Name of Paper	**: Mathematical Physics—II**
Name of the Course	**: B.Sc. (Hons.) Physics (CBCS)**
Semester	**: III**
Duration	**: 3 Hours**
Maximum Marks	**: 75**

2021

Attempt any four questions.
All questions carry equal marks.

Q. 1. Using method of separation of variables, solve 2-D equation $\frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$ **subjected to the conditions.**

$u(a, \theta, t) = 0,$

$u(r, \theta, 0) = 0$ **and**

$\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(r, \theta)$

Ans. Check similar questions for its solution. Q. 2. (Page 54) and Q. 5. (Page 56)

Q. 2. (*a*) Using one dimensional heat equation $\frac{\partial v}{\partial t} = h^2 \frac{\partial^2 v}{\partial x^2}$**, find the temperature V (*x*, *t*) in a bar of length which is perfectly insulated and whose ends are kept at temperature zero and the initial temperature is**

$f(x) = x$ **when** $0 < x < \frac{L}{2}$

$= L - x$ **when** $\frac{L}{2} < x < L$

Ans. Heat flow in : 1D

$$\frac{\partial v}{\partial t} = h^2 \frac{\partial^2 v}{\partial x^2} \qquad ...(1)$$

Use separation of variable method

$$V(x, t) = X(x)\, T(t) \qquad ...(2)$$

put (2) in (1) and divide by XT

$$= \frac{1}{h^2 T}\frac{dT}{dt} = \frac{d^2X}{dx^2} = -\lambda^2$$

$$\frac{1}{h^2 T}\frac{dT}{dt} = -\lambda^2$$

$$\frac{dT}{T} = -\lambda^2 h^2 dt$$

Integrating

$$\log T = -\lambda^2 h^2 t + \text{const}$$

$$T = Ae^{-\lambda 2h2t}$$

$$\frac{1}{x}\frac{\partial^2 x}{\partial x^2} = -l2$$

$$\frac{\partial^2 x}{\partial x^2} + \lambda^2 x = 0$$

$$X = B \sin \lambda x + C \cos \lambda x$$

$\therefore$

$$V(x, t) = XT$$

$$V(xt) = [AB \sin \lambda x + AC \cos \lambda x]e^{-\lambda 2h2t}$$

$$= [D \sin \lambda x + E \cos \lambda x]e^{-\lambda 2h2t}$$

At $x = 0$, $V = 0 \Rightarrow E = 0$

$$V(x, t) = D \sin \lambda x e^{-\lambda 2h2t}$$

At $x = L$, $V = 0 \Rightarrow D(\sin \lambda L)e^{-\lambda 2h2t} = 0$

$\Rightarrow$

$$\lambda L = n\pi$$

$$\lambda = \frac{n\pi}{L}$$

General Solution is

$$V(x, t) = \sum_{n=1}^{\infty}\left(D_n \sin \frac{n\pi}{L}\right)e^{\frac{-h^2 n^2 \pi^2 t}{L^2}}$$

To find Dn we have

$$V(x, t) = f(x) \text{ at } t = 0$$

so

$$f(x) = \sum D_n \sin \frac{n\pi x}{L} \qquad \text{...(3)}$$

A.T.Q. $f(x) = x \quad 0 \angle x \angle \frac{L}{2}$

$$= L - x \; \frac{\angle L}{2} \angle x \angle L$$

The requires the expansion of u in fourier series in the interval $x = 0$ and $x = \frac{L}{2}$ and from $x = \frac{L}{2}$ to $x = L$.

$$Dn = \frac{2}{L}\int_{O}^{L/2} x \sin\frac{n\pi x}{L}\,dx + \frac{2}{L}\int_{L/2}^{L} (L - x)\sin\frac{n\pi x}{L}\,dx$$

$$Dn = \frac{2}{L}\left[x\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) + \frac{\sin\frac{n\pi x}{L}}{\left(\frac{n\pi}{L}\right)^2}\right]_0^{L/2} + \frac{2}{L}\left[(L-x)\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - \frac{\sin\frac{n\pi x}{L}}{\left(\frac{n\pi}{L}\right)^2}\right]_{L/2}^{L}$$

solving, we get

$$Dn = \frac{2}{L}\left[0 + \left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi}{2}\right]$$

$$Dn = \frac{2L}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$Dn = 0 \text{ if } n = \text{even}$$

$$= \pm\frac{2L}{n^2\pi^2} \text{ if } n = \text{odd or } \frac{2L}{\pi^2}\frac{(-1)^{n+1}}{(2n-1)^2}$$

$$\therefore V(x, t) = \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)}{L}\pi x \; e^{\frac{-(2n-1)^2 n^2\pi^2 t}{L^2}}$$

