Elements of Modern Physics
Solved Paper – 2018

1. Answer any *five* of the following: $5 \times 3 = 15$

(a) Under favourable conditions human eye can detect 10^{-18} J of electromagnetic energy. How many 6000 Å photons does this represent?

(b) Define the terms: metastable states, optical pumping and population inversion.

(c) For what kinetic energy will a particle's de Broglie wavelength equal its Compton wavelength?

(d) If the lifetime of a particular excited state in an atom is 1.0×10^{-8} s, use the uncertainty principle to compute the line width of light emitted by the decay of this excited state.

(e) A nitrogen nucleus (mass = 14 × proton mass) emits a photon of energy 6.2 MeV. If the nucleus is initially at rest, what is the recoil energy of the nucleus in eV?

(f) Write the semi-empirical nuclear binding energy formula for a nucleus of mass number A, containing Z protons and N neutrons explaining each term used in the expression.

(g) An electron is confined to a one-dimensional infinite potential well of width L = 1.0 nm. Calculate the energies of the ground state and the first two excited states.

2. (a) Describe Davisson-Germer experiment. Explain how the experiment directly confirms the de Broglie hypothesis of matter wave. 5,2

(b) Show that the de Broglie wave group associated with a moving particle travels with the same velocity as the particle. 3

(c) What are the main features of photoelectric effect? Discuss how Classical Physics fails to explain these. 5

3. (a) In a two-slit experiment using electron, a working monitor that can tell through which slit the electron passes destroys the interference pattern on the screen. Explain the observation, using the uncertainty principle. 5,2

(b) Show, on the basis of uncertainty principle, that electrons cannot reside inside a nucleus. 3

(c) Use the uncertainty principle to estimate the minimum energy of a particle in a simple harmonic potential $\frac{1}{2} kx^2$. 5

4. (a) Discuss the Born probabilistic interpretation of the wave function. Write the conditions required for physical acceptability of wave function. 2,3

(b) Consider a wave function of the form

$$\psi(x) = Ae^{-p|x|}$$

Normalize the wave function. Find the corresponding wave function in momentum space. 4,4

(c) What are the dimensions of $\psi(x)$ and $\psi(p)$? 2

5. (a) A particle of energy $E<V_0$ is incident from left to right on a rectangular potential barrier of height V_0 and width a as defined below:

$$V(x) = 0 \qquad \text{for } x < 0, \text{ region I}$$

$= V_o$ for $0 < x < a$, region II

$= 0$ for $x > a$, region III

Write the Schrödinger equations and its physically acceptable solutions in all the three regions. Using these equations show that the reflection coefficient for the particle is given by:

$$R = \frac{\dfrac{V_o^2}{4E(V_o - E)}\sinh^2 \beta a}{1 + \dfrac{V_o^2}{4E(V_o - E)}\sinh^2 \beta a},$$

where $\beta = \sqrt{\dfrac{2m(V_0 - E)}{\hbar^2}}$ 8

(b) A particle of energy E strikes a potential step V_o. Taking $E > V_o$, distinguish between the classical and quantum behaviour of the particle. Obtain expressions for the reflection coefficient and transmission coefficient. 7

6. (a) Mention the similarities between an atomic nucleus and a liquid drop, on the basis of which the liquid drop model of the nucleus was proposed. Obtain the semi-empirical mass formula of a nucleus. 2,6

(b) Calculate the binding energy per nucleon for (i) Oxygen-16 (atomic mass = 15.99492 a.m.u.) and (ii) Silicon-29 (atomic mass = 28.97650 a.m.u.). Given that mass of proton = 1.00728 a.m.u. and mass of neutron = 1.00867 a.m.u. 4

(c) Estimate the mass of 1 mm^3 of nuclear matter of ^{227}Th nucleus. 3

7. (a) Radioactive material A (decay constant λ_a) decays into a material B (decay constant λ_b) which in turn decays into a stable substance C. Assuming that a sample contains only N_{a0} nuclei of material A at time $t = 0$, determine :

   i. the number of B nuclei remaining after a time t,

ii. the time at which the number of B nuclei is a maximum, and

iii. the number of C nuclei remaining after a time t. 6

(b) ^{226}Ra has a half-life of 1600 years. What is the activity of a sample of one gram of pure ^{226}Ra? What would be the activity of this sample at the end of 400 years? 4,2

