Free Study Material from

All Lab Experiments

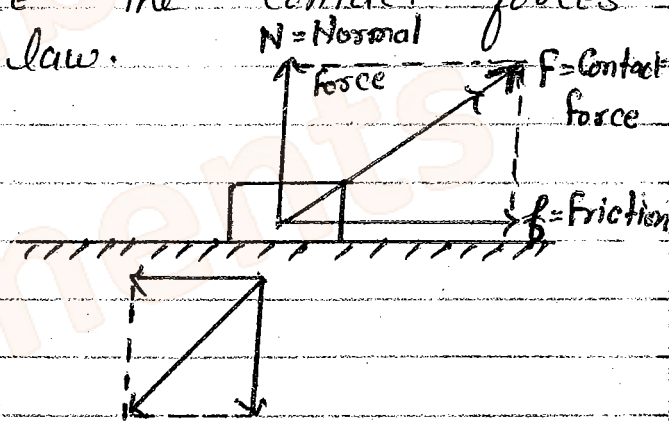
Classical Mechanics for NET/Gate Physical Sciences

>Frictional Force, Stability Analysis<

{Friction Force}

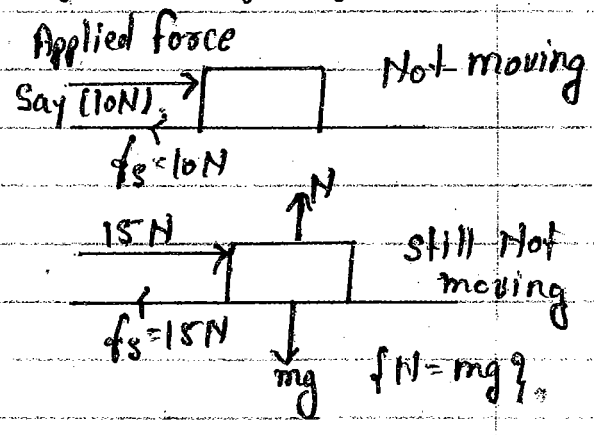
When two bodies are kept in contact, electro-magnetic forces act between the charged particles at the surfaces of the bodies. As a result each body exerts a contact force on the other.

The magnitude of the contact forces on the two bodies are equal but their direc^n are opposite and hence the contact forces obey Newton's IIIrd law.



Static Friction (f_s) :-

When two bodies do not slip on each other the force of friction is called static friction.



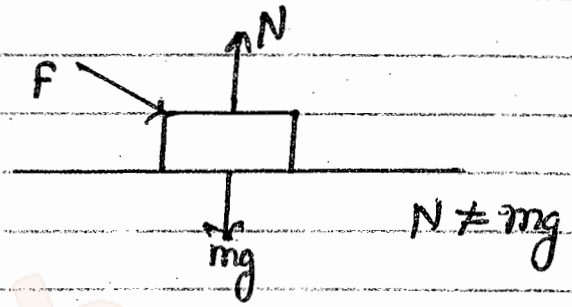
$$\therefore \boxed{f_s \leq \mu_s N}$$

N → Normal Reaction

μ_s → Coefficient of static friction.

The limiting value of static friction is -

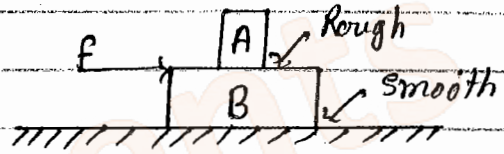
$$\boxed{(f_s)_{max} = \mu_s N}$$

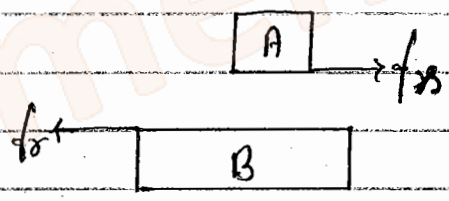


N may or may not be equal to mg.

⇒ friction may not be opposite to direction of motion always.

Ex. Here A will move with B since B is applying friction on A.



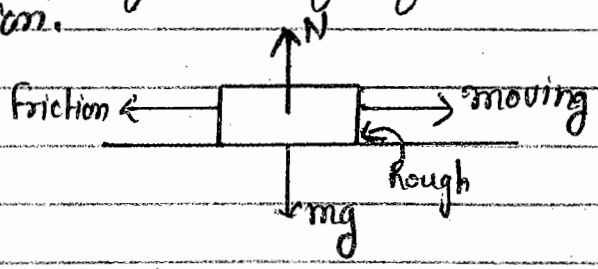


by Newton's Law.

Kinetic Friction :- (f_k) :-

When two solid bodies slip over each other, the force of friction is called kinetic friction.

$$\boxed{f_k = \mu_k N}$$



$\mu_k < \mu_s$ (slightly)

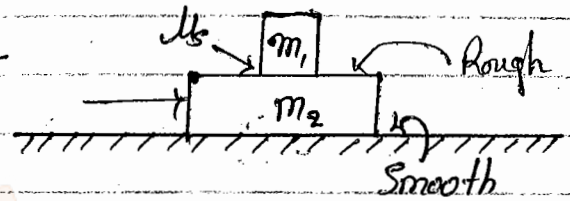
- Maximum force that can be applied without moving an object is $\mu_s N$.
- Minimum force required to move an object is almost equal to $\mu_k N$.

* Two Object Cases :-

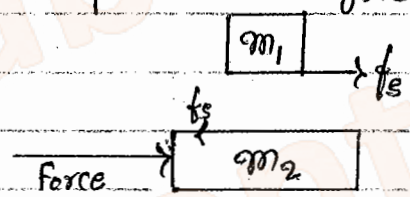
Q. What maximum force can be applied on lower block so that the two blocks move together (no relative motion between them).

Sol^n

The acceleration of upper block has a limit because it is moving due to static friction.



{ Upper block will move due to static friction



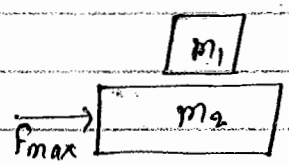
Maximum acceleration of upper block $= \frac{(f_s)_{max}}{m_1}$

$$= \frac{\mu_s N}{m_1}$$

$$= \frac{\mu_s \cancel{m_1} g}{\cancel{m_1}} = \mu_s g.$$

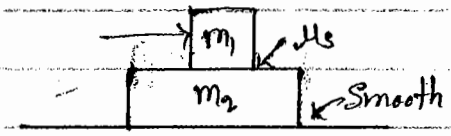
The two blocks will move together without any relative motion if their acceleration is less or equal to $\mu_s g$.

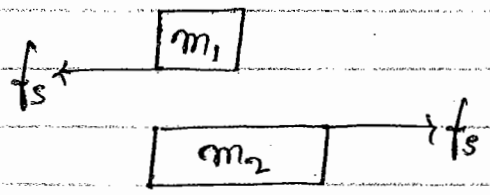
$$\boxed{F_{max} = (m_1 + m_2)\mu_s g}$$



$a_{max} = \mu_s g$

Q. What is maximum acc^n so that the two blocks move together.



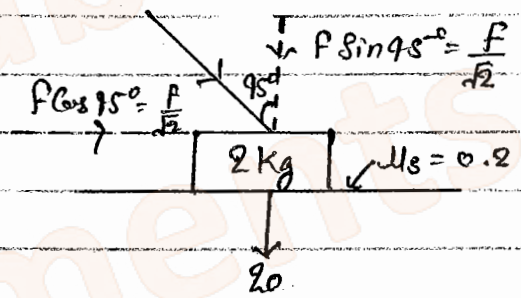


Maximum acc^n of $m_2 = \frac{(f_s)_{max}}{m_2} = \frac{\mu_s N}{m_2}$

$$\boxed{\text{Maximum acc}^n \text{ of } m_2 = \frac{\mu_s m_1 g}{m_2}}$$

Basic.
Q.15 A 2 kg block is lying on a rough surface. If coefficient of static friction between the block & ground is 0.2. What max^m force can be applied on the block without moving it.

Sol^n



The block will not move if -

$$\frac{F}{\sqrt{2}} = (f_s)_{max} = \mu_s N$$

$$\frac{F}{\sqrt{2}} = 0.2N \quad \text{—— (1)}$$

Vertical Equilibrium :-

$$20 + \frac{F}{\sqrt{2}} = N$$

Put N from (1)

$$20 + \frac{F}{\sqrt{2}} = \frac{F}{0.2\sqrt{2}}$$

$$\frac{F}{\sqrt{2}}\left[1 - \frac{1}{0.2}\right] = -20 \Rightarrow \frac{-F}{\sqrt{2}}\left[\frac{0.8}{0.2}\right] = -20$$

$$\Rightarrow F = \frac{20\sqrt{2}}{4} \Rightarrow F = 5\sqrt{2}$$

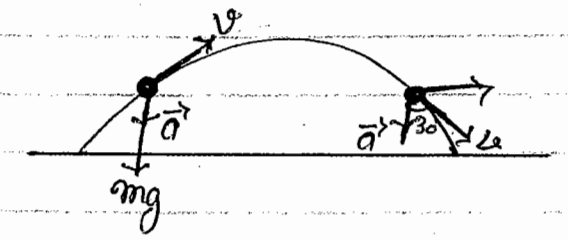
$$\Rightarrow \boxed{F \approx 7N}$$ Ans

To match the option

Basic

Q.10 A projectile is thrown at some angle. What is tangential acc^n of projectile at the instant its velocity vector makes 30° with its acc^n vector.