(*b*) Show that $\int_a^a x^{m-1}(a - x)^{n-1}\,dx = a^{m+n-1}\beta(m, n)$

Ans. See Back Q. 2. Page 47

$$\beta(m, n) = \frac{r(m)r(n)}{r(m+n)}$$

Ans. See Back Q. 4. Page 40

Q. 3. Given f(x) = x for 0 < x < 2

(a) Find the Fourier cosine series of the function in half range.

Ans. Half range cosine series for $f(x)$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) \qquad \text{...(1)}$$

$$a_0 = \frac{1}{2}\int_0^2 f(x)dx$$

$$= \frac{1}{2}\int_0^2 x dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 = 1$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}dx$$

$$a_n = \int_0^2 f\cos\frac{n\pi x}{2}dx$$

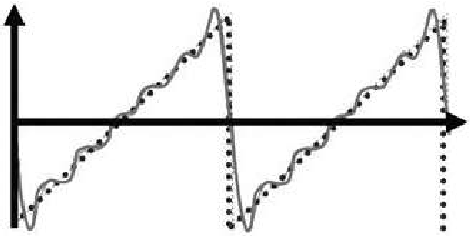
$$a_n = \left[\frac{x\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} - (1)\left(\frac{\frac{-\cos n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]^2$$

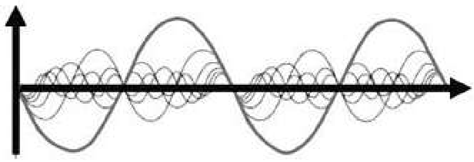
$$a_n = \frac{4}{n^2\pi^2}\left[(-1)^{n-1}\right]$$

from eq. (1)

$$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos\frac{n\pi x}{2}$$

Ans.





(*c*) Using Parseval's identity deduce that $\dfrac{n^4}{96} = \sum_{1}^{\infty} \dfrac{1}{n^4}$

Ans. Fourier cosine series function in part (*a*)

$$f(x) = 1 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2} \right] \cos \frac{n\pi x}{2}$$

$$X = 1 + \frac{4}{\pi^2} \left[\frac{-2}{1^2} \cos \frac{\pi x}{2} - \frac{2}{3^2} \cos \left(\frac{3\pi x}{2} \right) \right]$$

$$= 1 - \frac{8}{\pi^2} \left[\frac{1}{1^2} \cos \left(\frac{\pi x}{2} \right) + \frac{1}{3^2} \cos \left(\frac{3\pi x}{2} \right) + ... \right]$$

By Parseval's Identity

$$\frac{1}{2} \int_0^2 \left[f(x) \right]^2 dx = a_0^2 + \frac{1}{2} \sum a_n{}^2 \qquad ...(1)$$

Now,

$$\int_0^2 \left[f(x) \right]^2 dx = \int_0^2 x^2 dx = \frac{8}{3}$$

From eq. (1)

$$\frac{1}{2} \left(\frac{8}{3} \right) = 1 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{16}{x^4 \pi^4} \left[(-1)^n - 1 \right]^2$$

$$\frac{4}{3} = 1+\frac{1}{2}\left[\frac{64}{1^4\pi^4}+\frac{64}{3^4\pi^4}+.....\right]$$

$$\frac{4}{3}-1 = \frac{32}{\pi^4}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+....\right]$$

$$\frac{1}{3} = \frac{32}{\pi^4}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+....\right]$$

$$\frac{\pi^4}{96} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

Q. 4. (*a*) Find the complex form of the Fourier series of f(*x*) = exp (—*x*) for —1 ≤ *x* ≤ 1.

Ans. For a complex form of fourier series. The periodic function can be written in this way

$$= c_0 + \sum_{n=1}^{\infty} c_n e\frac{in\pi x}{l} + \sum_{n=1}^{\infty} c_{-n} e\frac{-in\pi x}{l}$$

