Elements of Modern Physics
Solved Paper – 2017

Q. 1. Attempt any five of the following : **(5×3=15)**

(a) Calculate the de Broglie wavelength of an electron accelerated through a potential difference of 100V and 1MV.

Ans. Planck's constant, $h = 6.62 \times 10^{-34}$ J sec.

Potential difference, V = (i) 100 V

= (ii) 1 MV

Mass of electron, $m = 9.1 \times 10^{-31}$ kg

$q = 1.6 * 10^{-19}$ C

Now,
$$\lambda = \frac{h}{\sqrt{2mqv}}$$

(i)
$$= \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 100 \times 1.6 \times 10^{-19}}}$$
$$= 1.226 \text{ Å}$$

(ii) For 1MV = 1×10^{6} V, the wavelength is 0.01226 A

(b) Determine the activity of 1 g of $_{92}U^{238}$ if its half life is 4.5×10^{9}y.

Ans. Activity. The activity of a sample of any radioactive nuclide is the rate at which the nuclei of its constituent atoms decay. If N is the number of nuclei present in the sample at a certain time, its activity A is given by $A = -\frac{dN}{dt}$. The

$$T_{1/2} = \frac{0.6931}{k}$$
$$T_{1/2} = 4.5 \times 10^{9} y$$

Other formula for the activity is A = k.N

$$= \frac{0.693}{T_{1/2}} \times \frac{\text{Avogadro's No.} \times \text{Mass of element}}{\text{Mass no. of element}}$$

$$= \frac{0.693 \times 6.023 \times 10^{23} \times 1}{4.5 \times 10^9 \times 238 \times 3.15 \times 10^7}$$

$$= \frac{4.17}{4.5} \times \frac{10^{23} \times 10^{-7}}{749.7}$$

$$= \frac{0.9267}{1} \times \frac{1000}{749.7} \times \frac{10^{20} \times 10^{-7}}{1}$$

$$= 0.9267 \times 1.334 \times 10^{13}$$

$$A = 1.236 \times 10^{13} \text{ dis s}^{-1}$$ Ans.

(c) The maximum energy of photoelectrons from Aluminum is 2.3 eV for radiation of 200 nm and 0.90 eV for radiation of 258 nm. Calculate the Planck's constant and the work function of Aluminium.

Ans. $\nu_1 = c/\lambda_1$

$$= \frac{3 \times 10^8}{200 \times 10^{-9}}$$

$$= 1.5 \times 10^{15} \text{ HZ}$$

$\nu_2 = c/\lambda_2$

$$= \frac{3 \times 10^8}{258 \times 10^{-9}}$$

$$= 1.163 \times 10^{15} \text{ HZ}$$

Now, $E_1 = \phi + KE_1$

$E_2 = \phi + KE_2$

$\therefore \quad E_1 - E_2 = KE_1 - KE_2$

or $\quad E_1 - E_2 = h\nu_1 - h\nu_2$

$$(2.3 - 0.9)\, 1.6 \times 10^{-19} = 6.65 \times 10^{-34} (1.5 - 1.163) 10^{15}$$

$$\therefore \quad h = \frac{1.4 \times 1.6 \times 10^{-34}}{0.337}$$

$$= 6.65 \times 10^{-34} \text{ J.s}$$ Ans.

We know, $\quad K.E. = h\nu - \phi$

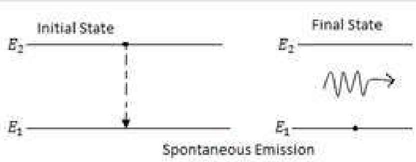
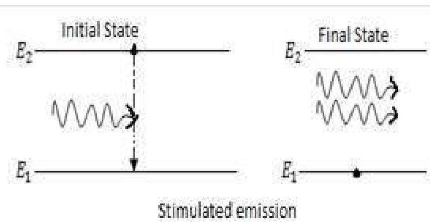
$$\therefore \quad 2.3 \times 1.6 \times 10^{-19} = \frac{6.65 \times 10^{-34} \times 3 \times 10^{8}}{200 \times 10^{-19}}$$

$$\therefore \quad \phi = 9.975 \times 10^{-19} - 3.680 \times 10^{-19}$$
$$= 6.295 \times 10^{-19} \, J$$
$$= \frac{6.295 \times 10^{-19}}{1.6 \times 10^{-19}} eV$$
$$= 3.934 \, eV.$$

Hence the work function of aluminium is 3.934 eV. **Ans.**

(d) Write two special characteristics of the light, emitted from a laser. Distinguish between spontaneous emission and stimulated emission.

Ans. The light emitted from a laser is monochromatic, that is, it is of one wavelength (color). In contrast, ordinary white light is a combination of many different wavelengths (colors). The difference between spontaneous and stimulated emission are given below-

SR No	Spontaneous emission	Stimulated emission
1	The transition of an electron from the excited state to the ground state happens as a result of the natural tendency of the electron without the action of any external agent. The radiation produced as a result of such transitions is called as spontaneous radiation.	Stimulated emission of radiation is the process whereby photons are used to generate other photons that have exact phase and wavelength as that of parent photon.
2	This phenomenon is found in LEDs, Fluorescent tubes.	This is the key process of formation of laser beam.
3	There is no population inversion of electrons in LEDs.	Population inversion is achieved by various ‘pumping’ techniques to get amplification giving the LASER its name “Light amplification by stimulated emission of radiation.”
4	No external stimuli required.	Thus stimulated emission is caused by external stimuli.
5		

(e) Why do we associate a wave packet and not a monochromatic de-Broglie wave with a particle ?

Ans. A wave packet is a type of wave motion comprising a group of waves, each with slightly different velocity and wave length, with phases and amplitude so chosen that they interfere constructively over only a small region of space where the particle can be located, outside of which they produce destructive interference so that the amplitude reduces to zero rapidly. In other words, the behaviour of the particle should be described by a wave function.

According to de-Broglie, the equation should be equally applicable to both the photons of radiation and material particles like electrons. This would not be possible if monochromatic light is used.

(f) Which of the following are ergen functions of the operator d^2/dx^2 ? Give the eigen values where appropriate, (i) cos x, (ii) e^{-ix}, (iii) $\sin^2 x$.

