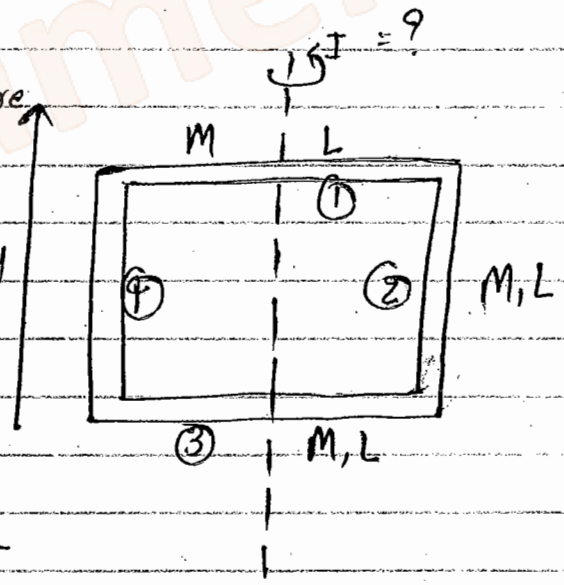
R = Radius of sphere

$$I_{tangent} = I_c + Md^2$$

$$= \frac{2}{5}MR^2 + MR^2$$

$$\boxed{I_{tangent} = \frac{7}{5}MR^2}$$

2 Calculate M.I. of square frame shown in fig. about an axis lying its plane and passing through the centre.



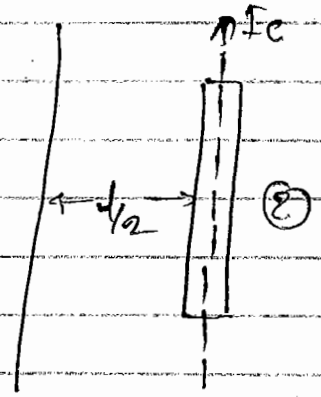
Sol^n M.I. about (1) Rod :-

$$I_{(1)} = \frac{ML^2}{12} \quad \text{—— (1)}$$

M. I. about (3) Rod :-

$$I_3 = \frac{ML^2}{12} \quad \text{—— (1)}$$

M.I. about ② Rod



$$I_{②} = I_c + Md$$

$$= 0 + \frac{ML^2}{4}$$

$$\boxed{I_{②} = \frac{ML^2}{4}} \quad \text{—— (III)}$$

Similarly

$$I_4 = \frac{ML^2}{4} \quad \text{—— (IV)}$$

So $I_{Net} = I_1 + I_2 + I_3 + I_4$

$$= \frac{ML^2}{12} + \frac{ML^2}{12} + \frac{ML^2}{4} + \frac{ML^2}{4}$$

$$= \frac{2ML^2}{12} + \frac{2ML^2}{4} = ML^2\left[\frac{\cancel{2}}{\cancel{12}_6} + \frac{\cancel{2}}{\cancel{4}_2}\right]$$

$$I_{net} = ML^2\left[\frac{2+6}{12}\right] = ML^2\left[\frac{\cancel{8}\,\cancel{4}_2}{\cancel{12}_{\cancel{6}\,3}}\right]$$

$$\boxed{I_{net} = \frac{2}{3}ML^2}$$ Ans

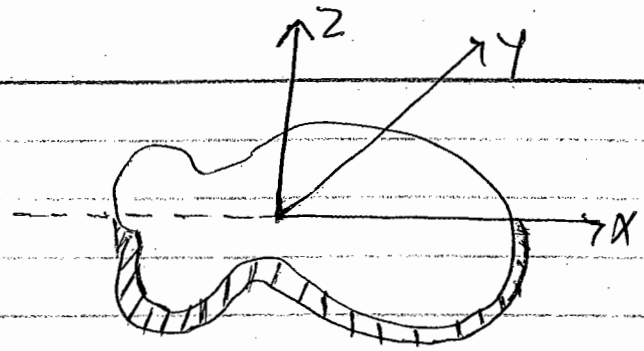
* Perpendicular Axis Theorem :-

Here we have three perpendicular axis (x, y, z). If x and y lie in the plane of object and z is perpendicular to its plane z must pass through point of intersection of x and y

⇒ It is not applicable in 3 dimension.

