**Mathematical physics – III**
**B.Sc. (Hons.) Physics**
**Solved Paper – 2018**

1. Attempt any **five** questions: (3x5 =15)

(a) Evaluate $(-27i)^{\frac{1}{3}}$

(b) Locate and name the singularities in the finite $z$ plane of the function

$$f(z) = \frac{ln(z+3i)}{z^2}$$

(c) Evaluate $\oint_C \frac{z^2-z+1}{z-2} dz$ over a circle C in the positive sense. C is described by $|z| = \frac{1}{2}$.

(d) Test the analyticity of the function $f(z) = z^2$.

(e) Show that $\delta(ax) = \frac{\delta(x)}{|a|}$ where $\delta(x)$ is the Dirac Delta function and $a$ is a constant.

(f) If $F(\omega)$ represents the Fourier transform of $f(t)$, then prove that the Fourier transform of $f(t)\cos at = \frac{1}{2}[F(\omega - a) + F(\omega + a)]$.

(g) Evaluate the Laplace transform of $f(t) = cos^2 2t$

(h) Determine the inverse Laplace transform of:

$$F(s) = \left\{\frac{e^{-2s}}{s^3}\right\}$$

## SECTION A

*Attempt any **two** questions from this Section.*

2 (a) Given a function $v(x,y) = e^x \sin y$. Find the function $u(x,y)$ such that $f(z) = u + i\,v$ is analytic. Express $f(z)$ in terms of $z$.

(b) Prove that

$$1 + \cos 72° + \cos 144° + \cos 216° + \cos 288° = 0$$

using complex analysis. (10, 5)

3. (a) Expand $f(z) = \sin z$ in a Taylor series about $z = \frac{\pi}{4}$ and determine the region of convergence of this series.

(b) Find the value of the integral $\oint_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3}\, dz$ over a circle C (in the positive sense) represented by $|z| = 1$.

(10, 5)

4. Using the method of contour integration prove any *two* of the following:

(a) $\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{4}$

(b) $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = \frac{\pi}{6}$

(c) $\int_0^\infty \frac{\cos mx}{x^2+1}\, dx = \frac{\pi}{2} e^{-m}, m > 0$ $\left(7\frac{1}{2}, 7\frac{1}{2}\right)$

## SECTION B

*Attempt **any two** questions from this Section.*

5 (a) Prove that the Fourier transform of a Gaussian function $\left(e^{-x^2}\right)$ is also a Gaussian function.

(b) Solve the one dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

for $t > 0$, $-\infty < x < \infty$.

$u(x,0) = f(x)$; $u_t(x,0) = 0$; where $u_t = \frac{\partial u}{\partial t}$ and $v$ is the velocity of the wave. (5, 10)

6 (a) Verify the convolution theorem ( Fourier transform) for

P. T. O.

$$f(t) = g(t) = \begin{cases} 1, & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$$

(b) Given that $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{t\sin t}{2}$, determine $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$, where the symbol $\mathcal{L}^{-1}$ represents the inverse Laplace transform operator. (10, 5)

7. (a) A semi-infinite rod ($x > 0$) is initially at temperature zero. At time $t = 0$, a constant temperature $T_o > 0$ is applied and maintained at the face $x = 0$. Using Laplace transform, find the temperature at any point of the rod at any later time $t > 0$.

$$\left[Given, \quad L^{-1}\left(\frac{e^{-x\sqrt{s/k}}}{s}\right) = erfc\left(\frac{x}{2\sqrt{kt}}\right)\right]$$

(b) Using Laplace transform, prove that,

$$\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\, dt = \ln\frac{2}{3}$$

**Ans 1(a)**

a) Let $f = (-27i)^{1/3}$

$\because \; i^3 = -i$

$f = (27)^{1/3} (-i)^{1/3}$

$= (3^3)^{1/3} (i^3)^{1/3}$

$= 3i$

$\therefore \; \boxed{\cancel{\text{Ans}}\; f = 3i}$

**Ans 1(b)**

1(b) $f(z) = \dfrac{\ln(z+3i)}{z^2}$.

$z = -3i$ is a branch point and is a non-isolated singularity. Also ~~since~~ $z=0$ is a pole of order 2 is isolated singularity.

**Ans 1(c)**

(c) $I = \oint \dfrac{z^2 - z + 1}{z-2}\, dz$.

$z - 2 = 0 \Rightarrow z = 2$ is a pole of order 1.

$z = 2$ lies outside $|z| = \frac{1}{2}$

$\therefore \; I = 0$.