Ans.	**Function**	**Eigen value**
(i)	$\cos x$	–1
(ii)	e^{-ix}	–1
(iii)	$\sin^2 x$	It is not an eigen function

(g) How does the uncertainty principle rule out the possibility of electron being present inside the nucleus ?

Ans.

On the basis of Heisenberg's uncertainty principle, it can be shown as to why electron cannot exist within the atomic nucleus. The radius of the atomic nucleus is of the order of 10^{-15} m. Now, if the electron were to exist within the nucleus, then the maximum uncertainty in its position would have been 10^{-15} m.

Now, $\Delta x \times \Delta p = \dfrac{h}{(4\pi)}$

or $\Delta x \quad m\Delta v = \dfrac{h}{(4\pi)}$

or $\Delta v = \dfrac{h}{4\pi m \Delta x}$

Mass of electron, m = 9.1 x 10^{-31} kg, Dx =1 x 10^{-15} m.

$$\therefore \quad \Delta v = \frac{6.6 \times 10^{-34}}{4 \times 3.14 \times 9.1 \times 10^{-31} \times 1 \times 10^{-15}} = 5.77 \times 10^{10}\ \text{ms}^{-1}$$

The value of uncertainty in velocity, Dv is much higher than the velocity of light (3.0 x 10^{8} ms^{-1}) and therefore, it is not possible. Hence an electron cannot be found within the atomic nucleus.

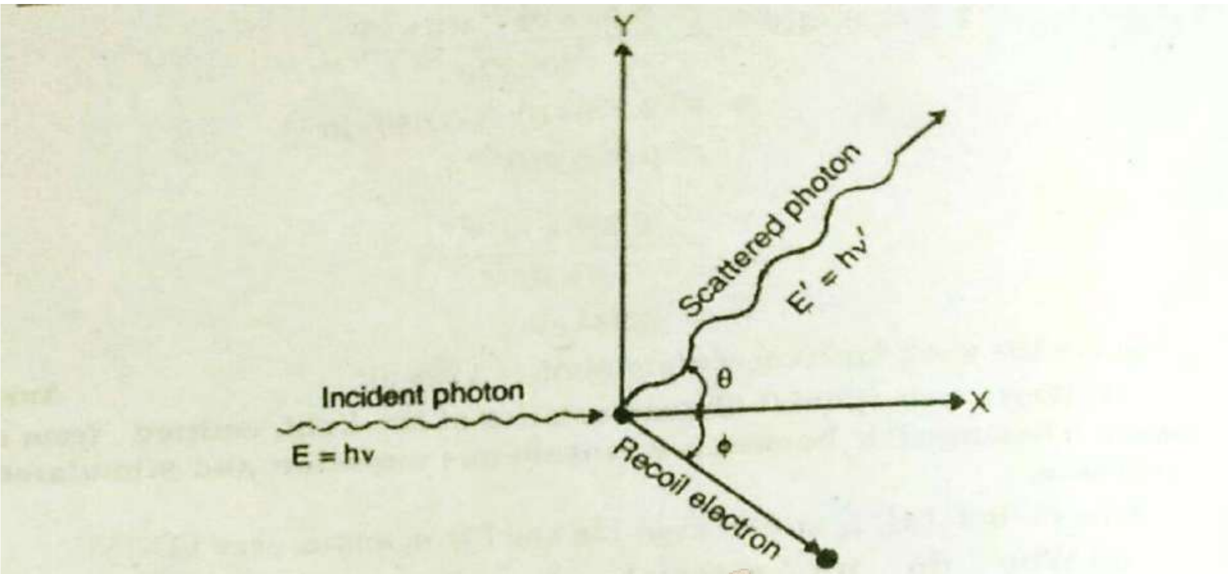
Q. 2. (a)What is Compton scattering? What is the origin of presence of the unmodified line at all scattering angles ?

Obtain the expression for change in wavelength

$$\Delta\lambda = \frac{h}{m_0 c}(1-\cos\phi)$$

where symbols have their usual meaning. (2,5)

Ans. Compton scattering is the scattering of a photon by a charged particle, usually an electron. It results in a decrease in energy (increase in wavelength) of the photon (which may be an X-ray or gamma ray photon), called the Compton effect. Part of the energy of the photon is transferred to the recoiling electron



Let a photon with an energy $h\upsilon$ and momentum $h\upsilon/c$ strikes an electron at rest. The initial momentum of the electron is zero and its initial energy is only the rest mass energy, m_0c^2. The scattered photon of energy $h\upsilon'$ and momentum $h\upsilon'/c$ moves off in a direction inclined at an angle θ to the original direction. The electron acquires a mo-mentum mv and moves at an angle φ to the original direction. The energy of the recoil electron is mc^2 (Fig.)

According to the principle of conservation of energy,

$$h\upsilon + m_0c^2 = h\upsilon' + mc^2 \qquad ...(1)$$

Considering the x and y components of the momentum and applying the principle of conservation of momentum,

$$\frac{h\upsilon}{c} = \frac{h\upsilon'}{c}\cos\theta + mc\cos\phi \qquad ...(2)$$

and

$$0 = \frac{h\upsilon'}{c}\sin\theta - m\upsilon\sin\phi \qquad ...(3)$$

From (2), $\quad \text{mvc} \cos\phi = h(\upsilon\text{-}\upsilon\cos\theta) \qquad ...(4)$

From (3), $\quad \text{mvc} \sin\phi = h\upsilon'\sin\theta \qquad ...(5)$

Squaring and adding (4) and (5),

$$m^2\upsilon^2c^2 = h^2(\text{v}^2 - 2\upsilon\upsilon'\cos\theta + \upsilon^2\cos^2 0) + h^2\upsilon^2\sin^2\theta$$
$$= h^2(\upsilon^2 - 2\upsilon\upsilon'\cos\theta) + h^2\upsilon^2 = h^2$$
$$(\upsilon^2 - 2\upsilon\upsilon'\cos\theta + \upsilon^2) \qquad ...(6)$$

From(l),

$$mc^2 = h(\upsilon - \upsilon') + m_0c^2$$
$$m^2c^4 = h^2(\upsilon^2 - 2\upsilon\upsilon' + \upsilon'^2) + 2h(\upsilon - \upsilon')m_0c^2 + m_0^2c^4 \qquad ...(7)$$