(c) How do we explain the emission of Beta-particles from radioactive nuclei even though they are not contained in them? What kind of observations on the energy spectrum of Beta-rays led 'Pauli' to propose the neutrino hypothesis in 1930? 3

1(a) Ans:

$E_{detect} := 10^{-18}$ $\qquad E_{photon}(\lambda) := \frac{h \cdot c}{\lambda}$ $\qquad N(\lambda) := \frac{E_{detect}}{E_{photon}(\lambda)}$

$N(600 \cdot 10^{-9}) = 3.017$

1(b) Ans:

Metastable states: It is excited state of an atom, nucleus, or other system that has a longer lifetime than the ordinary excited states and that generally has a shorter lifetime than the lowest, often stable, energy state, called the ground state. A metastable state may thus be considered a kind of temporary energy trap or a somewhat stable intermediate stage of a system the energy of which may be lost in discrete amounts. In quantum mechanical terms, transitions from metastable states are “forbidden” and are much less probable than the “allowed” transitions from other excited states.

Optical pumping: It is a process in which light is used to raise (or "pump") electrons from a lower energy level in an atom or molecule to a higher one. It is commonly used in laser construction, to pump the active laser medium so as to achieve population inversion.

Population inversion: A population inversion occurs while a system exists in a state in which more members of the system are in higher, excited states than in lower, unexcited energy states. The concept is of fundamental importance in laser science because the production of a population inversion is a necessary step in the workings of a standard laser.

1(c)Ans:

(c) $\frac{1}{2}mv^2 = \frac{p^2}{2m}$ is the Kinetic energy of a particle of mass m

By de Broglie's hypothesis

$$p = \frac{h}{\lambda} \Rightarrow \lambda = \frac{h}{p} \quad -① $$

Compton wavelength of the particle:

$$\lambda_c = \frac{h}{mc} \quad -② $$

∴ Equating ① and ② according to the question

$$\frac{h}{p} = \frac{h}{mc}$$

$$p = mc$$

$$\therefore KE = \frac{p^2}{2m} = \frac{m^2c^2}{2m} = \frac{mc^2}{2}$$

1(d)Ans:

(d) Given, $\Delta t = 10^{-8} s$

By uncertainty principle,

$$\Delta E . \Delta t \geq \frac{\hbar}{2}$$

$$\Delta E \geq \frac{\hbar}{2\Delta t} = \frac{1.054 \times 10^{-34}}{2 \times 10^{-8}}$$

$$\Delta E \geq 0.527 \times 10^{-26}$$

$$\Delta E \geq 0.33 \times 10^{-7} eV$$

1(e)Ans: Refer your textbook

1(f)Ans:

$$E_b(MeV) = a_V A - a_S A^{\frac{2}{3}} - a_C \frac{Z^2}{A^{\frac{1}{3}}} - a_A \frac{(A-2Z)^2}{A} \pm \delta(A,Z)$$

$$\delta(A,Z) = \begin{matrix} +\delta_0 \; for \; Z, N \; even \\ 0 \\ -\delta_0 \; for \; Z, N \; odd \end{matrix}$$

1(g)Ans:

(g) :-

$$\epsilon_n = \frac{n^2 \hbar^2 \pi^2}{2ma^2} \quad (n = 1, 2, 3 \ldots)$$

ground state energy, $\epsilon_0 = \frac{\hbar^2 \pi^2}{2ma^2}$

$$\Rightarrow \epsilon_0 = \frac{\hbar^2 \pi^2}{2mL^2}$$

$$\approx \frac{10 \times 10^{-68} J}{2 \times 10^{-30} \times 10^{-18}} \approx 5 \times 10^{-18} J$$

I[st] excited state ; $\epsilon_1 = \frac{2^2 \hbar^2 \pi^2}{2ma^2}$

$$\epsilon_1 = 4\epsilon_0$$

2[nd] excited state, $\epsilon_2 = \frac{3^2 \hbar^2 \pi^2}{2ma^2}$

$$\epsilon_2 = 9\epsilon_0$$

2(a) Ans

The experimental arrangement is shown in Figure 4.2. A narrow beam of electrons, accelerated through a potential difference V, was directed normally towards the surface of a nickel crystal. The electrons were scattered in all directions by the atoms in the crystal. The intensity of the scattered electrons was measured as a function of the latitude angle ϕ measured from the axis of the incident beam for different accelerating potentials. Figure 4.3 shows the polar graph of the variation of the intensity with ϕ for $V = 54$ volts. At each angle,