01/Sep./2014

$$\boxed{I_z = I_x + I_y}$$

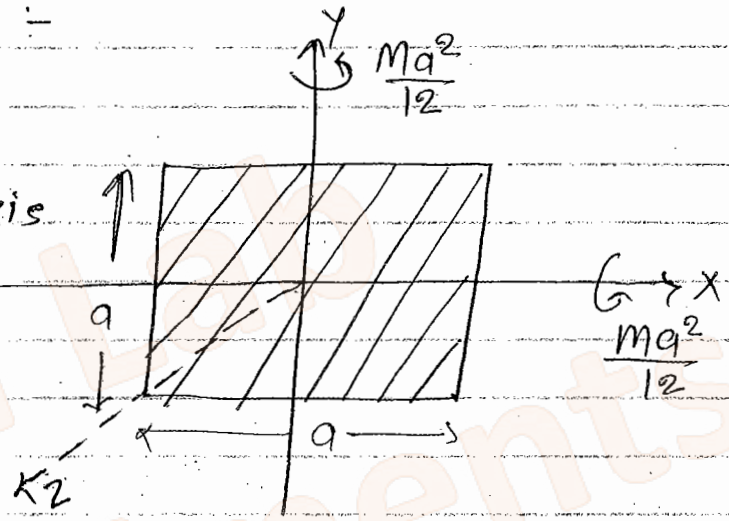


* Square Plate :-
{2-D object} :-

M.I. about Ier axis through center :-

$I_z = I_x + I_y$

$$\boxed{I_z = \frac{Ma^2}{6}}$$



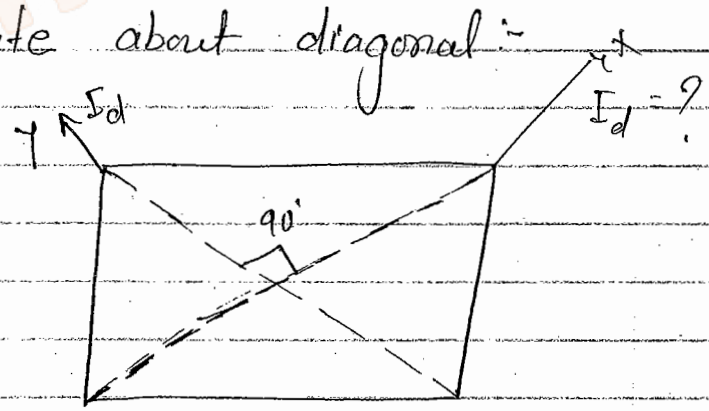
* M.I of square plate about diagonal :-

$I_z = I_x + I_y$

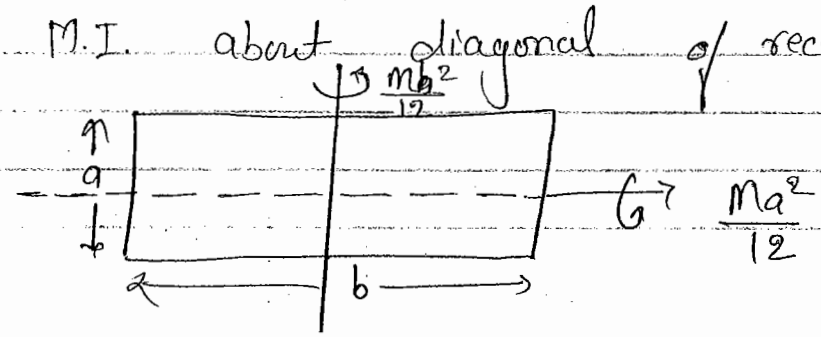
$\frac{Ma^2}{6} = I_d + I_d$

$2I_d = \frac{Ma^2}{6}$

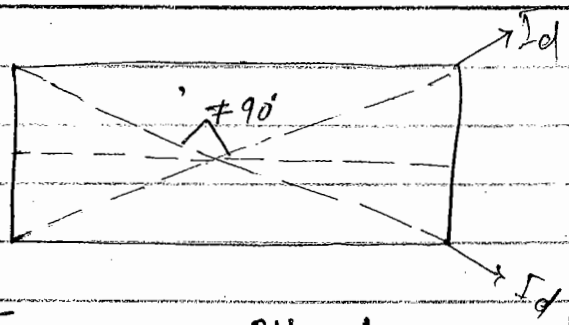
$$\boxed{I_d = \frac{Ma^2}{12}}$$



* What is the M.I. about diagonal of rectangle :-



I_d can not be calculated using $\perp^{or}$ axis theorem about its center.