Subtracting (6) from (7),

$$m^2c^2(c^2 - \upsilon^2) = -2h^2\upsilon\upsilon'(1 - \cos\theta) + 2h(\upsilon - \upsilon')m_0c^2 + m_0^2c^4 \qquad ...(8)$$

The value of $m^2c^2\,(c^2 - \upsilon^2)$ can be obtained from the relativistic formula

$$m = \frac{m_0}{\sqrt{\left(1 - \frac{\upsilon^2}{c^2}\right)}}\text{. Squaring,}$$

$$m^2 = \frac{m_0^2}{1 - \frac{\upsilon^2}{c^2}} = \frac{m_0^2 c^2}{c^2 - \upsilon^2}$$

$$m^2\,c^2\,(\,c^2 - \upsilon^2) = m_0^{\,2}\,c^4 \qquad \text{...(9)}$$

From (8) and (9),

$$m_0^{\,2}c^4 = -\,2h^2\,\upsilon\upsilon'\,(\,1 - \cos\theta) + 2h\,(\upsilon - \upsilon')\,m_0c^2 + m_0^{\,2}c^4$$

$$\therefore \quad 2h\,(\upsilon - \upsilon')\,m_0c^2 = 2h^2\,\upsilon\upsilon'\,(1 - \cos\theta)$$

or $$\frac{\upsilon - \upsilon'}{\upsilon\upsilon'} = \frac{h}{m_0c^2}(1 - \cos\theta)$$

or $$\frac{1}{\upsilon'} - \frac{1}{\upsilon} = \frac{h}{m_0c^2}(1 - \cos\theta)$$

or $$\frac{c}{\upsilon'} - \frac{c}{\upsilon} = \frac{h}{m_0c}(1 - \cos\theta)$$

or $$\lambda' - \lambda = \frac{h}{m_0c}(1 - \cos\theta) \qquad \text{...(10)}$$

$\therefore$ The change in wavelength $= d\lambda = \dfrac{h}{m_0c}(1 - \cos\theta)$

This relation shows that $d\lambda$ is independent of the wavelength of the incident radiations as well as the nature of the scattering substance, $d\lambda$ depends upon the angle of scattering only.

Case 1. When $\theta = 0$, $\cos\theta = 1$ and hence $d\lambda = 0$

Case 2. When $\theta = 90°$, $\cos\theta = 0$ and hence

$$d\lambda = \frac{h}{m_0c} = \frac{6.63 \times 10^{-34}}{(9.11 \times 10^{-31}) \times (3 \times 10^8)}\ m = 0.0243\ \text{Å}$$

This is known as *Compton wavelength*.

Case 3. When $\theta = 180°$, $\cos\theta = -1$ and hence $d\lambda = \dfrac{2h}{m_0c} = 0.0485$ A. $d\lambda$ has the maximum value at $\theta = 180°$.

(b) In a Compton scattering experiment. X-ray of wavelength 0.24 nm is scattered at an angle 60° relative to the incident beam. Find the wavelength of scattered X-ray. (3)

Ans. Wave length,

$$\lambda = 0.24 \text{ nm} = 0.24 \times 10^{-9} \text{ m}$$

Scattering angle $= 60°$

$$\therefore \text{The change in wavelength} = d\lambda = \frac{h}{m_0 c}(1-\cos\theta)$$

Wave length of modified ray, λ' is

Given by,

$$\lambda' = \lambda + \Delta\lambda$$

$$= 1 + \frac{h}{m_0 c} = (1-\cos\phi)$$

$$= 0.24 \times 10^{-9} + \frac{6.63 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^{8}}(1-\cos 60°)$$

$$= 0.24 \times 10^{-9} + 0.243 \times 10^{-11} \times \frac{1}{2}$$

$$= 24 \times 10^{-11} + 0.1215 \times 10^{-11}$$

$$= 24.12 \times 10^{-11}$$

$$\therefore \quad \lambda' = 2.412 \times 10^{-10} \text{ m}$$ Ans.

(c) Given a dispersion relation

$$w(k) = w(k_0) + (k-k_0)\left(\frac{\partial w}{\partial k}\right)_{k=k_0} + \frac{1}{2}(k-k_0)^2\left(\frac{\partial^2 w}{\partial k^2}\right)_{k=k_0}$$

show that a Gaussian wave packet

$$y(x,0) = \sqrt{\frac{2\pi}{\alpha}}e^{ik_0 x}e^{-x^2/2\alpha}$$

spreads as it propagates in time. (5)

Q. 3. (a) State Heisenberg uncertainty principle for measurement of position and momentum. Using Gamma ray microscope thought experiment proposed by Heisenberg, obtain an expression for the uncertainty relation. (2,5)

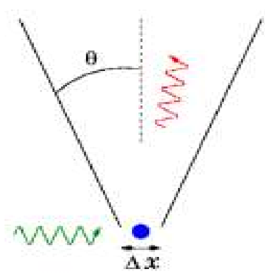
Ans. The Heisenberg uncertainty principle states that it is impossible to know simultaneously the exact position and momentum of a particle. That is, the more exactly the position is determined, the less known the momentum, and vice versa.

Heisenberg begins by supposing that an electron is like a classical particle, moving in the direction along a line below the microscope, as in the illustration to the right. Let the cone of light rays leaving the microscope lens and focusing on the electron make an angle with the electron. Let be the wavelength of the light rays. Then, according to the laws of classical optics, the microscope can only resolve the position of the electron up to an accuracy of

$$\Delta x = \frac{\lambda}{\sin \varepsilon}.$$

When an observer perceives an image of the particle, it's because the light rays strike the particle and bounce back through the microscope to their eye. However, we know from experimental evidence that when a photon strikes an electron, the latter has a Compton recoil with momentum proportional to , where is Planck's constant. It is at this point that Heisenberg introduces objective indeterminacy into the thought experiment. He writes that "the recoil cannot be exactly known, since the direction of the scattered photon is undetermined within the bundle of rays entering the microscope" . In particular, the electron's momentum in the direction is only determined up to

$$\Delta p_x \approx \frac{h}{\lambda} \sin \varepsilon.$$



Combining the relations for Δx and Δp_x, we thus have that[6]

$$\Delta x \Delta p_x \approx \left(\frac{\lambda}{\sin \varepsilon}\right)\left(\frac{h}{\lambda} \sin \varepsilon\right) = h,$$

which is an approximate expression of Heisenberg's uncertainty principle.