Sol^n ∵ Direction of acc^n will be toward force direction



$$\vec{a} = \frac{\vec{F}}{m} = \frac{mg}{m} = g$$

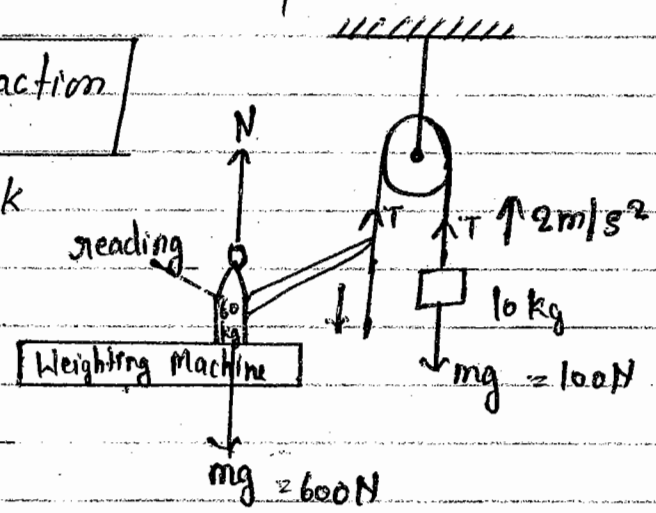
$$a_t = a \cos 30°$$

$$\boxed{a_t = g\frac{\sqrt{3}}{2}}$$ Ans.

Q.11 A person standing on a weighting machine as shown in the fig. pulls the string attached to a block in vertically downward direction. What is reading shown by the machine.

Sol^n

$$\boxed{\text{Reading} = \text{Normal Reaction}}$$



Eq^n of motion of block

$$T - mg = ma$$
$$T - 100 = 10 \times 2$$
$$T = 120$$

Consider equilibrium of person.

$$N + T = mg \Rightarrow N = Mg - T$$
$$= 600 - 120$$
$$= 480 \text{ Newton}$$

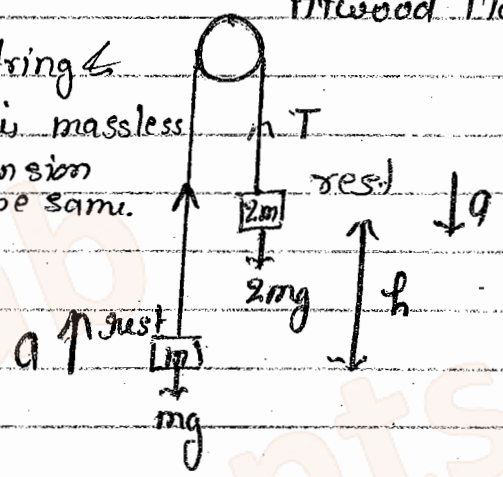
$$\boxed{N = 48 \text{ kg}}$$ Ans.

Q.7 In the fig. shown the two blocks are released from the position shown. After what time the two will cross each other. [Assume pulley & string to be light & smooth].

Sol^n

Equation of motion for m :-

Atwood Machine



$T - mg = ma$ —— (1)

{ If string & pulley is massless then tension will be same.

for 2m :-

$2mg - T = 2ma$ —— (2)

adding (1) & (2)

$\cancel{m}g = 3\cancel{m}a$

$$\boxed{a = \frac{g}{3}}$$

$s = ut + \frac{1}{2}at^2$

$\frac{h}{2} = 0 + \frac{1}{2}\frac{g}{3}t^2$ { The distance for crossing will be $h/2$

$t^2 = \frac{3h}{g}$

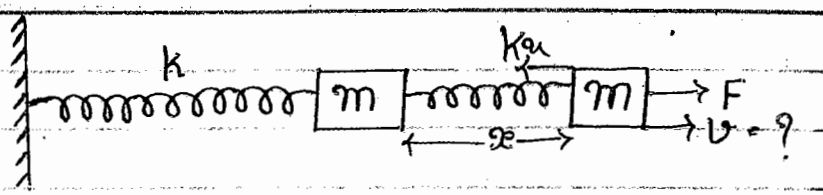
$$\Rightarrow \boxed{t = \sqrt{\frac{3h}{g}}}$$ Ans

A-3

Q.12 A spring mass system (Spring constant k & mass m) lies on a smooth horizontal surface. with one end of spring being rigidly fixed. At $t=0$ the mass is pulled with a constant horizontal force F. Speed of the mass after time t is :-

Sol^n

Equation of motion -



$$\vec{F} - kx = m\vec{a}$$

$$\boxed{\vec{a} = \frac{F}{m} - \frac{k}{m}x}$$ $\because$ $\vec{a}$ is variable

$\because$ We want to calculate - $v = f(t)$

So $F - kx = m\frac{dv}{dt}$

Steps :-

$$\boxed{\vec{a} = v\frac{dv}{dx} \xrightarrow{\text{integrating}} v = f(x) \xrightarrow{\text{integrate}} x = f(t) \xrightarrow{\text{differentiating}} v = f(t)}$$

So

$$F - kx = mv\frac{dv}{dx}$$

$$\frac{(F - kx)}{m}dx = v\,dv$$

$$\int \frac{F}{m}dx - \int \frac{k}{m}x\,dx = \int v\,dv$$

$$\frac{Fx}{m} - \frac{kx^2}{2m} = \frac{v^2}{2}$$

$$v^2 = \frac{2Fx}{m} - \frac{kx^2}{m} = \frac{2Fx - kx^2}{m} = \frac{2}{m}\left[Fx - \frac{kx^2}{2}\right]$$

$$v = \sqrt{\frac{2x[F - kx/2]}{m}}$$

$$\frac{dx}{dt} = \sqrt{\frac{2x[F - kx/2]}{m}}$$

$$\frac{dx}{\sqrt{2Fx - \frac{2kx^2}{2}}} = \frac{1}{\sqrt{m}}\int dt$$

Now integrating

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}$$

$$\Rightarrow \int \frac{dx}{[2Fx - kx^2]} = \frac{1}{\sqrt{m}}\int dt$$

let $\quad 2Fx - kx^2 = -[kx^2 - 2Fx]$

$$= -\left[(\sqrt{k}x)^2 - 2\sqrt{k}x \cdot \frac{F}{\sqrt{k}} + \left(\frac{F}{\sqrt{k}}\right)^2 - \left(\frac{F}{\sqrt{k}}\right)^2\right]$$

$$= -\left[\left(\sqrt{k}x - \frac{F}{\sqrt{k}}\right)^2 - \left(\frac{F}{\sqrt{k}}\right)^2\right]$$

$$= \underbrace{\left(\frac{F}{\sqrt{k}}\right)^2}_{(a)^2} - \underbrace{\left(\sqrt{k}x - \frac{F}{\sqrt{k}}\right)^2}_{(y)^2}$$

$$\sqrt{k}x - \frac{F}{\sqrt{k}} = y$$

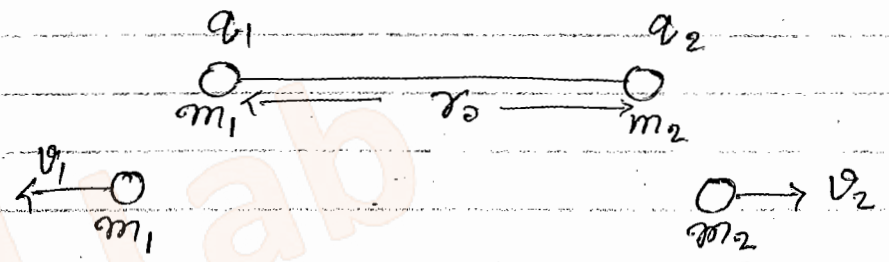
$$\sqrt{k}\,dx = dy$$

$$dx = \frac{dy}{\sqrt{k}}$$

Q.5 Two particles of masses m_1, m_2 and charges q_1, q_2 are placed r_0 distance apart on a smooth horizontal surface. Due to electrostatic repulsion they move away from each other. Ratio of their kinetic energy at a later time is-

Sol^n

$\vec{F}_{ext} = 0$



$$0 = m_2 v_2 - m_1 v_1$$

$$m_2 v_2 = m_1 v_1$$

$$\boxed{\frac{v_1}{v_2} = \frac{m_2}{m_1}} \quad \text{--- (1)}$$

$$\frac{K_1}{K_2} = \frac{\cancel{\frac{1}{2}} m_1 v_1^2}{\cancel{\frac{1}{2}} m_2 v_2^2}$$

$$\frac{K_1}{K_2} = \frac{m_1}{m_2}\left(\frac{v_1}{v_2}\right)^2$$

$$\frac{K_1}{K_2} = \frac{m_1}{m_2}\left(\frac{m_2}{m_1}\right)^2$$

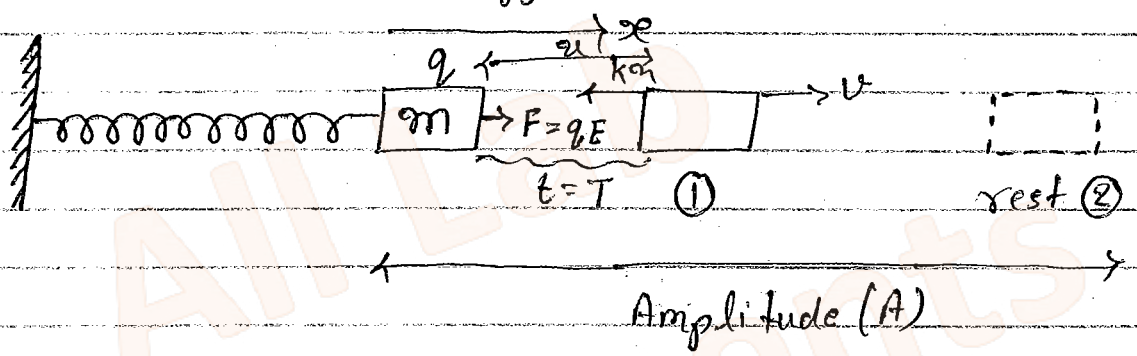
$$\boxed{\frac{K_1}{K_2} = \frac{m_2}{m_1}}$$