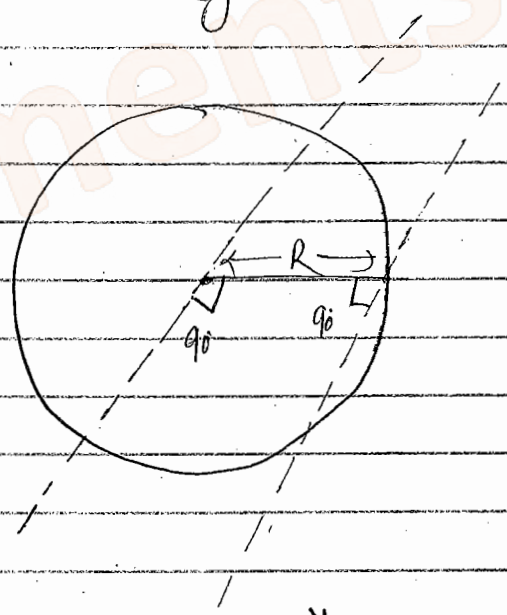
Note:- Inertia Tensor will be used.

* Disc of mass M and Radius R:-
{Assume Disc is uniform}

Q. M.I about an axis $\perp^{or}$ to its surface and passing through its edge?

Sol^n Method I:-

From $\parallel^{lel}$ Axis theorem:-

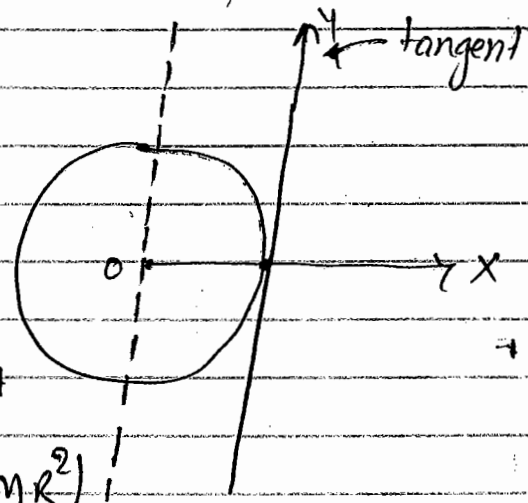


$$I = I_c + Md^2$$

$$= \frac{MR^2}{2} + MR^2$$

$$= \frac{3}{2} MR^2$$

Method IInd:-



$$I_z = I_x + I_y$$

$$I_{required} = I_{diameter} + I_{tangent}$$

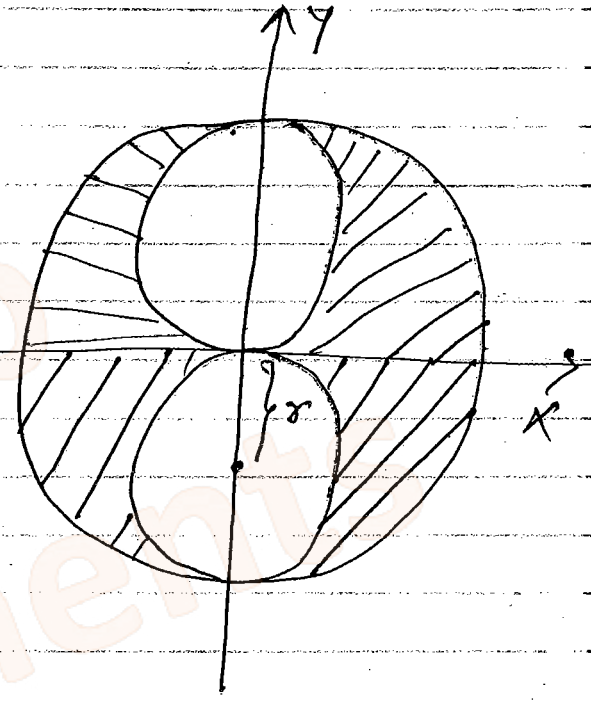
$$= \frac{MR^2}{4} + \left(\frac{MR^2}{4} + MR^2\right)$$

M.I. is a additive quantity.

$$= \frac{MR^2}{2} + MR^2 = \frac{3}{2} MR^2$$

So $\boxed{I_{required} = \frac{3}{2} MR^2}$ Ans

Gate Q.15



M.I. of remeaning object = M.I. of big sphere − M.I. of two small sphere..