(b) Show that the uncertainty principle can be expressed in the form $\Delta E \Delta t > h/2$, where ΔE is the uncertainty in the energy and Δt is the uncertainty in time. (3)

$$\Delta x \Delta p \sim \hbar \tag{1}$$

Later in the paper he considered specifically the case of a freely-moving Gaussian wavepacket in x space with a width Δx. Using the commutation relation

$$[\mathbf{x}, \mathbf{p}] = (\mathbf{xp} - \mathbf{px}) = -i\hbar \tag{2}$$

of Max Born [2], Jordan [3] showed that a transformation between x and p representations is effected by the function $\exp(ipx/\hbar)$ i.e. is a Fourier transformation. Using this result, Heisenberg could transform his wavepacket to momentum space and obtained a width Δp such that

$$\Delta x \Delta p = \hbar \tag{3}$$

It is important to note the equality in eq.(3), compared to the order of magnitude in eq.(1). Also it is clear that Heisenberg's derivation of the equality arises from the property of the Fourier transform relation between position and momentum free wavepackets and is not proved generally.

Heisenberg also discusses the classically conjugate variables of time and energy and defines a time operator through the, quote, "familiar relation"

$$[\mathbf{E}, \mathbf{t}] = (\mathbf{Et} - \mathbf{tE}) = -i\hbar \tag{4}$$

$$\Delta x \Delta p \sim \hbar \tag{1}$$

Later in the paper he considered specifically the case of a freely-moving Gaussian wavepacket in x space with a width Δx. Using the commutation relation

$$[\mathbf{x}, \mathbf{p}] = (\mathbf{xp} - \mathbf{px}) = -i\hbar \tag{2}$$

of Max Born [2], Jordan [3] showed that a transformation between x and p representations is effected by the function $\exp(ipx/\hbar)$ i.e. is a Fourier transformation. Using this result, Heisenberg could transform his wavepacket to momentum space and obtained a width Δp such that

$$\Delta x \Delta p = \hbar \tag{3}$$

It is important to note the equality in eq.(3), compared to the order of magnitude in eq.(1). Also it is clear that Heisenberg's derivation of the equality arises from the property of the Fourier transform relation between position and momentum free wavepackets and is not proved generally.

Heisenberg also discusses the classically conjugate variables of time and energy and defines a time operator through the, quote, "familiar relation"

$$[\mathbf{E}, \mathbf{t}] = (\mathbf{Et} - \mathbf{tE}) = -i\hbar \tag{4}$$

On the basis of this commutation relation, Heisenberg assumes a time-energy uncertainty relation (TEUR)

$$\Delta E \Delta t \sim \hbar \tag{5}$$

(c) **An electron of energy 200 eV is passed through a circular hole of radius 10^{-4} cm. What is the uncertainty introduced in the angle of emergence ?** **(5)**

Ans. We have $\frac{p^2}{2m} = E$

Here, $E = 200\, eV$

$= 200 \times 1.602 \times 10^{-19}$ Joules.

$\therefore \quad p = \sqrt{2mE}$

$= \sqrt{2 \times 9 \times 10^{-31} \times 200 \times 1.602 \times 10^{-19}}\, kgm\,\text{sec}$

$$\Delta p_x = \frac{h}{\Delta x}$$

$\Delta x = 2r$

Here $r = 10^{-4}$ cm

$= 10^{-6}$ m

$$\therefore \quad \Delta p_x = \frac{6.626 \times 10^{-34}}{2 \times 10^{-6}}$$

$= 3.316 \times 10^{-26}$

Let $\Delta\theta$ be the uncertainty in the angle of emergence.

$$\text{Then} \quad \Delta\theta = \frac{\Delta p_x}{p} = \frac{3.316 \times 10^{-26}}{\sqrt{2 \times 9 \times 10^{-31} \times 200 \times 1.602 \times 10^{-19}}}$$

$= 5.76 \times 10^{-6}$ radian. **Ans.**

Q. 4. (a) What differences will you observe on the screen of a two-slit experiment if you use

(i) photons from a monochromatic source,

(ii) electrons, from an electron gun, and

(iii)bullets from a machine gun? Interpret the results.

Ans. (i) A monochromatic source of light shows an intereference pattern on the wall, as it is electromagnetic wave shown in figure 1.

(ii) According to quantum mechanics, the electrons may also act as waves and show the same pattern of interference frindges after the slit (Figure 1)

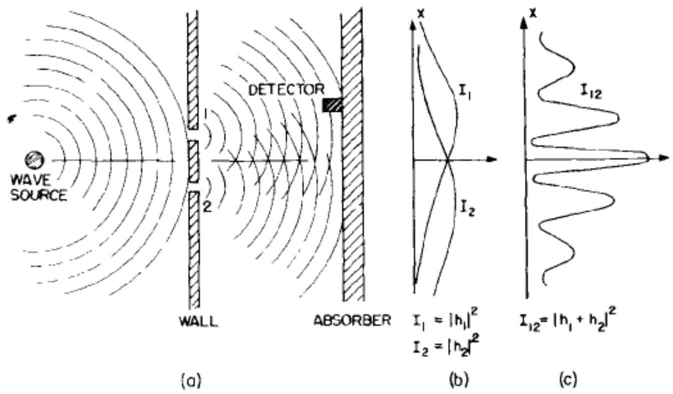


Figure 1

(iii) Bullets are classical objects, their wave nature is not noticeable. So, they will show a classic pattern of distribution as shown in figure 2.

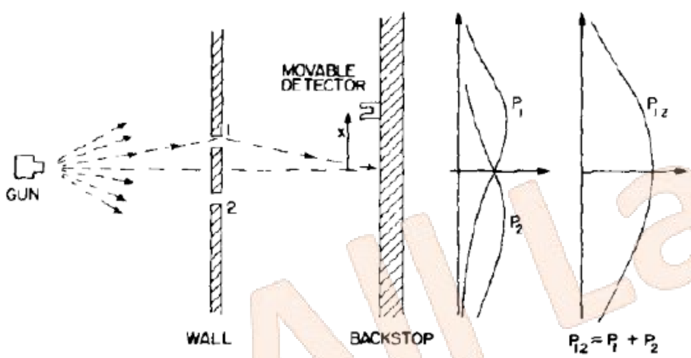


Figure 2

object.