**Ans 1(d)**

$$f(z) = z^2$$

$$z = x + iy$$

$$f(z) = (x+iy)^2 = x^2 - y^2 + 2ixy$$

$$= u + iv$$

$$u = x^2 - y^2 \qquad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2x$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad —(1)$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y$$

$$\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \quad —(2)$$

from (1) and (2), given function is analytic.

**Ans 1(e)**

(v) $\delta(ax) = \dfrac{\delta(x)}{|a|}$

Multiplying both the sides by $dx$ and integrating, we obtain

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \int_{-\infty}^{\infty} \frac{1}{|a|}\delta(x)\, dx$$

Consider $ax = y$. Thus $dx = \dfrac{dy}{a}$;

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \int_{-\infty}^{\infty} \delta(y)\, \frac{dy}{a}$$

For $a < 0$ the integral remains unaltered because $dy = -dx/a$ and the limits get interchanged so the net result remains unaffected and therefore

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(y)\, dy = \frac{1}{|a|}$$

Performing the same operation on the RHS of multiplying by $dx$ and integrating over $x$ results in $\dfrac{1}{|a|}\displaystyle\int_{-\infty}^{\infty} \delta(x)\, dx = \dfrac{1}{|a|}$

**Ans 1(f)**

**Solution.** Using linearity and shifting property of Fourier transform, we have

$$\begin{aligned}
F\{f(t)\cos at\} &= F\left\{f(t)\frac{e^{iat}+e^{-iat}}{2}\right\} \\
&= \frac{1}{2}F\{f(t)e^{iat}\} \\
&\quad + \frac{1}{2}F\{f(t)\, e^{-iat}\} \\
&= \frac{1}{2}F(\omega - a) + \frac{1}{2}F(\omega + a) \\
&= \frac{1}{2}[F(\omega - a) + F(\omega + a)].
\end{aligned}$$

**Ans 1(g)**

(g) $f(t) = \cos^2 at$.

let $g(t) = \cos^2 t$.

$\cos^2 t = \frac{1}{2}[\cos 2t + 1]$

$$L(\cos^2 t) = L\left[\frac{1}{2}(\cos 2t + 1)\right] = \frac{1}{2}[L(\cos 2t) + L(1)]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+(2)^2} + \frac{1}{s}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+4} + \frac{1}{s}\right] = \frac{s^2+2}{s(s^2+4)}$$

Now, for $f(t)$, By change of scale property, we have,

$$L\cos^2(at) = \frac{1}{a}\left[\frac{\left(\frac{s}{a}\right)^2 + 2}{\frac{s}{a}\left[\left(\frac{s}{a}\right)^2 + 4\right]}\right] = \frac{s^2 + 2a^2}{s(s^2+4a^2)}$$

Putting $a = 2$

$$L(\cos^2 2t) = \frac{s^2+8}{s(s^2+16)}$$

**Ans 1(h)** : do it yourself

**Ans 2(a)**

2) a. $v(x,y) = e^x \sin y$.

For analytic function,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial y} = e^x \cos y. = \frac{\partial u}{\partial x}$$

$$\Rightarrow \quad \partial u = e^x \cos y \, \partial x.$$

$$u = e^x \cos y.$$

$$\therefore \quad z = u + iv$$

$$= e^x \cos y + i e^x \sin y$$

$$= e^x (\cos y + i \sin y)$$

$$= e^x e^{iy} = e^{x+iy} = \underline{\underline{e^z}}. \text{ Ans}.$$



**Ans 2(b):** Solve it yourself.

**Ans 3(a):**

3 a) $f(z) = \sin z \qquad f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z . \qquad f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z \qquad f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$f'''(z) = -\cos z \qquad f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$\frac{\pi}{4}$ Taylor's series is given by:

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + f'''(a)\frac{(z-a)^3}{3!} + \cdots + f^n(a)\frac{(z-a)^n}{n!}$$

$$f(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right) + \left(-\frac{1}{\sqrt{2}}\right)\left(z-\frac{\pi}{4}\right)^2\frac{1}{2!} - \frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right)^3\frac{1}{3!} + \frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right)^4\frac{1}{4!} + \cdots$$

which is the required Taylor's series.

Region of convergence is $\left|z-\frac{\pi}{4}\right| < 1$.



**Ans 3(b)**

(b) $I = \oint \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$

Poles: $\left(z - \frac{\pi}{6}\right)^3 = 0$

$z = \frac{\pi}{6}$ order = 3.