Ans

Net - 2011

Q.10 A point particle of mass m carrying an electric charge q is attached to a spring of stiffness constant K. A constant electric field E along the direction of spring is switched on for a time interval T (where $T \ll \sqrt{\frac{m}{K}}$). Neglecting radiation loss, the amplitude of oscillation after the field is switched off is -

Sol^n



$$v = \frac{qE}{\sqrt{mK}} \sin\sqrt{\frac{K}{m}}\, T$$

$$\frac{dx}{dt} = \frac{qE}{\sqrt{mK}} \sin\sqrt{\frac{K}{m}}\, t$$

$$[x]_0^x = \frac{qE}{\sqrt{mK}} \left[-\cos\sqrt{\frac{K}{m}}\, t\right]_0^T \times \sqrt{\frac{m}{K}}$$

Elongation $\boxed{x = \frac{qE}{K}\left[1 - \cos\sqrt{\frac{K}{m}}\, T\right]}$

Conservation of energy :-

$$(K.E. + P.E.)_I = (K.E. + P.E.)_{II}$$

$$\left(\cancel{\tfrac{1}{2}} mv^2 + \cancel{\tfrac{1}{2}} Kx^2\right) = \left(0 + \cancel{\tfrac{1}{2}} KA^2\right)$$

Date
31/07/2019

$$\Rightarrow m \cdot \frac{q^2E^2}{mk}\sin^2 + \frac{k\,q^2E^2}{k^2}(1-\cos)^2 = kA^2$$

$$\Rightarrow \frac{q^2E^2}{k}\left[\sin^2 + 1 + \cos^2 - 2\cos\right] = kA^2$$

$$\Rightarrow 2\frac{q^2E^2}{k^2}\left[1-\cos\sqrt{\frac{k}{m}}\,T\right] = A^2$$

$$\Rightarrow \frac{4q^2E^2}{k^2}\left[\sin^2\sqrt{\frac{k}{m}}\cdot\frac{T}{2}\right] = A^2$$

$$\Rightarrow A = \frac{2qE}{k}\sin\sqrt{\frac{k}{m}}\,\frac{T}{2}$$

$$\Rightarrow A = \frac{2qE}{k}\sin\frac{T}{2\sqrt{\frac{m}{k}}} \longrightarrow \text{Very small}$$

$$\Rightarrow A = \frac{\cancel{2}qE}{k}\,\frac{T}{\cancel{2}\sqrt{\frac{m}{k}}}$$

$$= \frac{qE}{\cancel{k}}\sqrt{\frac{\cancel{k}}{m}}\,T$$

$$= \frac{qE}{\sqrt{km}}\,T$$

Ans

~~Ans~~ ~~Ques~~ ~~In previous question if m = Me, then time of fall will be -~~

A-3

Q.7 A small object of mass m falls from a height equal to radius of earth Re. If Me be mass of earth time taken by the particle to reach the earth's surface is (take $m << M_e$).

(a) $\left(\frac{\pi}{2}+1\right)\sqrt{\frac{2R_e^3}{GM_e}}$ (b) $\left(\frac{\pi}{2}+1\right)\sqrt{\frac{R_e^3}{GM_e}}$

(c) $\frac{\pi}{2}\sqrt{\frac{R_e^3}{GM}}$ (d) $\sqrt{\frac{2R_e^3}{GM}}$

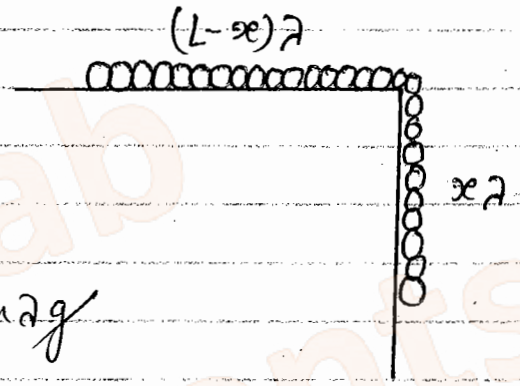
Soln

Basic

Ques A uniform rope of length 'L' is hanging off the edge of a rough table having coefficient of static friction μ. What should be minimum length of hanging hanging part so that the rope starts sliding down.

(a) $\frac{\mu L}{2}$ (b) $\frac{\mu L}{\mu+1}$ ✓ (c) $\left(\frac{1}{\mu}-1\right)L$ (d) $\frac{\mu L}{2-\mu}$

Sol^n



Equation of motion:-

$$x\lambda g = \mu (L-x)\lambda g$$

$$\Rightarrow x\lambda g = \mu L\lambda g - \mu x\lambda g$$

$$\Rightarrow x\lambda = \mu L\lambda - \mu x\lambda$$

$$\Rightarrow x + \mu x = \mu L$$

$$\Rightarrow x(1+\mu) = \mu L$$

So $\boxed{x = \frac{\mu L}{1+\mu}}$ Ans

31/July/2014

Stability Analysis

By stability analysis we mean finding equilibrium positions and investigating whether the given equilibrium is stable or unstable. It is easier to do stability analysis through potential rather than force. Therefore we will try to write potential of given system to discuss equilibrium whenever required.

Equilibrium criteria in one dimension :-

In such problem potential energy is given. If $V(x)$ be the potential under which a particle is moving then force acting on the particle is :

$$\boxed{\vec{F_x} = -\frac{dV(x)}{dx}}$$

If system (Particle) is in equilibrium, then net force on particle is zero.

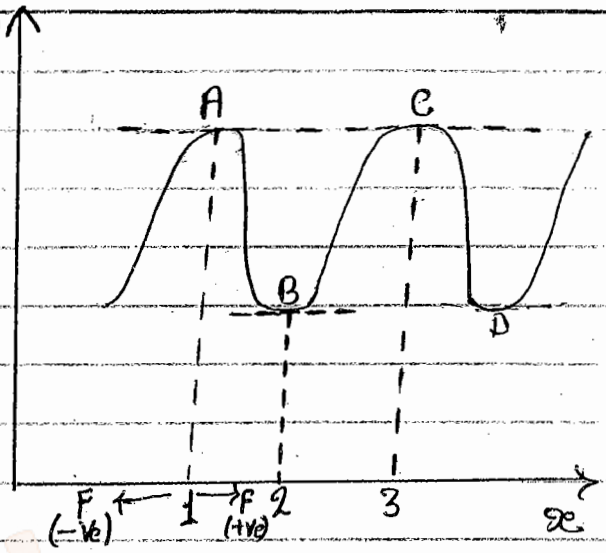
So, $\vec{F_n} = 0$

$$\Rightarrow \boxed{\frac{dV(x)}{dx} = 0}$$ (It gives equilibrium point (positions))

In $V(x)$ versus x graph $\frac{dV}{dx} = 0$ at the points where tangent to the curve is parallel to x-axis. Therefore in the figure shown below point A, B, C are equilibrium points.

Point 1, 2, and 3 are equilibrium points in space.

Point A is the point where slope = 0
" B " " " = 0
" C " " " = 0



Stable Equilibrium Point :-

A point is stable equilibrium point if, a particle at this point when displaced towards right experiences force towards left and vice-versa. That is the force tries to bring it back. Therefore at stable equilibrium point -

$$\frac{dF_x}{dx} < 0$$

or

$$\boxed{\frac{dF_x}{dx} = -V_e}$$

$\therefore \frac{d^2V(x)}{dx^2} > 0$ (Condition for minimum)

Thus, a stable equilibrium point is a minimum on V(x) versus x plot.
Therefore points B and D are stable point.

Unstable Equilibrium point :-

In this case a particle at equilibrium point is when displaced ~~towards~~ towards right it experiences force

also in rightward direction. That is the force tries to displace the particle away from equilibrium point. Therefore at unstable equilibrium point,

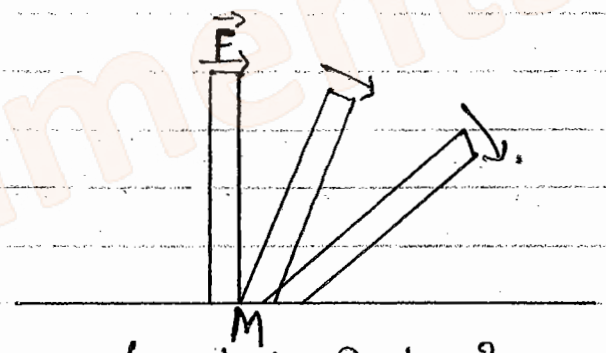
$$\frac{dF_x}{dx} > 0 \quad \text{or}$$

$$\boxed{\frac{dF_x}{dx} = +ve}$$

$$\therefore \quad \frac{d^2V(x)}{dx^2} < 0 \quad \text{(Condition of maximum)}$$

Thus, an unstable point is a maximum on $V(x)$ versus x plot. Therefore points A and C in the above figure are unstable points.