$$I_y = \frac{2}{5} MR^2 - \frac{2}{5} mr^2 \times 2$$

$$M = \rho \frac{4}{3} \pi R^3$$

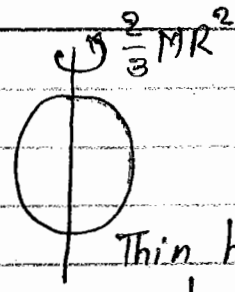
$$= \rho \pi \frac{4}{3} \pi r^3, \quad r = \frac{R}{2}$$

$$I_y =$$

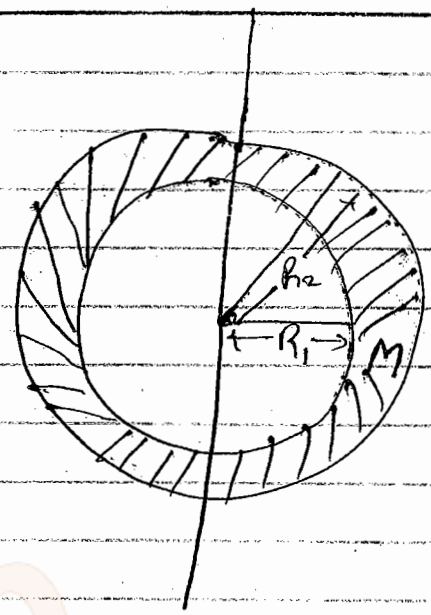
$$I_x = \frac{2}{5} MR^2 - \frac{2}{5} mr^2 \times 2$$

Q.9 A thick hollow sphere of mass 'M' has inner and outer radii R_1 and R_2. Moment of inertia of sphere about its diameter is -?

Sol^n



Thin hollow sphere.



To check options -

(c) $\frac{1}{3} m(R_1^2 + R_2^2)$

for the thin sphere -

$R_1 = R_2 = R$

$$\boxed{I = \frac{2}{3} MR^2}$$

But for solid sphere -

$R_1 = 0 \quad R_2 = R$

$$\boxed{I = \frac{1}{9} MR^2}$$ Which is not equal to the M.I. of solid sphere.

(d) $\frac{2}{5} m\left(\frac{R_2^5 - R_1^5}{R_2^3 - R_1^3}\right)$ $\left[\frac{0}{0} = \text{L' Hospital's rule. (लोपिटल)}\right]$

$R_1 = R$

$R_2 = R + x \qquad \lim_{x \to 0}$

$$I = \frac{2}{5} m\left(\frac{R_2^5 - R_1^5}{R_2^3 - R_1^3}\right)$$

$$= \lim_{x \to 0} m \frac{2}{\cancel{5}} \cancel{5}\left(\frac{(R+x)^4 - 0}{(R+x)^2 - 0}\right)$$

$$= \frac{2}{3} m \frac{R^4}{R^2} = \frac{2}{3} m R^2$$

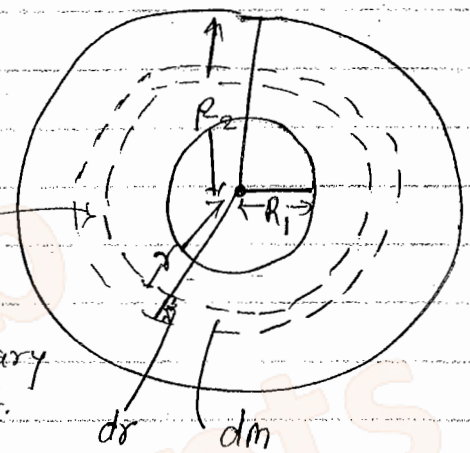
$$\boxed{I = \frac{2}{3} MR^2}$$ So it is correct.

* Moment of Inertia of a big object is to be calculated, we can first calculate M.I. of a small elementary object and integrate it.

Ex-

$$I = \int dI$$

$$dI = \frac{2}{3} dm\, r^2$$



Thin small elementary object.