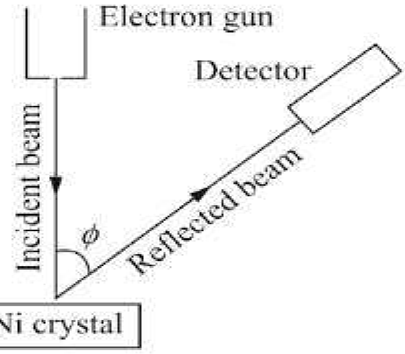

Figure 4.2 Schematic diagram of the Davisson-Germer experiment.

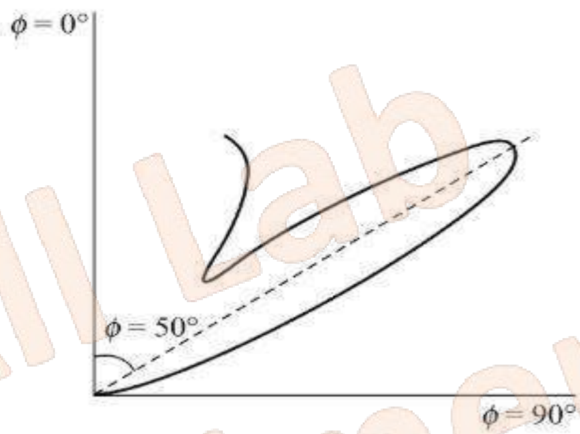

Figure 4.3 Polar plot of the intensity as a function of the scattering angle for 54 eV electrons.

the intensity is given by the distance of the point from the origin.

The Bragg condition for constructive interference is

$$n\lambda = 2d \sin\theta$$

where d is the spacing between the adjacent Bragg planes and n is an integer. The angle θ is shown in the figure. We have

$$\theta + \phi + \theta = 180°$$

or

$$\theta = \frac{180° - \phi}{2}$$

$$= 90° - (\phi/2)$$

From geometry,

$$d = D \sin \frac{\phi}{2}$$

where D is the interatomic spacing. Therefore,

$$n\lambda = 2D \sin \frac{\phi}{2} \sin \left(90° - \frac{\phi}{2}\right)$$

$$= 2D \sin \frac{\phi}{2} \cos \frac{\phi}{2}$$

$$= D \sin \phi$$

For nickel $D = 2.15$ Å. Assuming that the peak at $\phi = 50°$ corresponds to first order diffraction, we take $n = 1$. Therefore,

$$\lambda = 2.15 \times \sin 50°$$
$$= 1.65 \text{ Å}$$

Now, according to de Broglie's hypothesis, we have for electrons accelerated though a potential difference V (Equation 4.7),

$$\lambda = \frac{12.3}{\sqrt{V}} \text{ Å}$$

$$= \frac{12.3}{\sqrt{54}} = 1.66 \text{ Å}$$

The agreement between the two values is remarkably close.

2(b) Ans

The phase velocity of a wave associated with a particle comes out to be greater than the velocity of light which is impossible. This problem can be removed by considering *a moving particle is associated with a wave group or wave packet rather than a single wave.*

Suppose a particle of rest mass m_0 is moving with velocity v. The angular frequency and propagation constant is given by

$$\omega = 2\pi \text{v} = \frac{2\pi E}{h}$$

$$\omega = h \frac{2\pi m c^2}{h} \qquad \text{.......(1)} \; (\because E = m_0 c^2)$$

$$k = \frac{2\pi}{\lambda}$$

$$k = \frac{2\pi m \text{v}}{h} \qquad \text{.......(2)} \left(\lambda = \frac{h}{m \text{v}}\right)$$

From theory of relativity $m = \dfrac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$ ....(3)