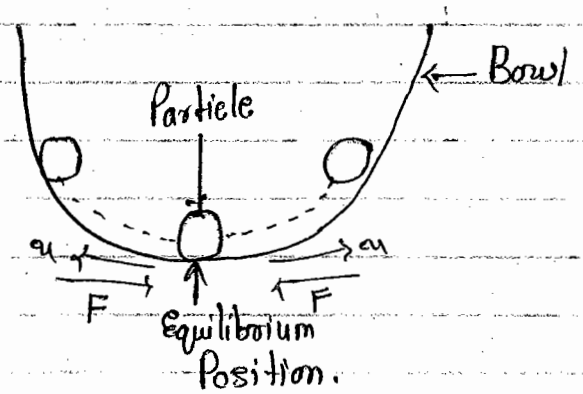
Example :-



{Unstable Position}

To understand equilibrium more closely we consider another example we take a hemispherical bowl and a marble like small sphere as shown in figure.

He we can see that the direction of force is opposite to the direction of displacement.



Now from $V(x)$ v/s x graph :-

⇒ At point 1 :-

Rightwards displacement :-

$$dx = +ve \quad , \quad dV = -ve$$

So $\vec{F} = -\frac{dV}{dx} = +ve$

So force is positive.

Leftwards displacement :-

$$dx = -ve \quad , \quad dV = -ve$$

$$\vec{F} = -\frac{dV}{dx} = +ve$$

So leftward displacement $= -ve$

Here

~~Rightwards Force ≠ Leftwards Force~~

Therefore we conclude that point 1 is unstable point.

∴ At unstable point $\frac{d^2V}{dx^2} < 0$ {for max^m V}

A condition of maxima.

⇒ At point 2 in $V(x)$ v/s x graph :-

Rightwards displacement :-

$$dV = +ve$$
$$dx = +ve$$
$$\vec{F} = -\frac{dV}{dx} = -ve$$

So for +ve displacement force is negative.

Leftwards Displacement :-

for $dx = -ve$

$dV = +ve$

$$\vec{F} = -\frac{dV}{dx} = +ve$$

So for negative displacement we get force is +ve

So we conclude that point 2 is stable point.

i.e. At stable point we must have -

$$\boxed{\frac{d^2V}{dx^2} > 0} \quad \{\text{for minimum } V\}$$

* Special Condition :-

If $\frac{d^2V}{dx^2} = 0$ then check higher even derivative. Then -

If,

$$\frac{d^nV}{dx^n} > 0 \longrightarrow (\text{Stable})$$

when

$$\frac{d^nV}{dx^n} < 0 \longrightarrow (\text{Unstable})$$

Here n is even, $n = 2, 4, 6, \ldots$ (any even value).

⇒ If all even higher derivatives are zero then the point neither "Stable" nor "Unstable".

Ex-1 $V(x) = \alpha x^4$ when $\alpha > 0$

find equilibrium point and check it is stable or unstable ?

Sol^n Let x_0 is equilibrium point.

$$\left(\frac{dV(x)}{dx}\right)_{x=x_0} = 0$$

$$4\alpha x_0^3 = 0$$

$$\Rightarrow x_0 = 0$$

So x_0 is a equilibrium point.

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 12\alpha x_0^2$$

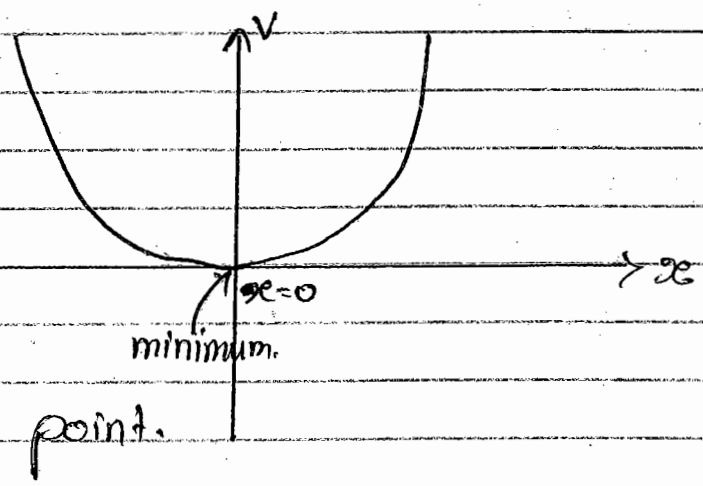
$$= 0 \quad \{\because x_0 = 0\}$$

further we check higher derivative.

$$\left.\frac{d^4V}{dx^4}\right|_{x=x_0} = 24\alpha > 0$$

So $x = x_0$ is stable point.

Graph of αx^4 :-

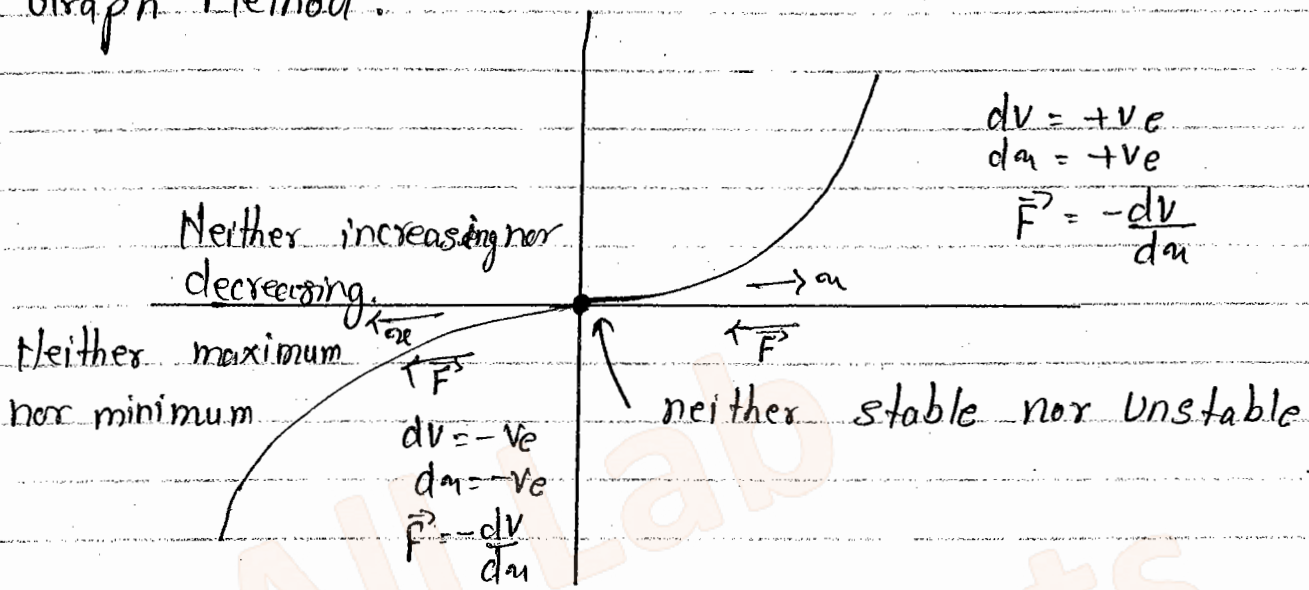


So,

$x = 0$ is a stable point.

Q. $V = \alpha x^3$, $x > 0$ find equilibrum point and check it is stable or unstable.

Sol Graph Method :-



Result :- Here for +ve displacement force is -ve but for -ve displacemut it is also negative. So it is neither stable nor unstable.

Second Method :-

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 3\alpha x^2$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 6\alpha x$$

$$\left.\frac{d^3V}{dx^3}\right|_{x=x_0} = 6\alpha$$

$$\left.\frac{d^4V}{dx^4}\right|_{x=x_0} = 0$$

So $x = x_0$ neither stable point nor unstable

Q. If $V(x) = x(x-2)^2$, how many stable and unstable points are there?

Sol^n

Condition of equilibrium (x_0) :-

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

$$\Rightarrow (x_0-2)^2 + 2x_0(x_0-2) = 0$$

$$\Rightarrow (x_0-2)\left[(x_0-2) + 2x_0\right]$$

$$\Rightarrow (x_0-2)\left[3x_0-2\right] = 0$$

$$\Rightarrow \left.\begin{array}{l} x_0 = 2 \\ x_0 = 2/3 \end{array}\right\} \text{Two equilibrium point.}$$

at $x_0 = 2$:-

$$\left.\frac{d^2V}{dx^2}\right|_{x_0=2} = 12-8 = 4 > 0$$

So $x_0 = 2$ is stable point.

$$\left.\frac{d^2V}{dx^2}\right|_{x_0=\frac{2}{3}} = 6\times\frac{2}{3} - 8 = -4 < 0$$

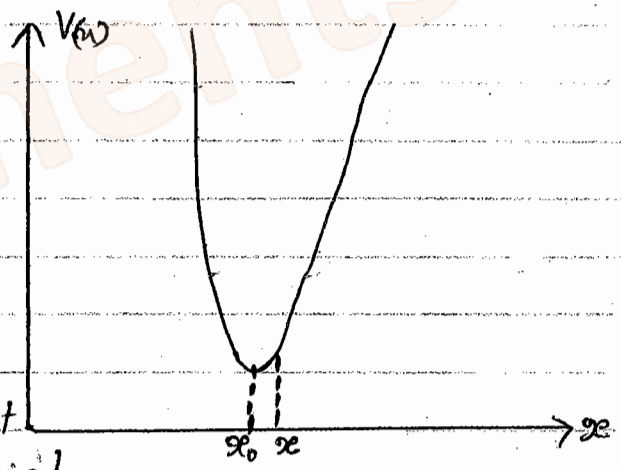
So $x_0 = \frac{2}{3}$ is a unstable point.