(b) The wave function for a particle moving along the positive x-direction is given as

$$\psi(x, t) = A \exp\left(i\left(\frac{px}{\hbar} - \frac{Et}{\hbar}\right)\right)$$

Using this derive expressions for momentum and kinetic energy operators in one dimension. (3)

Ans. $$\psi(x, t) = A \exp\left[i\left(\frac{px}{\hbar} - \frac{Et}{\hbar}\right)\right]$$

Momentum operator :

As Hamiltonian $$H = KE + PE = \frac{p^2}{2m} + V$$

Also $$H = \frac{\hbar^2}{2m}\nabla^2 + V$$

$$\therefore \quad \frac{p^2}{2m} + V = \frac{\hbar^2}{2m}\nabla^2 + V \text{ i.e., } p^2 = -\hbar^2\nabla^2$$

Thus the operator associated with momentum p_{op} *is*

$$p^2_{op} = -\hbar^2\nabla^2 = \frac{\hbar^2}{i^2}\nabla^2$$

i.e., $$p_{op} \text{ or } \hat{p} = \frac{\hbar}{i}\nabla$$

If p_x, p_y, p_z are components of momentum p.

the $$\left(\hat{i}p_x + \hat{j}p_y + \hat{k}p_z\right) = \frac{\hbar}{i}\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

∴ Therefore operators associated with momentum components are

$$(p_x)_{op} \text{ or } \hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}, \hat{p}_y = \frac{\hbar}{i}\frac{\partial}{\partial y}, \hat{p}_z = \frac{\hbar}{i}\frac{\partial}{\partial z}$$

Since the particle moves only in x-direction, only the term $\frac{\partial}{\partial x}$ will exist. The terms $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ will not be there. Please do the partial differentiation and solve.

Kinetic energy operator : As

$$H = KE + PE = T_{op} + V_{op}$$

Also $$H = -\frac{\hbar}{2m}\nabla^2 + V$$

∴ **Kinetic energy operator,** T_{op} or $\hat{T} = -\frac{\hbar^2}{2m}\nabla^2$

Please find $\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$. Put the value of $\frac{\partial^2}{\partial x^2}$ in the expression of kinetic energy operator.

(c) Normalize the wave function given below to find the constant 'A' for the Gaussian wavepacket given as

$$\psi(x, t) = A \exp\left(-\frac{\alpha^2 x^2}{2}\right) \exp(ikx) \text{ given that}$$

$$\int_{-\infty}^{\infty} \exp(-\alpha^2 x^2)\,dx = \sqrt{\frac{\pi}{\alpha}} \qquad \textbf{(5)}$$

Ans. $$y(x) = A \exp\left(-\frac{\alpha^2 x^2}{2}\right) \exp(ikx)$$

Here a is the normalising constant the normalisation condition is

$$\int_{-\infty}^{\infty} \psi^* \psi dx = 1$$

i.e., $$\int_{-\infty}^{\infty} \left\{ A^* e^{\frac{-\alpha^2 x^2}{2} - ikx} A e^{\frac{-\alpha^2 x^2}{2} + ikx} \right\} dx = 1$$

i.e, $$A^* A \int_{-\infty}^{\infty} e^{-\alpha^2 x^2} dx = 1$$

or $$A^* A \left(\sqrt{\frac{\pi}{\alpha}}\right) = 1$$

But $$A^* A = |A|^2$$

$\therefore$ $$|A|^2 \left(\sqrt{\frac{\pi}{\alpha}}\right) = 1$$

$\therefore$ $$|A|^2 = \frac{1}{\sqrt{\frac{\pi}{\alpha}}} = \sqrt{\frac{\alpha}{\pi}}$$

or $$|A| = \sqrt{\sqrt{\frac{\alpha}{\pi}}} = \left(\frac{\alpha}{\pi}\right)^{1/4}$$

$\therefore$ $$A = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{i\phi},$$

where ϕ is a phase factor.

Q. 5. (a) Consider a particle of mass *m* and energy $E < V_0$ approaching from the left, a one-dimensional potential step given by

$$V(x) = V_0 \quad \text{for} \quad x > 0$$
$$= 0 \quad \text{for} \quad x < 0;$$

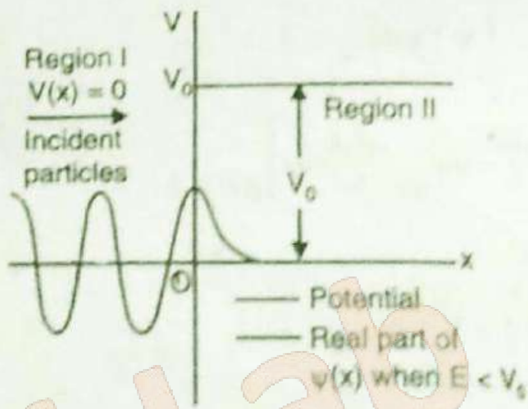
Show that the reflection coefficient is equal to 1. Explain how penetration into classically forbidden region is not in conflict with Classical Mechanics and find an expression for penetration depth.

(5,3)

Ans. Potential step. The potential function of a potential step is defined by

$$V(x) = 0 \qquad x < 0$$
$$= V_0 \qquad x > 0 \qquad ...(1)$$

Let electrons of energy E move from left to right, *i.e.*, along the positive direction of x-axis (Fig.). It is desired to find the eigenfunction solutions of the time-independent Schrodinger equation



For I region $V(x) = 0$. Therefore, the Schrodinger equation takes the form

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0 \qquad ...(3)$$

The solution of Eq. (3) is

$$\psi_1 = Ae^{ip_1x/\hbar} + Be^{-ip_1x/\hbar} \qquad ...(4)$$

where A and B are constants.

$$p_1 = \sqrt{(2mE)}\,.$$

Some particles may be reflected by the potential barrier, and some transmitted. The first and second terms respectively represent the *incident* and *reflected* particles.