Residue: $a_z\left(z=\frac{\pi}{6}\right) = \lim_{z\to\frac{\pi}{6}} \frac{1}{(3-1)!} \frac{d^2}{dz^2}\left(z-\frac{\pi}{6}\right)^3 \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3}\Bigg|_{z=\frac{\pi}{6}}$

$= \lim_{z\to\frac{\pi}{6}} \frac{1}{2!} \frac{d^2}{dz^2}(\sin^6 z)$

$= \lim_{z\to\frac{\pi}{6}} \frac{1}{2} \frac{d}{dz}\left[6\sin^5 z \cos z\right]$

$= \lim_{z\to\frac{\pi}{6}} \frac{1}{2}\left[6(5)\sin^4 z\cos^2 z + 6\sin^5 z(-\sin z)\right]$

$= \lim_{z\to\frac{\pi}{6}} \frac{1}{2}\left[30\sin^4 z\cos^2 z - 6\sin^6 z\right]$

$= \lim_{z\to\frac{\pi}{6}} \frac{1}{2}\left(30\sin^4 z - 36\sin^6 z\right)$

$= \lim_{z\to\frac{\pi}{6}} \left(15\sin^4 z - 18\sin^6 z\right)$

$= 15\left(\frac{1}{2}\right)^4 - 18\left(\frac{1}{2}\right)^6$

$$= \left(\frac{1}{2}\right)^4 \left[15 - \frac{18}{4}\right] = \frac{21}{16}$$

$$\therefore \quad I = 2\pi i \,(\text{Residues})$$

$$= 2\pi i \times \frac{21}{16} = \frac{21\pi i}{8}$$

**Ans 4(a) :**

Consider $\int_C f(z)\,dz$, where $f(z) = \dfrac{1}{z^4+1}$

taken around the closed contour consisting of real axis and upper half $C_R$, *i.e.* $|z| = R$. Poles of $f(z)$ are given by

$$z^4 + 1 = 0 \text{ i.e. } z^4 = -1 = (\cos\pi + i\sin\pi)$$

$$\Rightarrow \qquad z^4 = [\cos(2n+1)\pi + i\sin(2n+1)\pi]$$

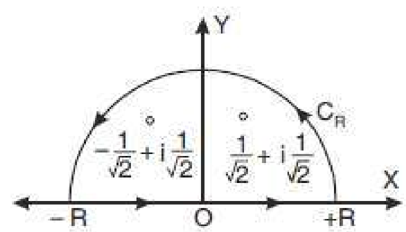
$$z = [\cos(2n+1)\pi + i\sin(2n+1)\pi]^{\frac{1}{4}} = \left[\cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4}\right]$$

If $\quad n = 0, \quad z_1 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = e^{i\frac{\pi}{4}}$

$n = 1, \quad z_2 = \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = e^{i\frac{3\pi}{4}}$

$n = 2, \quad z_3 = \left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$n = 3, \quad z_4 = \left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$




There are four poles, but only two poles at $z_1$ and $z_2$ lie within the contour.

$$\text{Residue}\left(\text{at } z=e^{\frac{i\pi}{4}}\right)=\left[\frac{1}{\frac{d}{dz}(z^4+1)}\right]_{z=e^{\frac{i\pi}{4}}}=\left[\frac{1}{4z^3}\right]_{z=e^{\frac{i\pi}{4}}}=\frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3}=\frac{1}{4e^{i\frac{3\pi}{4}}}$$

$$=\frac{1}{4}e^{-i\frac{3\pi}{4}}=\frac{1}{4}\left[\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right]=\frac{1}{4}\left[-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right]$$

$$\text{Residue}\left(\text{at } z=e^{\frac{3i\pi}{4}}\right)=\left[\frac{1}{\frac{d}{dz}(z^4+1)}\right]_{z=e^{\frac{3i\pi}{4}}}=\frac{1}{[4z^3]_{z=\frac{3i\pi}{4}}}=\frac{1}{4\left(e^{i\frac{3\pi}{4}}\right)^3}=\frac{1}{4e^{i\frac{9\pi}{4}}}$$

$$=\frac{1}{4}e^{-i\frac{9\pi}{4}}=\frac{1}{4}\left(\cos\frac{9\pi}{4}-i\sin\frac{9\pi}{4}\right)=\frac{1}{4}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)$$

$$\int_C f(z)\,dz=2\pi i\ \text{(sum of residues at poles within c)}$$

$$\int_{-R}^{R} f(z)\,dz+\int_{C_R} f(z)\,dz=2\pi i\ \text{(sum of the residues)}$$

$$\int_{-R}^{R}\frac{1}{x^4+1}\,dx+\int_{C_R}\frac{1}{z^4+1}\,dz=2\pi i\ \text{(sum of the residues)}$$