Putting the value of *m* in (1) and (2) we get

$$\omega = \frac{2\pi m_0 c^2}{h\sqrt{1 - \frac{v^2}{c^2}}} \quad ....(4)$$

$$k = \frac{2\pi m_0 v}{h\sqrt{1 - \frac{v^2}{c^2}}} \quad ....(5)$$

Differentiating (4) and (5) we get

$$\frac{d\omega}{dv} = \frac{2\pi m_0 v}{h\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \quad ....(6)$$

$$\frac{dk}{dv} = \frac{2\pi m_0}{h\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \quad ....(7)$$

But group velocity is given by

$$v_g = \frac{d\omega}{dk}$$

Dividing equation (6) by (7) we get

$$\frac{d\omega}{dk} = \frac{d\omega}{dv} \Big/ \frac{dk}{dv}$$

$$v_g = v$$

Hence, wave packet associated with a moving particle travels with the same velocity as the particle.

2(c) Ans:

The following interesting results were obtained in the study:

(1) No electrons are emitted if the incident radiation has a frequency less than a *threshold* value ν_0. The value of ν_0 varies from metal to metal.

(2) The kinetic energy of the emitted electrons varies from zero to a maximum value. The maximum value of energy depends on the frequency and not on the intensity of radiation. It varies linearly with the frequency.

(3) The number of photoelectrons emitted per second, or the photoelectric current, is proportional to the intensity of radiation but is independent of the frequency.

(4) The photoelectric emission is an instantaneous process, *i.e.*, there is negligible time lag between the incidence of radiation and the emission of electrons, regardless of how low the intensity of radiation is.

Failure of classical physics:

These results, except number three, cannot be explained if we consider radiation to be wave-like, obeying classical electromagnetic theory. Classically, the maximum energy of the emitted electrons should increase with the intensity of incident radiation. The frequency of radiation has nothing to do with it. The reason is that the force exerted on the electrons in the metal should be proportional to the magnitude of the electric field $\boldsymbol{E}$ of the incident wave, and the magnitude of $\boldsymbol{E}$ increases when the intensity of the radiation is increased. Contrary to this, it is observed that the energy of the photoelectrons is independent of the intensity of light but depends on the frequency. Further, classically, electromagnetic energy is absorbed by the electron gradually and the electron can be ejected only when this energy becomes more than the *work function*† of the metal. Therefore, there may be a time lag between the onset of the radiation and the emission of the electron. The lag will be longer when the intensity of radiation is decreased. No such time lags have ever been observed, even with radiation of very low intensity. All observed time lags have been less than or equal to 10^{-9} seconds.

3(a) Ans: refer your textbook

3(b) Ans: According to the Heisenberg's uncertainty principle,

$$\Delta x \Delta p \geq \frac{h}{4\pi}$$

Where m is the mass of electron,

Substituting, p=mv, we get,

$$\Delta x \Delta v \geq \frac{h}{4\pi m}$$

Now, if we consider the radius of the atomic nucleus to be 10^{-15}m and mass of electron, 'm' to be $9.1*10^{-31}$ kg, we get

$$10^{-15}\Delta v \geq \frac{6.626*10^{-34}}{4*3.14*9.1*10^{-31}}$$

$$\Delta v \geq \frac{6.626*10^{-34}}{4*3.14*9.1*10^{-31}*10^{-15}}$$

$$\Delta v \geq 5.79*10^{10} \text{ m/s}$$

Calulating Δv, we get a value of
$5.79*10^{10}$ m/s which contradicts the theory of relativity. What this means is that if an electron exists in the nucleus, it has to travel with a speed of $5.79*10^{10}$ m/s. An object can only travel faster than light if it has no mass but electrons do have mass, hence they can't travel faster than the speed of light which is precisely 299,792,458 m/s.
Hence, an electron can't exist in the nucleus.

3(c) Ans: The energy of the quantum harmonic oscillator must be at least

$$E = \frac{(\Delta p)^2}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2 \qquad \Delta x = \text{position uncertainty} \qquad \Delta p = \text{momentum uncertainty}$$

Taking the lower limit from the uncertainty principle

$$\Delta x \Delta p = \frac{\hbar}{2}$$

Then the energy expressed in terms of the position uncertainty can be written

$$E = \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x)^2$$

Minimizing this energy by taking the derivative with respect to the position uncertainty and setting it equal to zero gives

$$-\frac{\hbar^2}{4m(\Delta x)^3} + m\omega^2 \Delta x = 0$$

Solving for the position uncertainty gives

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$$

Substituting gives the minimum value of energy allowed.

$$E_0 = \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x)^2 = \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4} = \frac{\hbar\omega}{2}$$

4(a) Ans:

If there is a wave associated with a particle, then there must be a function to represent it. This function is called wave function.