The Schrodinger wave equation for II region is

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\psi = 0 \qquad ...(5)$$

The solution of Eq. (5) is

$$\psi_2 = Ce^{ip_2x/\hbar} + De^{-ip_2x/\hbar} \qquad ...(6)$$

where $p_2 = \sqrt{[2m(E - V_0)]}$; Cand Dare constants.

In Eq. (6), the first term represents the *transmitted wave*. The second term represents a wave coming from $+\infty$ in the negative direction. Clearly for $x > 0$ no particles can flow to the left and D must be zero. Therefore, Eq. (6) becomes

$$\psi_2 = Ce^{ip_2x/\hbar} \quad ...(7)$$

The continuity of ψ implies that $\psi_1 = \psi_2$ at $x = 0$

$$\therefore \quad A + B = C \quad ...(8)$$

The continuity of $\frac{d\psi}{dx}$ implies that $\frac{d\psi_1}{dx} = \frac{d\psi_2}{dx}$ at $y = 0$

$$\therefore \quad p_1(A - B) = p_2C \quad ...(9)$$

Solving (8) and (9) we get

$$B = \frac{p_1 - p_2}{p_1 + p_2}A \quad ...(10)$$

and

$$C = \frac{2p_1}{p_1 + p_2}A \quad ...(11)$$

B and C represents the amplitudes of *reflected* and *transmitted* beams respectively in terms of the amplitude of the incident wave.

The *reflectance* and the *transmittance* at the potential discontinuity may be defined as follows:

Reflectance $$R = \frac{\text{magnitude of reflected current}}{\text{magnitude of incident current}}$$

Transmittance $$T = \frac{\text{magnitude of transmitted current}}{\text{magnitude of incident current}}$$

Case (ii) : $E < V_0$. When $E < V_0$, $p_2 = \sqrt{[2m(E - V_0)]}$ **is imaginary.**

Hence $$p_2 = i\sqrt{[2m(V_0 - E)]}$$

and $$p_2^* = -i\sqrt{[2m(V_0 - E)]} = -p_2$$

The probability current in this is given by

$$(J_x) = \frac{2\hbar}{2im}\left[\psi_2 * \frac{d\psi_2}{dx} - \psi_2 \frac{d\psi_2^*}{dx}\right]$$

$$= \frac{\hbar}{2im}\left[C^* e^{-ip_2x/\hbar}\left(\frac{ip_2}{\hbar}\right)Ce^{ip_2x/\hbar} - Ce^{ip_2x/\hbar}\left(\frac{-ip_2^*}{h}\right)C^* e^{-ip_2x/\hbar}\right]$$

Substituting $p_2^* = -p_2$ we get,

$$J_x = \frac{\hbar}{2im}\left[C^* e^{-ip_2x/\hbar}\left(\frac{ip_2}{\hbar}\right)Ce^{ip_2x/\hbar} - CC^*\left(\frac{ip_2}{h}\right)e^{-ip_2x/\hbar}e^{ip_2x/\hbar}\right]$$

Thus the transmitted current is zero.

$$T = \frac{\text{magnitude of transmitted current}}{\text{magnitude of incident current}} = 0$$

...(19)

$\therefore \quad T = 0$

By definition, $R + T = 1$

$\therefore \quad R = 1$

Hence Proved.

$$D = \frac{1}{2\alpha} = \frac{\hbar}{2\sqrt{2m(V-E)}} \qquad (17)$$

The penetration depth for particles incident on a potential step with $E < V$

(b) Estimate the penetration distance Ax for a small dust particle of radius r = 10^{-6} m and density $p = 10^4$ kg/m^5, moving at very low velocity $v = 10^{-2}$ m/sec, if the particle impinges on a potential step of height equal to twice its kinetic energy in the region left of the step. (2)

(c) A particle of mass m is confined within a one-dimensional field-free region between two perfectly elastic and impenetrable walls at $x = 0$ and $x = a$. Obtain the energy eigenvalues and normalized eigenfunctions for the particle. (5)

Ans.

First we will consider a free particle moving in 1D so $V(x) = 0$. The TDSE now reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

which is solved by the function

$$\psi = Ae^{ikx}$$

where

$$k = \pm\frac{\sqrt{2mE}}{\hbar}$$

A general solution of this equation is

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

where A and B are arbitrary constants. It can also be written in terms of sines and cosines as

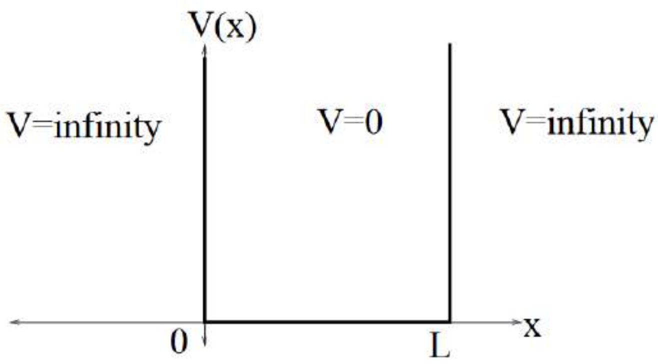
$$\psi(x) = C\sin(kx) + D\cos(kx)$$

The constants appearing in the solution are determined by the boundary conditions. For a free particle that can be anywhere, there is no boundary conditions, so k and thus $E = \hbar^2k^2/2m$ can take any values. The solution of the form e^{ikx} corresponds to a wave travelling in the +x direction and similarly e^{-ikx} corresponds to a wave travelling in the -x direction. These are eigenfunctions of the momentum operator. Since the particle is free, it is equally likely to be anywhere so $\psi^*(x)\psi(x)$ is independent of x. Incidently, it cannot be normalized because the particle can be found anywhere with equal probability.

Now, let us confine the particle to a region between $x = 0$ and $x = L$. To do this, we choose our interaction potential $V(x)$ as follows

$$\begin{aligned} V(x) &= 0 \quad \text{for } 0 \le x \le L \\ &= \infty \quad \text{otherwise} \end{aligned}$$

It is always a good idea to plot the potential energy, when it is a function of a single variable, as shown in Fig.1. The TISE is now given by



$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

First consider the region outside the box where $V(x) = \infty$. Since $V(x)\psi(x)$ has to be finite for finite energy, we insist that $\psi(x) = 0$. In other words, the particle cannot go outside the box.