Now, $\left|\int_{C_R}\frac{1}{z^4+1}dz\right|\le\int_{C_R}\frac{1}{|z^4+1|}|dz|$

$$\le\int_{C_R}\frac{1}{(|z^4|-1)}|dz|\qquad [\text{Since } z=Re^{i\theta}, |dz|=|Re^{i\theta}\,i\,d\theta|=R\,d\theta]$$

$$\le\int_0^{\pi}\frac{1}{R^4-1}R\,d\theta\le\frac{R}{R^4-1}\int_0^{\pi}d\theta$$

$$\le\frac{R\pi}{R^4-1}=\frac{\pi/R^3}{1-1/R^4}\qquad \text{which}\to 0$$

as $R\to\infty$.

Hence, $\int_{-R}^{R} \frac{1}{x^4+1}\,dx = 2\pi i$ (Sum of the residues within contour)

As $R \to \infty$

Hence, $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = 2\pi i$ (Sum of the residues within contour) ... (1)

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = 2\pi i\left[\frac{1}{4}\left(-\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right) + \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi}{2}\,i\left(-\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right) = \frac{\pi i}{2}\left(-i\,\frac{2}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$$

Hence, the given integral = $\frac{\pi}{\sqrt{2}}$ **Ans.**

**Ans 4(b):**

Let $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$

$= \text{Real part of } \int_0^{2\pi} \frac{\cos 2\theta + i\sin 2\theta}{5+4\cos\theta}\,d\theta$

$= \text{Real part of } \int_0^{2\pi} \frac{e^{2i\theta}}{5+2\,(e^{i\theta}+e^{-i\theta})}\,d\theta$

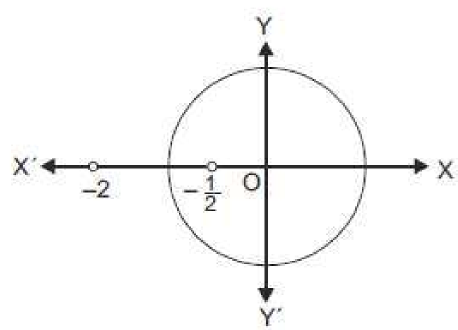
$$\left[\begin{aligned} e^{i\theta} &= z \\ \Rightarrow\quad i.e.^{i\theta}\,d\theta &= dz \\ \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} &= \frac{dz}{iz} \end{aligned}\right]$$

$= \text{Real part of } \oint_C \frac{z^2}{5+2\left(z+\frac{1}{z}\right)}\,\frac{dz}{iz}$ [$C$ is the unit circle $|z| = 1$]

$= \text{Real part of } \oint_C \frac{z^2}{5z+2z^2+2}\,\frac{dz}{i}$

$= \text{Real part of } \oint_C \frac{-iz^2}{2z^2+5z+2}\,dz$

$= \text{Real part of } \oint_C \frac{-iz^2}{(2z+1)(z+2)}\,dz$



Poles are determined by putting denominator equal to zero.

$$(2z+1)(z+2) = 0 \quad \Rightarrow \quad z = -\frac{1}{2}, -2$$

The only simple pole at $z = -\frac{1}{2}$ is inside the contour.

Residue at $\left(z = -\frac{1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) f(z) = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) \frac{-i z^2}{(2z+1)(z+2)}$

$$= \lim_{z \to -\frac{1}{2}} \frac{-i z^2}{2(z+2)} = \frac{-i\left(-\frac{1}{2}\right)^2}{2\left(-\frac{1}{2}+2\right)} = \frac{-i}{12}$$

By Cauchy's Integral Theorem

$$\int_C f(z)\, dz = 2\pi i \quad \text{(Sum of the residues within } C\text{)}$$

$$= 2\pi i \left(\frac{-i}{12}\right) = \frac{\pi}{6}, \text{ which is real}$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = \frac{\pi}{6}$$

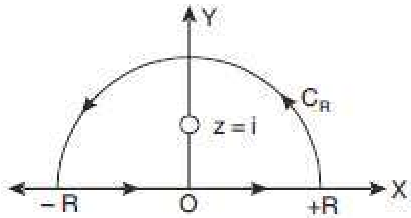
$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\int_0^{\infty} \frac{dx}{x^4+1}$$

$$\int_0^{\infty} \frac{dx}{x^4+1} = \frac{1}{2}\left(\frac{\pi}{\sqrt{2}}\right) = \frac{\pi}{2\sqrt{2}} = \frac{\sqrt{2}\,\pi}{4}$$

**Ans 4(c) :**

Consider the integral $\int_C f(z)\,dz$, where

$f(z) = \dfrac{e^{imz}}{z^2+1}$, taken round the closed contour $c$ consisting of the upper half of a large circle $|z| = R$ and the real axis from $-R$ to $R$.