Wave function is defined as that quantity whose variations make up matter waves. It is represented by Greek symbol ψ(psi), ψ consists of real and imaginary parts.

Ψ=A+iB

PHYSICAL SIGNIFICANCE OF WAVE FUNCTIONS (BORN'S INTERPRETATION):

Born's interpretation

The wave function ψ itself has no physical significance but the square of its absolute magnitude $|\psi^2|$ has significance when evaluated at a particular point and at a particular time $|\psi^2|$ gives the probability of finding the particle there at that time.

The wave function ψ(x,t) is a quantity such that the product

$P(x,t)=\psi^*(x,t)\psi(x,t)$

Is the probability per unit length of finding the particle at the position x at time t.

P(x,t) is the probability density and $\psi^*(x,t)$ is complex conjugate of ψ(x,t)

Hence the probability of finding the particle is large wherever ψ is large and vice-versa.

Condiitions required for the acceptability of a wave function are as follows:

NORMALIZATION CONDITION

The probability per unit length of finding the particle at position x at time t is

$P=\psi^*(x,t)\psi(x,t)$

So, probability of finding the particle in the length dx is

$Pdx=\psi^*(x,t)\psi(x,t)dx$

Total probability of finding the particle somewhere along x-axis is

$\int pdx =\int \psi^*(x,t)\psi(x,t)dx$

If the particle exists , it must be somewhere on the x-axis . so the total probability of finding the particle must be unity i.e.

$$\int\psi^*(x,t)\psi(x,t)dx=1 \qquad (1)$$

This is called the normalization condition . So a wave function $\psi(x,t)$ is said to be normalized if it satisfies the condition(1)

ORTHOGONAL WAVE FUNCTIONS

Consider two different wave functions ψ_m and ψ_n such that both satisfies Schrodinger equation.These two wave functions are said to be orthogonal if they satisfy the conditions.

Or $\int \psi_n^* (x,t)\, \psi_m(x,t)\, dV=0$ for $n\neq m$] (1)

$\int \psi_n^* (x,t)\, \psi_m(x,t)\, dV=0$ for $m\neq n$]

If both the wave functions are simultaneously normal then

$\int \psi_m\, \psi_m^*\, d\,V=1=\int \psi_n \psi_n^*\, dV$ (2)

Orthonormal wave functions:

The sets of wave functions, which are both normalized as well as orthogonal are called orthonormal wave functions.

Equations (16) and (17) are collectively written as

$\int\psi^*_m\psi_n dV=\{$ 0 if $m\neq n$

$=[1$ if $m=n$

Ans 4b: refer Ans 4a

Ans 5a

Consider a beam of particles of mass m that are sent from the left on a potential barrier

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & 0 \leq x \leq a, \\ 0, & x > a. \end{cases}$$

Classically, we would expect total reflection: every particle that arrives at the barrier ($x = 0$) will be reflected back; no particle can penetrate the barrier, where it would have a negative kinetic energy.

We are now going to show that the quantum mechanical predictions differ sharply from their classical counterparts, for the wave function is not zero beyond the barrier. The solutions of the Schrödinger equation in the three regions yield expressions that are similar to (4.36) except that $\psi_2(x) = Ce^{ik_2x} + De^{-ik_2x}$ should be replaced with $\psi_2(x) = Ce^{k_2x} + De^{-k_2x}$:

$$\psi(x) = \begin{cases} \psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x}, & x \leq 0, \\ \psi_2(x) = Ce^{k_2x} + De^{-k_2x}, & 0 < x < a, \\ \psi_3(x) = Ee^{ik_1x}, & x \geq a, \end{cases} \tag{4.48}$$

where $k_1^2 = 2mE/\hbar^2$ and $k_2^2 = 2m(V_0 - E)/\hbar^2$. The behavior of the probability density corresponding to this wave function is expected, as displayed in Figure 4.3, to be oscillatory in the regions $x < 0$ and $x > a$, and exponentially decaying for $0 \leq x \leq a$.