* Frequency of Oscillation about stable equilibrium point {for small oscillation}:-

We know a particle of mass m moving under potential $V(x) = V_0 + \frac{1}{2}kx^2$ has freq. of oscillation $\omega = \sqrt{\frac{k}{m}}$, about stable equilibrium point, then we can expand $V(x)$ about x_0 using Taylor's series expansion as,

$$V(x) = V(x_0) + (x-x_0)\left.\frac{dV}{dx}\right|_{x=x_0} + \frac{(x-x_0)^2}{\lfloor 2}\left.\frac{d^2V}{dx^2}\right|_{x=x_0} + \frac{(x-x_0)^3}{\lfloor 3}\left.\frac{d^3V}{dx^3}\right|_{x=x_0} + \cdots\cdots$$

Let a particle of mass m is moving under a potential $V(x)$. Let x_0 be a stable point.



In the above expansion- $(x-x_0)$ is a displacement from stable equilibrium point.

At the equilibrium position -

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

therefore, if $(x-x_0)$ be small then higher power terms are neglected.

$$V(x) = V(x_0) + \frac{(x-x_0)^2}{\lfloor 2}\left.\frac{d^2V}{dx^2}\right|_{x=x_0}$$

it is greater than 0 So let it is = k.

$$\boxed{V(x) = V(x_0) + \frac{1}{2} K (x - x_0)^2}$$

So force :-

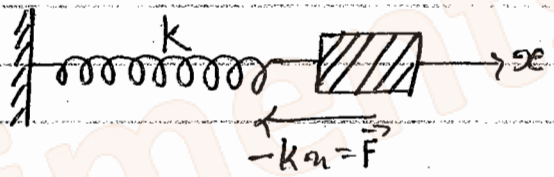
$$\vec{F} = -\frac{dV}{dx}$$

$$\vec{F} = -(x - x_0) K$$

$$\boxed{\vec{F} = -K(x - x_0)}$$

Here force is $(-K)$ times displacement and this is a case of S.H.M.

An example of S.H.M:-



So in the case of S.H.M.

$$\Rightarrow -Kx = ma$$

$$\Rightarrow \boxed{\ddot{x} = -\left(\frac{K}{m}\right) \cdot x}$$

$$\omega^2 = \frac{K}{m}$$

So $\boxed{\omega = \sqrt{\frac{K}{m}}}$

ω : freq. of oscillation
m : mass of particle
K : Force Constant.

Here $\boxed{K = \left.\frac{d^2V}{dx^2}\right|_{x = x_0}}$

x_0 : stable point.

Here frequency of oscillation is independent of displacement. So it is true for small oscillation ω is independent of amplitude. If oscillation is not small then ω is dependent of amplitude (energy).

Conclusions :-

⇒ $$\omega = \sqrt{\frac{k}{m}} \quad \text{where } k = \left.\frac{d^2V}{dx^2}\right|_{x=x_0}$$

x_0 is stable point

⇒ It is not applicable in the case of -

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 0$$

⇒ If $\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 0$. Then we will write -

$$V(x) = V(x_0) + \frac{(x-x_0)^3}{\lfloor 3} \left.\frac{d^3V}{dx^3}\right|_{x=x_0}$$

$$V(x) = V(x_0) + 3\frac{(x-x_0)^2}{\lfloor 3} \cdot k$$

$$\boxed{V(x) = V(x_0) - \frac{(x-x_0)^2}{\lfloor 2} \cdot k}$$

$$\boxed{\vec{F} = -\frac{dV}{dx}}$$

$$\boxed{\vec{F} = -\frac{(x-x_0)^2}{\lfloor 2} \cdot k}$$

Here $\vec{F} \propto (x-x_0)^2$ So it is not in S.H.M.

⇒ If $V = V(\theta)$ then -

$$\omega = \sqrt{\frac{k}{I}}, \quad k = \left.\frac{d^2V}{d\theta^2}\right|_{x=\theta_0}$$

Where θ_0 = Stable point.

I = Moment of Inertia.

CSIR - 2012

Ques 9 Consider the motion of a classical particle in a one dimensional double-well potential $V(x) = \frac{1}{4}(x^2-2)^2$. If the particle is displaced infinitesimally from the minimum of the positive X-axis (and friction is neglected), then-

(a) the particle will execute simple harmonic motion in the right well with an angular freq. $\omega = \sqrt{2}$

(b) the particle will execute simple harmonic motion in the right with an angular frequency $\omega = 2$.

(c) the particle will switch between the right and left well. {Not possible in classical mechanics.}

(d) the particle will approach the bottom of the right well and settle there {Here no settle because there is no friction}.

Sol^n

$$V(x) = \frac{1}{4}(x^2-2)^2$$

$$\therefore \omega = \sqrt{\frac{K}{m}}, \quad K = \left.\frac{d^2V}{dx^2}\right|_{x=x_0}$$

At equilibrium point -

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

$$x_0(x_0^2-2) = 0$$

$$\Rightarrow \left.\begin{array}{l} x_0 = 0 \\ x_0 = \pm\sqrt{2} \end{array}\right\} \text{ there is 3 equilibrium points.}$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 3x_0^2 - 2$$

at $x = x_0 = 0$

$$\frac{d^2V}{dx^2} = -2 < 0 \quad (\text{So } x_0 = 0 \text{ is unstable point})$$

at $x = x_0 = \pm\sqrt{2}$

$$\left.\frac{d^2V}{dx^2}\right|_{x_0=\pm\sqrt{2}} = 6-2 = 4 > 0 \quad \left\{ \begin{array}{l} x_0 = \pm\sqrt{2} \text{ is} \\ \text{stable point} \end{array} \right.$$

So there are two stable points $(\pm\sqrt{2})$.

$\therefore \boxed{k = 4}$

$\therefore \omega = \sqrt{\frac{4}{1}}$ $\left\{ \begin{array}{l} \text{take } m = 1 \\ \text{Unit mass} \end{array} \right\}$

Angular freq. $\boxed{\omega = 2}$

---

* How to Plot a graph {V(x) v/s x} :-

Steps :-

(i) Put $V(x) = 0$ and find the value of x.

(ii) Put $dV/dx = 0$ and find the value of x.

(iii) find position of maxima and minima i.e. check second derivative.
If second derivative is zero then check higher derivative.

(iv) Connect all the points where $V(x) = 0$ by making appropriate maxima or minima.

(V) Consider $x \longrightarrow \pm\infty$

Ex- $V(x) = \frac{1}{4}(x^2-2)^2$

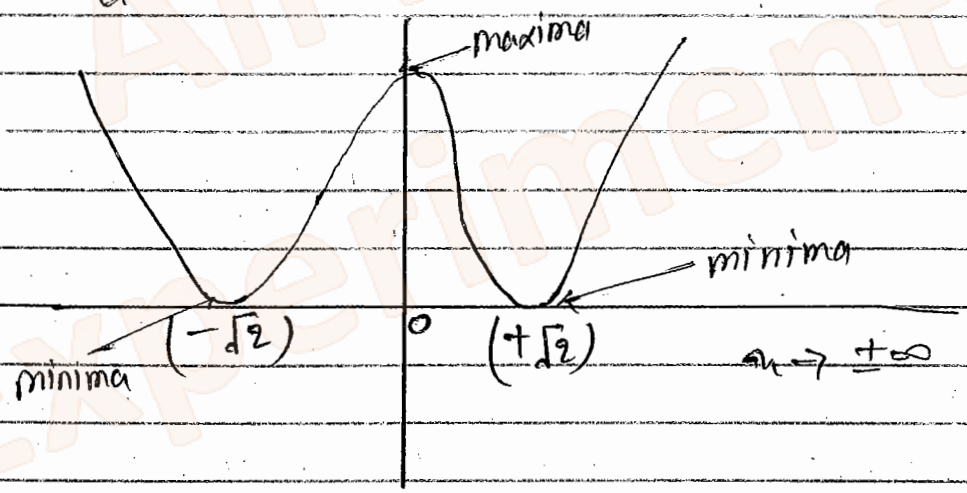
Sol^n $V(x) = 0 \Rightarrow (x^2-2)^2 = 0$

$x = \pm\sqrt{2}$

$\frac{dV}{dx} = 0$ find $x = 0, \pm\sqrt{2}$

$\frac{d^2V}{dx^2} > 0$ at $x = \pm\sqrt{2}$ i.e. minima

$\frac{d^2V}{dx^2} < 0$ at $x = 0$ i.e. it is maxima



A-4
Q.14 A particle of mass 'm' is moving under potential $V(x) = ax^3 - bx$, frequency of oscillation about the stable equilibrium posion is -

(a) $\left[\frac{12ab}{m^2}\right]^{1/4}$ ✓ (b) $\left[\frac{6ab}{m^2}\right]^{1/4}$ (c) $\left[\frac{9ab}{m^2}\right]^{1/4}$ (d) $\left[\frac{3ab}{m^2}\right]^{1/4}$,

Sol^n

$V(x) = ax^3 - bx$

let x_0 is equilibrium point then

$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 3ax_0^2 - b = 0$$

$$3ax_0^2 = b$$

$$x_0^2 = \frac{b}{3a}$$

$$\boxed{x_0 = \pm\sqrt{\frac{b}{3a}}}$$

Now $\frac{d^2V}{dx^2} = 6ax$

$= \sqrt{\frac{b}{3a}} \cdot 6a > 0$ stable (minima)

$= -6a\sqrt{\frac{b}{3a}} < 0$ Unstable (maxima).