In the box, we have the TISE given by the free particle term

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

now subjected to the boundary conditions given by

$$\psi(0) = \psi(L) = 0$$

Thus, we take the general solution

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

If we put $x = 0$, we get $\psi(0) = B = 0$. If we now put $\psi(L) = 0$, we get

$$A\sin(kL) = 0 \qquad \text{or} k = \frac{n\pi}{L}$$

where n is any integer. Clearly $n = 0$ is not valid as the wavefunction vanishes. Also, we see that changing the sign of n simply changes the sign of the wavefunction and as we said before, it does not produce a new wavefunction.

Thus the solution of the TISE that satisfies the boundary condition is written as

$$\psi_n(x) = A\sin(\frac{n\pi x}{L}) \qquad \text{where } n = 1, 2, 3, ...$$

The constant A is determined by the normalization condition to be $\sqrt{2/L}$. The corresponding energy is given by

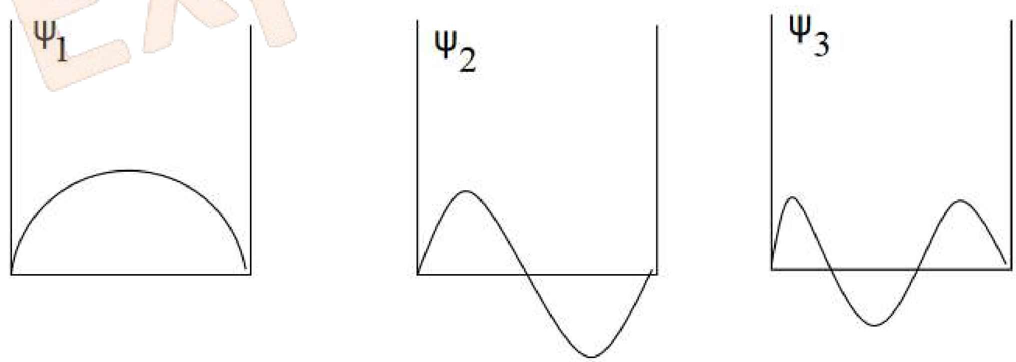
$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{n^2 h^2}{8mL^2}$$

so we have quantization of energy with

$$E_1 = \frac{\hbar^2 k^2}{2m} \qquad E_2 = \frac{4\hbar^2 k^2}{2m}$$

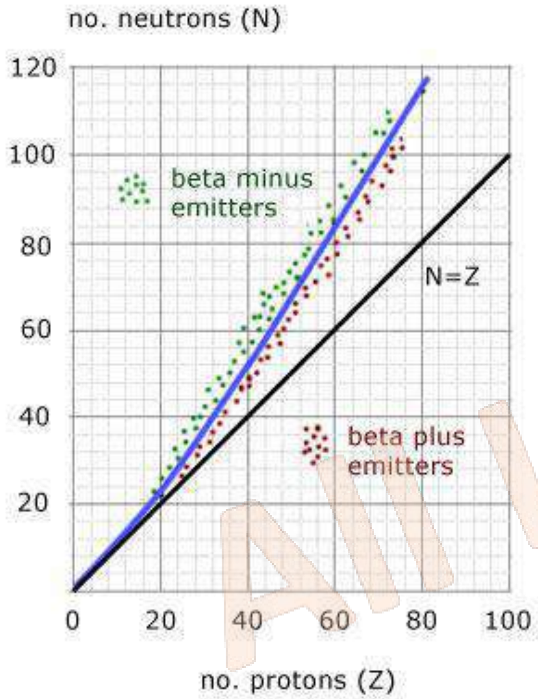
and so on. Notice that the lowest possible energy is not zero. This is referred to as zero point energy. The first few wavefunctions are plotted schematically as shown below. Notice that as the quantum number increases, the wavefunction becomes more oscillatory. For $n = 2$, the wavefunction is zero at the midpoint of the box $x = L/2$. This point is a node of this wavefunction. A node refers to a point (other than boundary points) where the wavefunction goes to zero. For the particle in a 1D box, we see that the number of nodes is equal to $n - 1$.

Though the particle in a 1D box is a simple model system, it illustrates the important features of a quantum mechanical description. It is a very useful first approximation to the behavior of π electrons in conjugated alkenes.



Q. 6. (a) Discuss the nature of nuclear force. Plot the N-Z graph for stable nuclei. Why do stable nuclei usually have more neutrons than protons? (3,1,1)

Ans. **Nuclear force -** The nuclear force (or nucleon–nucleon interaction or residual strong force) is a force that acts between the protons and neutrons of atoms. Neutrons and protons, both nucleons, are affected by the nuclear force almost identically. Since protons have charge +1 e, they experience an electric force that tends to push them apart, but at short range the attractive nuclear force is strong enough to overcome the electromagnetic force. The nuclear force binds nucleons into atomic nuclei. The nuclear force is powerfully attractive between nucleons at distances of about 1 femtometre (fm, or 1.0×10^{-15} metres), but it rapidly decreases to insignificance at distances beyond about 2.5 fm. At distances less than 0.7 fm, the nuclear force becomes repulsive. This repulsive component is responsible for the physical size of nuclei.



The combination of an even number of protons and an even number of neutrons is prefered by nature for stable nucleus. The odd-odd combination of stable nucleids is found only in the light elements. The number of even odd combinations is about the same.

For larger values of Z, the coulomb electrostatic repulsion becomes important, and the number of neutrons must be greater than the nunumber of protons.

(b) A sample of the isotope ^{131}I, which has a half-life of 8.04 days, has an activity of 5 mCi at the time of shipment. Upon receipt of the ^{131}I in a medical laboratory, its activity is 4.2 mCi. How much time has elapsed between the two measurements ? Calculate the mean life of sample.

(4,1)

Ans.

$$t_{1/2} = 8.04 \text{ days}$$

$$= 8.04 \times 24 \times 3600 \text{ sec.}$$

Mean life $= \frac{1}{\lambda}$, where

$$\lambda = \frac{0.693}{t_{1/2}}$$

$$\therefore \quad \text{mean life} = \frac{t_{1/2}}{0.693}$$

$$= \frac{8.04}{0.693}$$

$$= 11.602 \text{ days.}$$

Now, $1\ Ci = 3.7 \times 10^{10}$ disintegrations/sec.