To find the reflection and transmission coefficients,

$$R = \frac{|B|^2}{|A|^2}, \qquad T = \frac{|E|^2}{|A|^2}, \tag{4.49}$$

we need only to calculate B and E in terms of A. The continuity conditions of the wave function and its derivative at $x = 0$ and $x = a$ yield

$$A + B = C + D, \tag{4.50}$$

$$ik_1(A - B) = k_2(C - D), \tag{4.51}$$

$$Ce^{k_2a} + De^{-k_2a} = Ee^{ik_1a}, \tag{4.52}$$

$$k_2\left(Ce^{k_2a} - De^{-k_2a}\right) = ik_1Ee^{ik_1a}. \tag{4.53}$$

The last two equations lead to the following expressions for C and D:

$$C = \frac{E}{2}\left(1 + i\frac{k_1}{k_2}\right)e^{(ik_1-k_2)a}, \qquad D = \frac{E}{2}\left(1 - i\frac{k_1}{k_2}\right)e^{(ik_1+k_2)a}. \tag{4.54}$$

Inserting these two expressions into the two equations (4.50) and (4.51) and dividing by A, we can show that these two equations reduce, respectively, to

$$1+\frac{B}{A}=\frac{E}{A}e^{ik_1a}\left[\cosh(k_2a)-i\frac{k_1}{k_2}\sinh(k_2a)\right], \tag{4.55}$$

$$1-\frac{B}{A}=\frac{E}{A}e^{ik_1a}\left[\cosh(k_2a)+i\frac{k_2}{k_1}\sinh(k_2a)\right]. \tag{4.56}$$

Solving these two equations for B/A and E/A, we obtain

$$\frac{B}{A}=-i\frac{k_1^2+k_2^2}{k_1k_2}\sinh(k_2a)\left[2\cosh(k_2a)+i\frac{k_2^2-k_1^2}{k_1k_2}\sinh(k_2a)\right]^{-1}, \tag{4.57}$$

$$\frac{E}{A}=2e^{-ik_1a}\left[2\cosh(k_2a)+i\frac{k_2^2-k_1^2}{k_1k_2}\sinh(k_2a)\right]^{-1}. \tag{4.58}$$

Thus, the coefficients R and T become

$$R=\left(\frac{k_1^2+k_2^2}{k_1k_2}\right)^2\sinh^2(k_2a)\left[4\cosh^2(k_2a)+\left(\frac{k_2^2-k_1^2}{k_1k_2}\right)^2\sinh^2(k_2a)\right]^{-1}, \tag{4.59}$$

$$T=\frac{|E|^2}{|A|^2}=4\left[4\cosh^2(k_2a)+\left(\frac{k_2^2-k_1^2}{k_1k_2}\right)^2\sinh^2(k_2a)\right]^{-1}. \tag{4.60}$$

We can rewrite R in terms of T as

$$R=\frac{1}{4}T\left(\frac{k_1^2+k_2^2}{k_1k_2}\right)^2\sinh^2(k_2a). \tag{4.61}$$

Since $\cosh^2(k_2a)=1+\sinh^2(k_2a)$ we can reduce (4.60) to

$$T=\left[1+\frac{1}{4}\left(\frac{k_1^2+k_2^2}{k_1k_2}\right)^2\sinh^2(k_2a)\right]^{-1}. \tag{4.62}$$

Now since

$$\left(\frac{k_1^2+k_2^2}{k_1k_2}\right)^2=\left(\frac{V_0}{\sqrt{E(V_0-E)}}\right)^2=\frac{V_0^2}{E(V_0-E)}, \tag{4.63}$$

we can rewrite (4.61) and (4.62) as follows:

$$R = \frac{1}{4}\frac{V_0^2 T}{E(V_0 - E)} \sinh^2\left(\frac{a}{\hbar}\sqrt{2m(V_0 - E)}\right), \tag{4.64}$$

$$T = \left[1 + \frac{1}{4}\frac{V_0^2}{E(V_0 - E)} \sinh^2\left(\frac{a}{\hbar}\sqrt{2m(V_0 - E)}\right)\right]^{-1}, \tag{4.65}$$

or

$$R = \frac{T}{4\varepsilon(1-\varepsilon)} \sinh^2\left(\lambda\sqrt{1-\varepsilon}\right), \tag{4.66}$$

$$T = \left[1 + \frac{1}{4\varepsilon(1-\varepsilon)} \sinh^2\left(\lambda\sqrt{1-\varepsilon}\right)\right]^{-1}, \tag{4.67}$$

where $\lambda = a\sqrt{2mV_0/\hbar^2}$ and $\varepsilon = E/V_0$.