Poles of $f(z)$ are given by

$$z^2 + 1 = 0 \text{ i.e. } z^2 = -1 \text{ i.e. } z = \pm i$$

The only pole which lies within the contour is at $z = i$.
The residue of $f(z)$ at $z = i$

$$= \lim_{z \to i} \frac{(z-i)e^{imz}}{(z^2+1)} = \lim_{z \to i} \frac{e^{imz}}{z+i} = \frac{e^{-m}}{2i}$$

Hence by Cauchy's residue theorem, we have

$$\int_C f(z)\,dz = 2\pi i \times \text{ sum of the residues}$$

$$\Rightarrow \quad \int_C \frac{e^{imz}}{z^2+1} dz = 2\pi i \times \frac{e^{-m}}{2i} \qquad \Rightarrow \quad \int_{-R}^{R} \frac{e^{imx}}{x^2+1} dx = \pi e^{-m}$$

Equating real parts, we have

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1} dx = \pi e^{-m} \qquad \Rightarrow \int_0^{\infty} \frac{\cos mx}{x^2+1} dx = \frac{\pi e^{-m}}{2}$$

**Ans.**

**Ans 5a)**

The Fourier transform of a Gaussian function $f(x) \equiv e^{-a x^2}$ is given by

$$\begin{aligned} \mathcal{F}_x\left[e^{-a x^2}\right](k) &= \int_{-\infty}^{\infty} e^{-a x^2}\, e^{-2\pi i k x}\, dx \\ &= \int_{-\infty}^{\infty} e^{-a x^2} \left[\cos(2\pi k x) - i \sin(2\pi k x)\right] dx \\ &= \int_{-\infty}^{\infty} e^{-a x^2} \cos(2\pi k x)\, dx - i \int_{-\infty}^{\infty} e^{-a x^2} \sin(2\pi k x)\, dx. \end{aligned}$$

The second integrand is odd, so integration over a symmetrical range gives 0. The value of the first integral is given by Abramowitz and Stegun , so

$$\mathcal{F}_x\left[e^{-a x^2}\right](k) = \sqrt{\frac{\pi}{a}}\, e^{-\pi^2 k^2 / a},$$

so a Gaussian transforms to another Gaussian.

**Ans 5b)**

The one-dimensional wave equation is given by

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}.$$

In order to specify a wave, the equation is subject to boundary conditions

$$\psi(0,t)=0$$
$$\psi(L,t)=0,$$

and initial conditions

$$\psi(x,0)=f(x)$$
$$\frac{\partial \psi}{\partial t}(x,0)=g(x).$$

d'Alembert devised his solution in 1746, and Euler subsequently expanded the method in 1748. Let

$$\xi \equiv x - v\,t$$
$$\eta \equiv x + v\,t.$$

By the chain rule,

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial \xi^2} + 2\frac{\partial^2 \psi}{\partial \xi\,\partial \eta} + \frac{\partial^2 \psi}{\partial \eta^2}$$
$$\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial \xi^2} - 2\frac{\partial^2 \psi}{\partial \xi\,\partial \eta} + \frac{\partial^2 \psi}{\partial \eta^2}.$$

The wave equation then becomes

$$\frac{\partial^2 \psi}{\partial \xi\,\partial \eta} = 0.$$

Any solution of this equation is of the form

$$\psi(\xi,\eta) = f(\eta) + g(\xi) = f(x+v\,t) + g(x-v\,t),$$

where $f$ and $g$ are *any* functions. They represent two waveforms traveling in opposite directions, $f$ in the negative $x$ direction and $g$ in the positive $x$ direction.