$\therefore$ $1 \text{ m } Ci = 10^{-3} \times 3.7 \times 10^{10}$

$= 3.7 \times 10^{7}$ dis./sec.

Now,
$$A = \frac{0.693}{t_{1/2}} \times N$$

$$A_1 = 5 \times 3.7 \times 10^{7}$$

$$\therefore \quad 5 \times 3.7 \times 10^{7} = \frac{0.693}{8.04 \times 24 \times 3600} \times N_1$$

$$\therefore \quad N_1 = \frac{5 \times 3.7 \times 10^{7} \times 8.04 \times 24 \times 3600}{0.693}$$

$$= 1.854 \times 10^{14}$$

$$A_2 = 4.2 \times 3.7 \times 10^{7}$$

$$\therefore \quad N_2 = \frac{4.2 \times 3.7 \times 10^{7} \times 8.04 \times 24 \times 3600}{0.693}$$

$$= 1.557 \times 10^{14}$$

Now
$$\lambda = \frac{0.693}{t_{1/2}}$$

We have,
$$N = N_0\, e^{-\lambda t}$$

$$\therefore \quad \lambda t = \log_e \frac{N_0}{N}$$

$$\therefore \quad t = \frac{1}{\lambda} \log_e \frac{N_0}{N}$$

Here
$$N_0 = 1.854 \times 10^{14}$$
$$N = 1.557 \times 10^{14}$$

Please put these values and solve.

(c) Calculate binding energy per nucleon for (i) $_5B^{10}$ **with mass number 10.0161 a.m.u. (ii)** $_{14}Si^{29}$ **with mass number 28.9857** a.m.u. Using **this calculation find which atom is more stable. Given that** mass of **proton is 1.0081a.m.u. and mass of neutron is 1.0089** a.m.u. (4,1)

Ans. (i) The concerned element is $_5B^{10}$

No. of proton = 5

No. of neutrons = 10 – 5 = 5

Mass of 5 Protons + Mass of 5 Neutrons

$= 5 \times 1.0081 + 5 \times 1.0089$

$= 5.0405 + 5.0445$

$= 10.0850$ a.m.u

Mass no. = 10.0161 a.m.u

∴ mass defect, $\Delta m = 10.0850 - 10.0161$

$= 0.0689$ a.m.u

$$\text{Binding energy} = \frac{\text{Mass defect} \times 931.5}{\text{No. of Nucleous}} MeV$$

$$= \frac{0.0689 \times 931.5}{10}$$

$= 6.418\ MeV$

$= 6.418 \times 1.602 \times 10^{-19}$ J

$= 1.028 \times 10^{-18}$ J

(ii) Pl. try similarly for $_{14}Si^{29}$

Q. 7. (a) Define mean life and half life of a radioactive substance. **Derive expressions for mean life and half life in terms of** radioactive **constant.** (2,5)

Ans. Pl. Ref. I.Q. 3, and I.Q. 4, unit -6 page 124 -126.

(b) A nuclear reactor of 20% efficiency and an output of 700 MW **uses** $_{92}U^{235}$ **as fuel. Each fission reaction gives 200 MeV of** energy. **Calculate (i) the number of uranium atoms undergoing fission** per day **(ii) mass of uranium consumed by the reactor per day.** Given that **Avogadro's number is 6,023 ×10**^{23}. (3,2)

Ans. Output of reactor = 700 Mw

Efficiency of reactor = 20%

$$\therefore \text{Actual output} = \frac{700 \times 20}{100}$$

= 140 Mw

$= 140 \times 10^6$ w

$= 1.4 \times 10^8\ Js^{-1}$

Amount of energy needed/sec = 1.4×10^8 J

Amount of energy needed in one day

$$= 24 \times 3600 \times 1.4 \times 10^8 \text{ J}$$
$$= 1209.6 \times 10^{10} \text{J}$$
$$= 1.2096 \times 10^{13} \text{J}$$

Energy released per fission = 200 MeV

$$= 200 \times 1.602 \times 10^{-19} \text{J}$$
$$= 320.4 \times 10^{-19} \text{ J}$$
$$= 3.204 \times 10^{-17} \text{ J}$$

$$\text{No. of fusions} = \frac{1.209 \times 10^{13}}{3.204 \times 10^{-17}}$$
$$= 3.77 \times 10^{29}$$

This is also the no. of atoms of uranium.

Convert it into no. of moles and then find its mass to get the answer.

(c) What makes large nuclei (A>210) unstable? Why do they tend to stabilize by the emission of α-panicles rather than protons ? (1,2)

There are two forces operating inside the nucleus of an atom: (i) the electrostatic force which causes the repulsion between various protons and tends to make the nucleus unstable, and (ii) the nuclear force or strong force which causes attraction between protons and protons, protons and neutrons, and neutrons and neutrons, and tends to make the nucleus stable.

If an atom is big, then its nucleus will also be quite big. Now, in a big nucleus, the distances between the nuclear particles (protons and neutrons) are comparatively large due to which the nuclear force of attraction or strong force of attraction between them becomes weak. It should be noted that in a big nucleus, the electrostatic force of repulsion also decreases but this decrease is much less as compared to the decrease in nuclear force of attraction.

So, in a big nucleus, the electrostatic force of repulsion becomes slightly greater than the nuclear force of attraction because of which the big nucleus becomes unstable. From the above discussion we conclude that it is the atoms having large nucleus (having a large mass number) which are unstable. The uranium-235 atom is one such big atom having a large nucleus of mass number 235, which is unstable (235 is the total number of protons and neutrons in the nucleus of this uranium atom. It includes 92 protons and 143 neutrons).

Further, the α- particle has a high binding energy. To escape from a nucleus, a particle must have K.E. available. The α- particle is smaller than its constituent nucleons for such energy to be available. It has been found that the emission of an α -paricle is energetically possible. other decay modes would require energy to be supplied from outside the nucleus. The α-decay in ${}_{92}4^{232}$ is accompanied by the release of 5.4 MeV of energy . But 6.1 MeV of energy would have to be supplied from an external source if a proton is to be emitted.