$\therefore \left.\frac{d^2V}{dx^2}\right|_{x=x_0} = K = 6a\sqrt{\frac{b}{3a}} = \left[\frac{36a^2b}{3a}\right]^{1/2}$

So angular freq $(\omega) = \sqrt{\frac{K}{m}}$
about stable point.

So $\omega = \sqrt{\left(\frac{\cancel{36}^{12}a^{\cancel{2}}b}{\cancel{3}\cancel{a}}\right)^{1/2} \cdot \frac{1}{m}}$

$$\omega = \sqrt{\left(\frac{12ab}{m^2}\right)^{1/2}}$$

$$\boxed{\omega = \left(\frac{12ab}{m^2}\right)^{1/4}}$$

Ans

Note:-

In a particle in which speed is to be calculated & Potential energy is given.
Then we apply conservation of energy.

Ex- $V(x) = A - Bx + Cx^3$

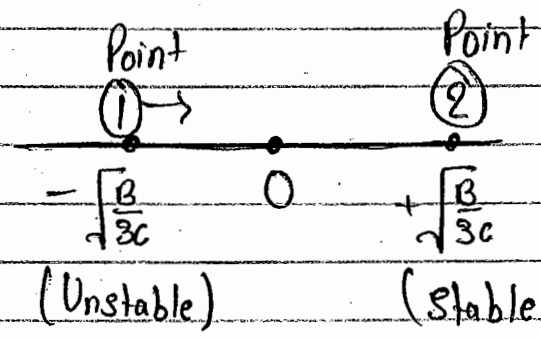
For equilibrium point -

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

$$-B + 3Cx_0^2 = 0$$

$$\boxed{x_0 = \pm\sqrt{\frac{B}{3C}}}$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 6Cx_0$$



If system is conservative then conservation of energy.

$$(P.E. + K.E.)_{\text{at point 1}} = (P.E. + K.E.)_{\text{at point 2}}$$

$$\cancel{A} + B\sqrt{\frac{B}{3C}} - \frac{CB}{3C}\sqrt{\frac{B}{3C}} + 0 = \cancel{A} - B\sqrt{\frac{B}{3C}} + \frac{CB}{3C}\sqrt{\frac{B}{3C}} + \frac{1}{2}mv^2$$

$$2B\sqrt{\frac{B}{3C}} - \frac{2CB}{3C}\sqrt{\frac{B}{3C}} = \frac{1}{2}mv^2$$

$$\sqrt{\frac{B}{3C}}\left[2B - \frac{2\cancel{C}B}{3\cancel{C}}\right] = \frac{1}{2}mv^2$$

$$2B\sqrt{\frac{B}{3C}}\left[1 - \frac{B}{3}\right] = \frac{1}{2}mv^2$$

$$\boxed{\left\{\frac{4B}{m}\sqrt{\frac{B}{3C}}\left[1 - \frac{B}{3}\right]\right\}^{1/2} = V}$$

Ans

June 2013

Q. There is a particle of mass m moving with potential $V(x) = ax - bx^3$ initially particle is at rest at stable point what is minimum speed should be given to it. so that its motion becomes unstable.

Sol^n

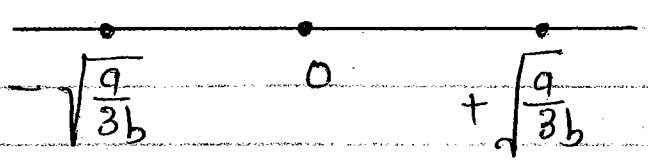
$$V(x) = ax - bx^3$$

$$\left.\frac{dV(x)}{dx}\right|_{x_0} = a - 3bx_0^2 = 0$$

$$\Rightarrow 3bx_0^2 = a$$

$$\Rightarrow x_0 = \pm\sqrt{\frac{a}{3b}}$$

So that equilibrium point is; $+\sqrt{\frac{a}{3b}}$ and $-\sqrt{\frac{a}{3b}}$



$$\frac{d^2V(x)}{dx^2} = -6bx_0 < 0$$

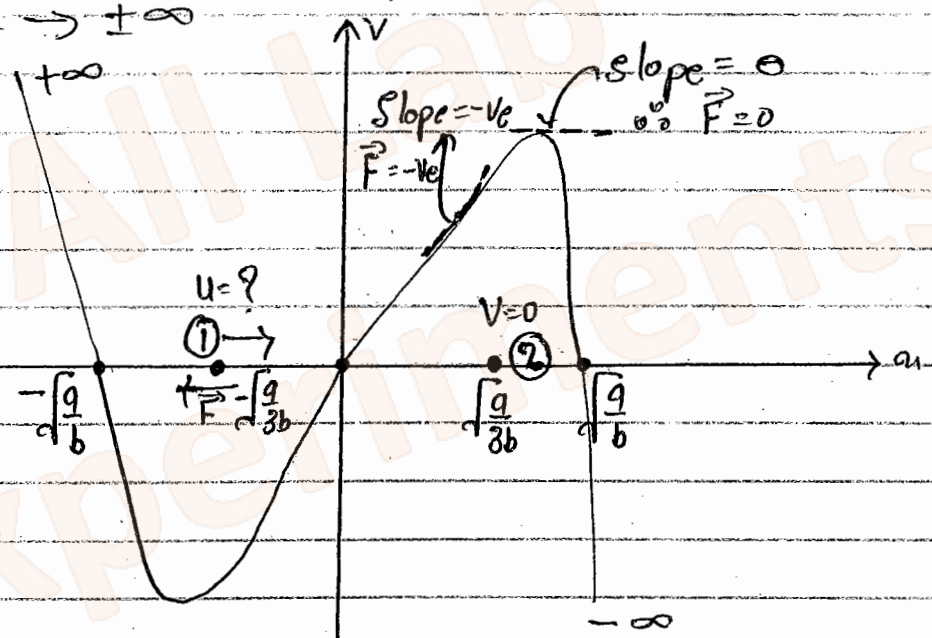
So $-\sqrt{\frac{a}{3b}}$ is stable point (min^m potential point.

$V(x) = 0$

$\Rightarrow x(a - bx^2) = 0$

$x = 0, \quad x = \pm\sqrt{\frac{a}{b}}$

and $x \to \pm\infty$



for minimum speed that particle reaches unstable point and stop there.

Applying Energy Conservation at point ① and ②:

$$(P.E. + K.E.)_I = (P.E. + K.E.)_{II}$$

$$\Rightarrow \frac{1}{2}mu^2 - a\sqrt{\frac{a}{3b}} + b\left(\frac{a}{3b}\right)^{3/2} = 0 + a\sqrt{\frac{a}{3b}} - b\left(\frac{a}{3b}\right)^{3/2}$$

$$\Rightarrow \frac{1}{2}mu^2 = 2a\sqrt{\frac{a}{3b}} - 2b\left(\frac{a}{3b}\right)^{3/2}$$

$$\frac{1}{2} mu^2 = 2\sqrt{\frac{a}{3b}}\left\{a - \cancel{b}\left(\frac{a}{3\cancel{b}}\right)\right\}$$

$$mu^2 = 4\sqrt{\frac{a}{3b}}\left\{\frac{3a-a}{3}\right\} = 4\sqrt{\frac{a}{3b}}\left\{\frac{2a}{3}\right\}$$

$$mu^2 = \sqrt{\frac{a}{3b}}\left\{\frac{8a}{3}\right\} = \left\{\frac{64a^2 \cdot a}{27b}\right\}^{1/2}$$

$$u^2 = \left\{\frac{64a^3}{27m^2b}\right\}^{1/2}$$

$$\boxed{u = \left\{\frac{64a^3}{27m^2b}\right\}^{1/4}}$$

Q.6 A particle of mass $m=4$ moves along the x-axis under the influence of the potential $V(x) = 2(e^{-\frac{3}{2}x} - 2e^{-\frac{3}{4}x})$, if the particle oscillates with small amplitude around the minimum potential, what is the period of the oscillation.

(a) 0.12 (b) 1.33 (c) 8.37 ✓ (d) 11.17

Sol^n

$$V(x) = 2\left[e^{-\frac{3}{2}x} - 2e^{-\frac{3}{4}x}\right]$$

$$\omega = \sqrt{\frac{k}{m}}, \quad k = \left.\frac{d^2V}{dx^2}\right|_{x=x_0} \text{(stable)}$$

$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

$$2\left[-\frac{3}{2}e^{-\frac{3}{2}x_0} + \frac{3}{2}e^{-\frac{3}{4}x_0}\right] = 0$$

$$e^{-\frac{3}{2}x_0} = e^{-\frac{3}{4}x_0}$$

Taking ln on both side.

$$\cancel{-}\frac{\cancel{3}}{2}x_0 = \cancel{-}\frac{\cancel{3}}{4}x_0$$

$$\left(\frac{1}{2}-\frac{1}{4}\right)x_0 = 0$$

$$\because \left(\frac{1}{2}-\frac{1}{4}\right) \neq 0$$

$$\therefore \boxed{x_0 = 0}$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 2\left[\frac{9}{4}e^{-\frac{3}{2}x_0} - \frac{9}{8}e^{-\frac{3}{4}x_0}\right]$$

$$= 2\left[\frac{9}{4}-\frac{9}{8}\right]$$

$$= 2\times\frac{9}{8}$$

$$= \frac{9}{4} = k = +ve$$

So $x_0 = 0$ is stable point.

$$\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$\frac{2\times 3.14}{T} = \sqrt{\frac{9}{4\times 4}}$$

$$\frac{2\times 3.14}{T} = \frac{3}{4}$$

$$T = \frac{8\times 3.14}{3} = \frac{25.12}{3}$$

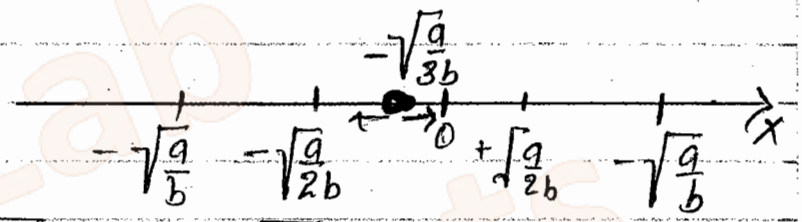
$$\boxed{T = 8.3733}$$

Q.7 Force acting on a particle along x-axis is given to be $F(x) = 2bx^3 - ax$. If the particle is released from point $x = -\sqrt{\frac{a}{3b}}$ it will move towards which of the following points.