**Ans 6a)** do it yourself

**Ans 6b)**

**Solution.** We know that

$$L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{1}{a}\sin at$$

Hence by convolution theorem

$$L^{-1}\frac{1}{(s^2+a^2)}\frac{1}{(s^2+a^2)}=\int_0^t\frac{1}{a}\sin ax\frac{1}{a}\sin a(t-x)\,dx$$

$$=\frac{1}{a^2}\int_0^t\frac{1}{2}[\cos(ax-at+ax)-\cos(ax+at-ax)]\,dx\quad\left\{\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]\right\}$$

$$=\frac{1}{2a^2}\int_0^t[\cos(2ax-at)-\cos at]\,dx=\frac{1}{2a^2}\left[\frac{1}{2a}\sin(2ax-at)-x\cos at\right]_0^t$$

$$=\frac{1}{2a^2}\left[\frac{1}{2a}\sin(2at-at)-t\cos at-\frac{1}{2a}\sin(-at)\right]=\frac{1}{2a^2}\left[\frac{1}{2a}\sin at-t\cos at+\frac{1}{2a}\sin at\right]$$

$$=\frac{1}{2a^2}\left[\frac{2}{2a}\sin at-t\cos at\right]=\frac{1}{2a^3}[\sin at-at\cos at]$$ **Ans.**

Where a=1.

**Ans 7a)**

~~EXAMPLE 3.1~~ For a semi infinite bar, $x>0$, we consider the 1-d form of the heat diffusion equation. Let $u(x,t)$ represent the temperature at a distance $x$ from one end of the bar at time $t$. The boundary condition is that $u(0,t)=u_0$ and the initial condition is $u(x,0)=0$.

As it is a semi infinite bar, we solve this problem using Laplace transform ($x>0$). For an infinite bar, we employ Fourier transform.

Let $\frac{k}{c\rho}=\alpha$, the thermal diffusivity. We consider the Laplace transform on both the sides of the heat diffusion equation

$$\mathcal{L}\left\{\alpha\frac{\partial^2u(x,t)}{\partial x^2}\right\}=\mathcal{L}\left\{\frac{\partial u(x,t)}{\partial t}\right\}\quad\text{or}$$

$\alpha\int_0^\infty\frac{\partial^2u(x,t)}{\partial x^2}e^{-st}\,dt=sU(x,s)-u(x,0)$ with $U(x,s)$ as the Laplace transform of $u(x, t)$.

The LHS integration is with respect to $t$ while the double differentiation is with respect to $x$.

$$\alpha\int_0^\infty \frac{\partial^2 u(x,t)}{\partial x^2} e^{-st}\, dt = \alpha \frac{\partial^2}{\partial x^2}\int_0^\infty u(x,t)\, e^{-st}\, dt$$

$$= \alpha \frac{\partial^2 U(x,s)}{\partial x^2}$$

Using the initial condition $u(x, 0) = 0$, we obtain

$\alpha \dfrac{d^2U(x,s)}{dx^2} = sU(x, s)$, let $\dfrac{s}{\alpha} = p^2$

$U(x, s) = Ae^{px} + Be^{-px}$ with $A$ and $B$ as constants to be evaluated. $u(x, t) = 0$ as $x \to \infty$ as for the boundedness of the temperature. Thus, $U(x, s) = 0$ as $x \to \infty$ which implies that $A = 0$.

Since $u(0, t) = u_0$ therefore $U(0, s) = \mathcal{L}\{u_0\} = \dfrac{u_0}{s}$ with $u_0$ as the constant temperature at $x = 0$.

$\dfrac{u_0}{s} = B$

and the complete solution is $U(x, s) = \dfrac{u_0}{s} e^{-px} = \dfrac{u_0}{s} e^{-x\sqrt{s/\alpha}}$

The solution $u(x, t)$ is the Laplace inverse of $U(x, s)$.

$$\mathcal{L}^{-1}\left\{\frac{u_0}{s}e^{-px}\right\} = u_0\left[1 - \mathrm{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)\right]$$

erf is error function

**Ans 7b)**

7(b): $\displaystyle\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\, dt$

$\mathcal{L}(\cos 6t - \cos 4t) = \mathcal{L}(\cos 6t) - \mathcal{L}(\cos 4t)$

$= \dfrac{s}{s^2+6^2} - \dfrac{s}{s^2+4^2}$

$\mathcal{L}\dfrac{(\cos 6t - \cos 4t)}{t} = \displaystyle\int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\log 1 - \frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$= 0 - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

where $a = 6$ $b = 4$

$$\therefore \int_0^\infty \frac{\cos 6t - \cos 4t}{t} = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Putting $s = 0$ $a = 6$ $b = 4$

$$I = \frac{1}{2}\log\left(\frac{4}{6}\right)^2 = \log\frac{2}{3}$$