Ans 5b

It is easy to infer the quantum mechanical study from the treatment of the potential step presented in the previous section. We need only to mention that the wave function will display an oscillatory pattern in all three regions; its amplitude reduces every time the particle enters a new region (see Figure 4.3):

$$\psi(x) = \begin{cases} \psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x}, & x \le 0, \\ \psi_2(x) = Ce^{ik_2x} + De^{-ik_2x}, & 0 < x < a, \\ \psi_3(x) = Ee^{ik_1x}, & x \ge a, \end{cases} \tag{4.36}$$

where $k_1 = \sqrt{2mE/\hbar^2}$ and $k_2 = \sqrt{2m(E - V_0)/\hbar^2}$. The constants B, C, D, and E can be obtained in terms of A from the boundary conditions: $\psi(x)$ and $d\psi/dx$ must be continuous at $x = 0$ and $x = a$, respectively:

$$\psi_1(0) = \psi_2(0), \qquad \frac{d\psi_1(0)}{dx} = \frac{d\psi_2(0)}{dx}, \tag{4.37}$$

$$\psi_2(a) = \psi_3(a), \qquad \frac{d\psi_2(a)}{dx} = \frac{d\psi_3(a)}{dx}. \tag{4.38}$$

These equations yield

$$A + B = C + D, \qquad ik_1(A - B) = ik_2(C - D), \tag{4.39}$$

$$Ce^{ik_2a} + De^{-ik_2a} = Ee^{ik_1a}, \qquad ik_2\left(Ce^{ik_2a} - De^{-ik_2a}\right) = ik_1Ee^{ik_1a}. \tag{4.40}$$

Solving for E, we obtain

$$\begin{aligned} E &= 4k_1k_2Ae^{-ik_1a}[(k_1+k_2)^2\,e^{-ik_2a} - (k_1-k_2)^2\,e^{ik_2a}]^{-1} \\ &= 4k_1k_2Ae^{-ik_1a}\left[4k_1k_2\cos(k_2a) - 2i\left(k_1^2+k_2^2\right)\sin(k_2a)\right]^{-1}. \end{aligned} \tag{4.41}$$

The transmission coefficient is thus given by

$$\begin{aligned} T &= \frac{k_1|E|^2}{k_1|A|^2} = \left[1 + \frac{1}{4}\left(\frac{k_1^2-k_2^2}{k_1k_2}\right)^2 \sin^2(k_2a)\right]^{-1} \\ &= \left[1 + \frac{V_0^2}{4E(E-V_0)}\sin^2\left(a\sqrt{2mV_0/\hbar^2}\sqrt{E/V_0-1}\right)\right]^{-1}, \end{aligned} \tag{4.42}$$

because

$$\left(\frac{k_1^2-k_2^2}{k_1k_2}\right)^2 = \frac{V_0^2}{E(E-V_0)}. \tag{4.43}$$

Using the notation $\lambda = a\sqrt{2mV_0/\hbar^2}$ and $\varepsilon = E/V_0$, we can rewrite T as

$$T = \left[1 + \frac{1}{4\varepsilon(\varepsilon-1)}\sin^2(\lambda\sqrt{\varepsilon-1})\right]^{-1}. \tag{4.44}$$

Similarly, we can show that

$$R = \frac{\sin^2(\lambda\sqrt{\varepsilon-1})}{4\varepsilon(\varepsilon-1)+\sin^2(\lambda\sqrt{\varepsilon-1})} = \left[1 + \frac{4\varepsilon(\varepsilon-1)}{\sin^2(\lambda\sqrt{\varepsilon-1})}\right]^{-1}. \tag{4.45}$$

Ans 6a: Refer your book
Ans 6b: use the formula:

$$BE = (Zm_p)c^2 + (Nm_n)c^2 - M_{nuc}c^2$$

Binding energy per nucleon= BE/A
Ans 6c: Do it yourself

Ans 7a:

7a) $A \longrightarrow B \longrightarrow C.$

(i)

Rate of increase of b nuclei :

no. of b produced by a − no. of b nuclei decaying

For every a nucleus that decays one b nucleus is formed, so that b nuclei are formed at the rate of

$$-\frac{dN_a}{dt} = \lambda_a N_a$$

$$\frac{dN_b}{dt} = \lambda_a N_a - \lambda_b N_b$$

$$= \lambda_a N_{a0} e^{-\lambda_a t} - \lambda_b N_b.$$

$$\frac{dN_b}{dt} + \lambda_b N_b = \lambda_a N_{a0} e^{-\lambda_a t}.$$

This is first order differential equation.

$\therefore \quad N_b = c e^{-\lambda_b t}$.

where c is a constant.

Particular solution is

$$N_b = D e^{-\lambda_a t}$$

$$(-\lambda_a + \lambda_b) D e^{-\lambda_a t} = \lambda_a N_{a0} e^{-\lambda_a t}$$

$$D = \frac{\lambda_a N_{a0}}{\lambda_b - \lambda_a}$$

$\therefore$ complete sol^n is:

$$N_b = C e^{-\lambda_b t} + \frac{\lambda_a N_{a0}}{\lambda_b - \lambda_a} e^{-\lambda_a t}.$$

for constant c, $N_b = N_{b0}$ $t = 0$

$$N_{b0} = c + \frac{\lambda_a N_{a0}}{\lambda_b - \lambda_a}$$

$$N_b = N_{b0} e^{-\lambda_b t} + \frac{\lambda_a N_{a0}}{\lambda_b - \lambda_a} \left[e^{-\lambda_a t} - e^{-\lambda_b t} \right]$$

(ii)

With $N_{bo} = 0$, we ~~kn~~ have

$$N_b = \frac{\lambda_a N_{ao}}{\lambda_b - \lambda_a}\left[e^{-\lambda_a t} - e^{-\lambda_b t}\right]$$

for maximum,

$$\frac{\partial N_b}{\partial t} = \frac{\lambda_a N_{ao}}{\lambda_b - \lambda_a}\left[-\lambda_a e^{-\lambda_a t} + \lambda_b e^{-\lambda_b t}\right] = 0$$

$$\Rightarrow \quad t = \frac{1}{\lambda_a - \lambda_b} \ln\left(\frac{\lambda_a}{\lambda_b}\right)$$

(iii) The total No. of nuclei present at any time will be:

$$N_c = N_{ao} - N_a - N_b$$

$$= N_{ao} - N_{ao} e^{-\lambda_a t} - \frac{\lambda_a N_{ao}}{\lambda_b - \lambda_a}\left[e^{-\lambda_a t} - e^{-\lambda_b t}\right]$$

$$= N_{ao}\left[1 - \frac{\lambda_b}{\lambda_b - \lambda_a} e^{-\lambda_a t} + \frac{\lambda_a}{\lambda_b - \lambda_a} e^{-\lambda_b t}\right]$$

Ans 7b:

7 b) The number of atoms in 1 gm of Ra:

$$N = (1g)\left(\frac{1\,g\,mole}{226\,g}\right)\left(6.025 \times 10^{23}\,\frac{atoms}{g\,mole}\right)$$

$$= 2.666 \times 10^{21}$$

Decay constant :

$$\lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{1600 \text{ years}} = 1.373 \times 10^{-11} \text{ sec}^{-1}$$

$$\text{Activity}, A = \lambda N = 1.373 \times 10^{-11} \times 2.666 \times 10^{21}$$

$$= 3.66 \times 10^{10} \text{ disintegrations/sec.}$$

$$\text{Activity}, A(t) = A_0 e^{-\lambda t}.$$

~~$= 3.66 \times 10^{10} e^{-1.373 \times 10^{-11}}$~~

$$= 3.66 \times 10^{10} e^{-\frac{0.693}{1600} \times 400}$$

$$= 3.66 \times 10^{10} e^{-0.173}$$

$$= 3.077 \times 10^{10} \text{ disintegrations/sec}$$

Ans 7c: Beta particles (not a ray) can be ejected from a nucleus because they are from the decay of a neutron into a proton. The beta particle itself is not an electron but a positron, in other words an antimatter electron.

check full explanation in the study material