$$\frac{dm}{dV} = \rho = \frac{M}{\frac{4}{3}\pi(R_2^3 - R_1^3)}$$

$$dm = \frac{3M \cdot dV}{4\pi(R_2^3 - R_1^3)}$$

$$\boxed{dm = \frac{3M}{4\pi(R_2^3 - R_1^3)} dV}$$

$$dV = 4\pi r^2 dr$$

$$dm = \frac{3M}{\cancel{4\pi}(R_2^3 - R_1^3)} \cancel{4\pi} r^2 dr$$

$$= \frac{3M r^2 dr}{(R_2^3 - R_1^3)}$$

$$= \frac{2}{3} M \int_{R_1}^{R_2} \frac{3r^2 dr}{(R_2^3 - R_1^3)} r^2 = \frac{M}{(R_2^3 - R_1^3)} \times \frac{2}{\cancel{3}} \times \cancel{3} \left[\frac{r^5}{5}\right]_{R_1}^{R_2}$$

$$= \frac{2M}{5(R_2^3 - R_1^3)} \left[(R_2^5 - R_1^5)\right]$$

$$= \frac{2}{5} M \left(\frac{(R_2^5 - R_1^5)}{(R_2^3 - R_1^3)} \right)$$

Ans

* Dynamics of rigid body :- A rigid body can have following types of motion.

1 Pure Rotation :-

At least one point of object remains at rest.

In this case -

$$K.E \neq \frac{1}{2} m v^2$$

Here, $\boxed{K.E = \frac{1}{2} I \omega^2}$

2. Pure Translation :- It means there is no rotation i.e. $\boxed{\omega = 0}$

$$L_{axis} = I_{axis}\, \omega$$

$$\boxed{\tau = I_{axis}\, \alpha}$$ eq^n of motion.

$$\boxed{\frac{dL_{axis}}{dt} = J_{axis}}$$

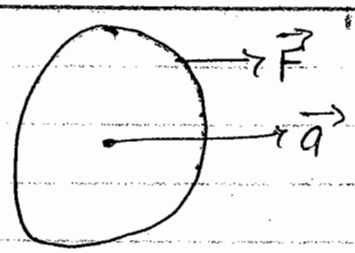
2. Pure Translation :-

It means there is no rotation i.e. $\omega = 0$

$$\boxed{K.E. = \frac{1}{2} m v^2}$$

Equation of motion:-

$$\boxed{F_{net\ force} = ma}$$



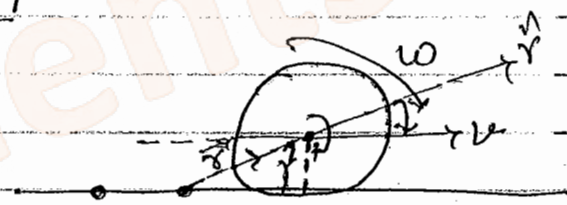
$$\boxed{L = pr_\perp}$$

3. General Motion:-

Translation + Rotation

$$\boxed{K.E. = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2}$$

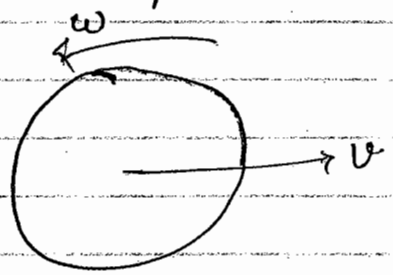
$$\boxed{L = I\omega \pm pr_\perp}$$



$r_\perp = R$ [If L is calculated about a point on path of round object]

$$\boxed{L = I\omega + pr_\perp}$$ { ∵ direction of ω and $r \times v$ is same.

$$\boxed{L = I\omega - pr_\perp}$$ { ∵ direction of ω and $r \times v$ is opposite }



$v = \omega r$ - velocity of any particle which is at the distance R at the center, due to rotation.

* Rolling Motion :-

It is a special case of general motion.

Translational velocity or translational accelaration of center of mass and rotational (Angular) velocity or rotational accelaration about center of mass are related as -

$$\boxed{V_{cm} = \omega R}$$

Diff. w.r. to t.

$$\frac{dv}{dt} = \alpha R$$

→ HET

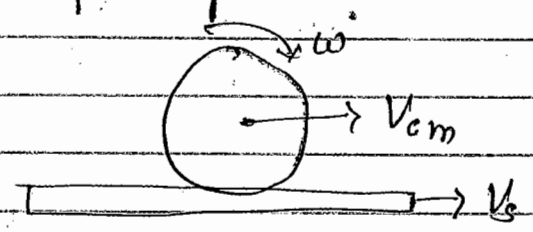
On a stationary surface.

$$V_{cm} - \omega R = V_s$$

$$a_{cm} - \alpha R = a_s$$

→ Crate

On a moving surface.



V_{cm} = Velocity of com.

* Physical Pendulum {Compound Pendulum} :-

It is a rigid body (rod, disc, ring etc) oscillating about a horizontal axis under the effect of gravity. Time period oscillation.