(a) 0 ✓ (b) $+\sqrt{\frac{a}{3b}}$ (c) $-\sqrt{\frac{a}{2b}}$ ✓ (d) $-\sqrt{\frac{a}{b}}$

Sol^n

$\vec{F}(x) = 2bx^3 - ax$



$$F(x_1) = -2\cancel{b} \cdot \frac{a}{3\cancel{b}}\sqrt{\frac{a}{3b}} + a\sqrt{\frac{a}{3b}}$$

$$= \left(\frac{-2a}{b} + a\right)\sqrt{\frac{a}{3b}}$$

$$= \frac{a}{3b}\sqrt{\frac{a}{3b}} \quad (+ve)$$

So it move in the direction of O.

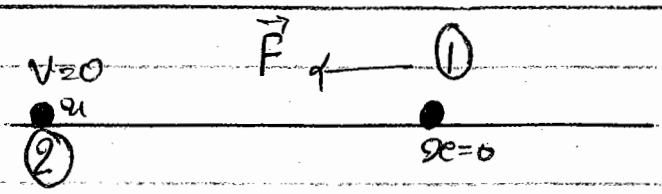
Q.12 A particle of mass 2Kg is released at $x = 0$ m. The particle is acted upon by a force whose potential is expressed as $V(x) = 2x - 4x^3$ joule. The force on the particle at the point where it will again come to rest for the first time is -

(a) -4 Newton (b) +4 Newton ✓ (c) zero Newton

(d) +2 Newton.

Sol^n

$V(x) = 2x - 4x^3$



$\vec{F} = -\frac{dV}{dx} = 12x^2 - 2$

Put $x = 0$

$F = -2$ So F is -ve

Apply conservation of energy at ① & ②

$(K.E. + P.E.)_I = (K.E. + P.E.)_{II}$

$(0+0) = (0 + 2x - 4x^3)$ (Put $x=0$ in $V(x)$)

$x = 0, \pm\frac{1}{\sqrt{2}}$

$-\frac{1}{\sqrt{2}}$ is the point where particle is in rest for first time.

So $\vec{F} = 12x^2 - 2 = 12\left(-\frac{1}{\sqrt{2}}\right)^2 - 2$

$= 6 - 2 = 4$ Newton. Ans

A-4

Q.10 The total energy E of the particle of mass m executing small oscillations about the origin along on the x-direction is given by, $E = \frac{1}{2}mv^2 + V_0 \cosh\left(\frac{x}{L}\right)$, where V_0 and L are positive constants. The time period T of oscillation is -

(a) $T = \frac{1}{2\pi}\sqrt{\frac{m}{V_0}}$ (b) $T = 2\pi\sqrt{\frac{L}{m}}$ (c) $T = \pi L\sqrt{\frac{m}{E}}$

(d) $T = 2\pi\sqrt{\frac{mL^2}{V_0}}$

Sol^n

$$E = \underbrace{\frac{1}{2}mv^2}_{K.E.} + \underbrace{V_0 \cosh\left(\frac{x}{L}\right)}_{P.E.}$$

$$V(x) = V_0 \cosh\left(\frac{x}{L}\right)$$

$\because$ Origin is stable point (given)

$$x = 0$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$k = \left.\frac{d^2V}{dx^2}\right|_{x=0} = \frac{V_0}{L^2}\left(\cosh\frac{x}{L}\right)_{x_0}$$

$$\boxed{k = \frac{V_0}{L^2}}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{V_0}{L^2 m}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{V_0}{L^2 m}}$$

$$\boxed{T = 2\pi\sqrt{\frac{L^2 m}{V_0}}}$$ Ans

Q.11 A particle of mass 'm' is moving under a one dimensional potential $V(x) = ax^3 - bx$. Due to the force acting on the particle its kinetic energy changes as the particle moves from one point to the other. What can be maximum change in K.E. of the particle in this case -

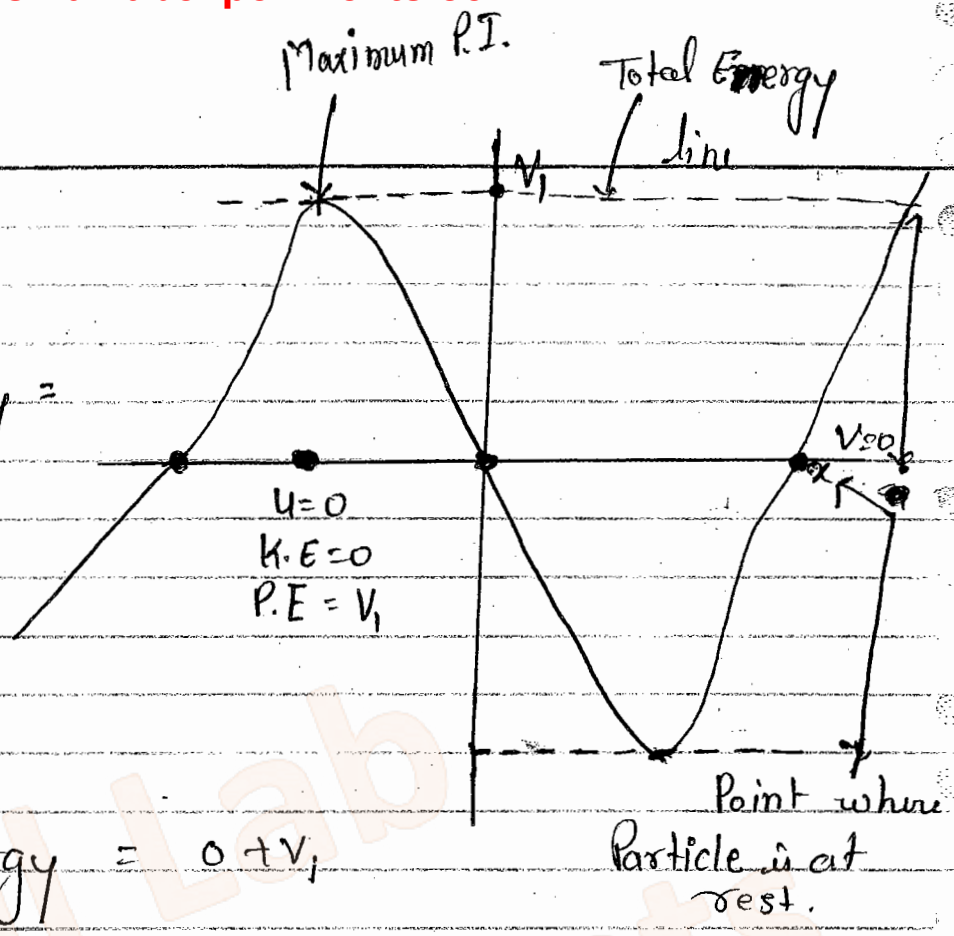
(a) $\sqrt{\frac{16b^3}{3a}}$ (b) $\sqrt{\frac{8b^3}{3a}}$ (c) $\sqrt{\frac{4b^3}{3a}}$ (d) $\sqrt{\frac{8b^3}{3a}}$

Sol^n

$V(x) = ax^3 - bx$

$\because$ Total Energy =

K.E + P.E.



Sol^n

Total Energy $= 0 + V_1$

Total Energy $= V_1$

Maximum Change in K.E. = Max^m change in P.E.

$= V_{max} - V_{min}$

Q.14 A particle of mass 'm' is moving under potential $V(x) = ax^3 - bx$, frequency of oscillation about the stable equilibrium position is -

(a) $\left[\frac{12ab}{m^2}\right]^{1/4}$ (b) $\left[\frac{6ab}{m^2}\right]^{1/4}$ (c) $\left[\frac{9ab}{m^2}\right]^{1/4}$ (d) $\left[\frac{3ab}{m^2}\right]^{1/4}$

Sol^n

$$V(x) = ax^3 - bx \Rightarrow \left.\frac{dV}{dx}\right|_{x=x_0} = 3ax_0^2 - b$$

$$3ax_0^2 - b = 0 \Rightarrow 3ax_0^2 = b \Rightarrow x_0^2 = \frac{b}{3a} \quad \boxed{x_0 = \pm\sqrt{\frac{b}{3a}}}$$

So $\pm\sqrt{\frac{b}{3a}}$ are two equilibrium point.

$$\left.\frac{d^2V}{dx^2}\right|_{x_0=\sqrt{\frac{b}{3a}}} = 6ax_0 = 6a\sqrt{\frac{b}{3a}} > 0$$

So $x_0 = \sqrt{\frac{b}{3a}}$ is minima or stable point.

$$\left.\frac{d^2V}{dx^2}\right|_{x_0=-\sqrt{\frac{b}{3a}}} = 6ax_0 = -6a\sqrt{\frac{b}{3a}} < 0$$

So $x_0 = -\sqrt{\frac{b}{3a}}$ is maxima or unstable point.

$\because$ $K = \left.\frac{d^2V}{dx^2}\right|_{x=x_0}$ where x_0 is stable point

$$\text{So } K = 6a\sqrt{\frac{b}{3a}} = \sqrt{\frac{36a^2b}{3a}} = \sqrt{\frac{36ab}{3}}$$

$$\because \omega = \sqrt{\frac{K}{m}} = \sqrt{\left(\frac{36ab}{3m^2}\right)}^{1/2}$$

$$\omega = \sqrt{\left(\frac{12ab}{m^2}\right)}^{1/2} = \left(\frac{12ab}{m^2}\right)^{1/4}$$

$$\boxed{\omega = \left(\frac{12ab}{m^2}\right)^{1/4}}$$ Ans

.15 In the previous question force on the particle is maximum at –

(a) $x = 0$ (b) $x = \sqrt{\frac{b}{a}}$ (c) $x = \sqrt{\frac{b}{3a}}$ (d) $x = -\sqrt{\frac{b}{3a}}$

Sol^n

$$V(x) = ax^3 - bx$$

$$\vec{F} = -\frac{dV}{dx} = -(3ax^2 - b) = b - 3ax^2$$

for $\vec{F}$ to be max^m or min^m

$$\frac{dF}{dx} = 0$$

$$-6ax = 0$$

$$\because -6a \neq 0$$

$$\therefore \boxed{x = 0}$$

$$\frac{d^2F}{dx^2} = -6a \quad (\text{min}^m)$$

$$\therefore F_{max} = b - 0 = b \quad \text{at } \boxed{x = 0}$$

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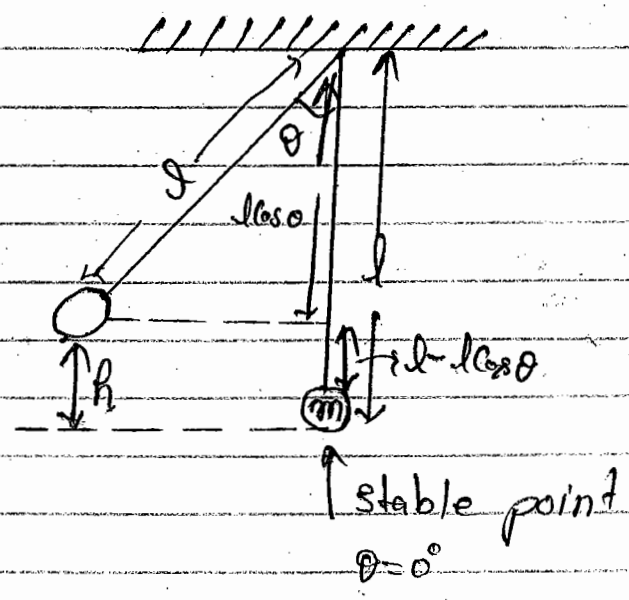
* Time Period of Simple Pendulum :-



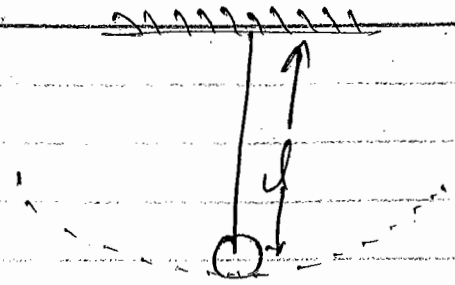
$$P.E = mgh$$

$$\boxed{P.E = mgl(1 - \cos\theta)}$$

$$k = \left.\frac{d^2V}{d\theta^2}\right|_{\theta=\theta_0}$$

$$= mgl\cos\theta \Big|_{\theta=0^\circ}$$

$$\boxed{K = mgl}$$



$$I = \sum_i m_i r_i^2 = ml^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{\cancel{m}g\cancel{l}}{\cancel{m}l^{\cancel{2}}}}$$

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

Note:-

⇒ K in Linear Case:-

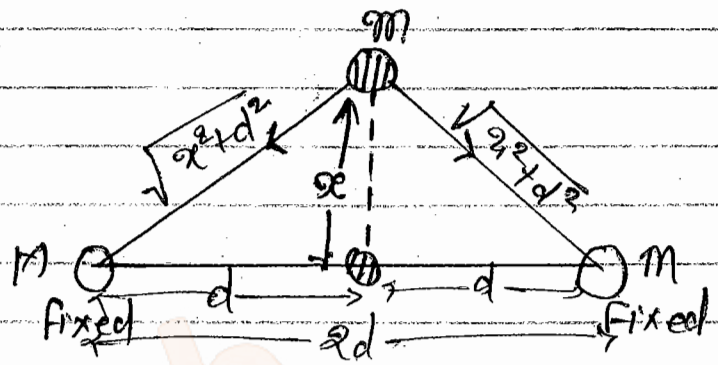
$$K = \left.\frac{d^2V}{dx^2}\right|_{x=0} \qquad \omega = \sqrt{\frac{K}{m}}$$

⇒ K in Angular Case:-

$$K = \left.\frac{d^2U}{d\theta^2}\right|_{\theta=\theta_0} \qquad \omega = \sqrt{\frac{K}{I}}$$

Ques If m is slightly displaced along perpendicular bisector, what is its frequency of oscillation (Use P.E. method.)

Soln



P.E of the system $(V_m) = -\frac{2GMm}{\sqrt{x^2+d^2}} - \frac{GM^2}{2d}$

Stable point is $x=0$

$$K = \left.\frac{d^2V}{dx^2}\right|_{x=0}$$

$$\frac{dV}{dx} = 2GMm \frac{d}{dx}(x^2+d^2)^{-1/2}$$

$$= \frac{2GMm}{(x^2+d^2)^{3/2}}$$

$$\frac{d^2V}{dx^2} = 2GMm \frac{d}{dx}\frac{x}{(x^2+d^2)^{3/2}}$$

$$= 2GMm\left[\frac{1.(x^2+d^2)^{3/2} - x.\frac{3}{2}\times(x^2+d^2).2x}{(x^2+d^2)^3}\right]$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=0} = \frac{2GMm\,d^3}{d^6} = \frac{2GMm}{d^3}$$

$$\omega = \sqrt{\frac{K}{m}}$$

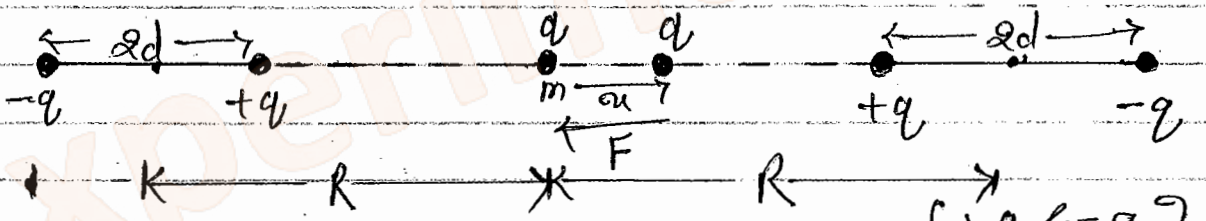
$$\omega = \sqrt{\frac{2GM\cancel{m}}{d^3\cancel{m}}} = \sqrt{\frac{2GM}{d^3}}$$

Angular frequency $\boxed{\omega = \sqrt{\frac{2GM}{d^3}}}$ Ans

CSIR - 2013 (June)

Q. If particle of mass m and charge +q is slightly displaced from given position what will be its frequency of oscillation ($d << R$).

Sol^n



$\{+q \,\&\, -q$ are fixed$\}$

$$V(x) = \frac{1}{4\pi\epsilon_0} Q.q \left[\frac{1}{(R-d+x)} - \frac{1}{(R+d+x)} + \frac{1}{R-d-x} - \frac{1}{R+d-x}\right]$$

$$\frac{d^2V}{dx^2} = \frac{Qq}{4\pi\epsilon_0}\left[\frac{2}{(R-d+x)^3} - \frac{2}{(R+d+x)^3} + \frac{2}{(R-d-x)^3} - \frac{2}{(R+d-x)^3}\right]$$

$\therefore$ $x = 0$ is stable point.

$$\left.\frac{d^2V}{dx^2}\right|_{x=0} = \left[\frac{qQq}{4\pi\epsilon_0}\left\{\frac{1}{(R-d)^3} - \frac{1}{(R+d)^3}\right\}\right]$$

$\because d << R$

So $$K = \frac{qQq}{4\pi\epsilon_0 R^3}\left[\left(1-\frac{d}{R}\right)^{-3} + \left(1+\frac{d}{R}\right)^{-3}\right]$$

$$= \frac{\cancel{qQq}}{\cancel{4\pi\epsilon_0 R^3}}\left[\cancel{1} + \frac{3d}{R} - \cancel{1} + \frac{3d}{R}\right]$$

$$= \frac{6Qqd}{\pi\epsilon_0 R^4}$$

$$\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{6dqQ}{\pi\epsilon_0 R^4}}$$

$$\boxed{\omega = \sqrt{\frac{6dQq}{\pi\epsilon_0 R^4}}}$$ Ans