Where

$$c_0 = \frac{a_0}{2} = \frac{1}{2}\frac{1}{l}\int_0^{2l} f(x)dx$$

$$c_n = \frac{1}{2l}\int_0^{2l} f(x)e^{\frac{-in\pi x}{1}}dx$$

$$c_{-n} = \frac{1}{2l}\int_0^{2l} f(x)e^{\frac{in\pi x}{1}}dx$$

In this way, the solution for exp (—x) will be

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n(1-in\pi)}{1+n^2p^2}\sinh 1.e^{in\pi x}$$

(*b*) Show that

(*i*) $(x^2 - 1)\, P'_n(x) = n(X P_n(x) - P_{n-1}(x))$

Ans. $P_n' - xP'_{n-1} = nP_{n-1}$ ...(1) [Recurrence Formula *III*]

$xPn' - P'n-1 = nPn$ ...(2) [Recurrence Formula *II*]

Multiplying (2) by x and subtracting from (1), we get

$$(1 - x^2)\, P_n' = n(P_{n-1} - xP_n)$$ **Proved.**

(*ii*) $x J'_n(x) = -n J_n(x) + x J_{n-1}(x)$

Ans. We have $J_n = \sum_{r=0}^{\infty} \frac{(-1)}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

Differentiating w.r.t. 'x' we get $J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{2}$

$$xJ_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} = \sum_{r=0}^{\infty} \frac{(-1)^r \left[(2n+2r)-n\right]}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r)-n}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} - n\sum_{r=0} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r 2}{r!\,\Gamma(n+r)}\left(\frac{x}{2}\right)^{n+2r} - nJ_n = x\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma\left[(n-1)+r+1\right]}\left(\frac{x}{2}\right)^{(n-1+2r)} - nJ_n$$

$= xJ_{n-1} - nJ_n$ **Proved.**

Q. 5. (*a*) Discuss the nature of singularity at $x = 1$ of the differential equation $(x^2 - 1)\, y + x\, y - y = 0$.

Ans. $(x^2 - 1)\, y^{11} + xy^1 - y = 0$

$$y^{11} + \frac{x}{(x^2-1)} y^1 - \frac{1}{(x^2-1)} y = 0$$

$$P(x) = \frac{x}{x^2-1} \quad ; Q(x) = \frac{-1}{x^2-1}$$

Now, find

$$\lim_{x \to x_0} (x - x_0)\, P(x) \text{ and } \lim_{x \to x_0} (x - x_0)\, Q(x)$$

$$\lim_{x \to 1} (x-1)\frac{x}{x^2-1} \Rightarrow \lim_{x \to !} x-1 \frac{x}{(x-1)(x+1)} = \frac{1}{2}$$

$$\lim_{x \to 1} (x-1)^2 \left(\frac{-1}{x^2-1}\right) = \frac{0}{0} = 0$$

$\therefore$ Since both limit exists, $x = 1$ is a regular singular point.

(*b*) Solve the differential equation $(x - x^2)\dfrac{\partial^2 y}{\partial x^2} + (1 - 5x)\dfrac{\partial y}{\partial x} - 4y = 0$ using

Ans. Given equation :

$$(x-x^2)^{11}+(1-5x)y^1-4y=0 \qquad ...(1)$$

$x=0$ is a singular pt.

then solution of eq. (1) be,

$$y=\sum_{n=0}^{\infty} a_n.x^{m+n}=a_0x^m+a_1x^{m+1}+a_2x^{m+2}+....$$

$$y=x^m\,[a_0+a_1x^1+a_2x^2+.....]$$

$$y^1=ma_0x^{m-1}+(m+1)\,a_1x^m+(m+2)a_2x^{m+1}+.....$$

$$y^{11}=m(m-1)\,a_0x^{m-2}+(m+1)\,ma_1x^{m-1}+(m+2)(m+1)\,a_2x^m+....$$

Substitute $y_1\,y^1$ and y^{11} in eq. (1),

$(x-x^2)\,[m(m-1)\,a_0x^{m-2}+(m+1)ma_1x^{m-1}+(m+2(m+1)a_2x^m+....]$

$+(1-5x)\,[ma_0x^{m-1}+(m+1)\,a_1x^m+(m+2)\,a_2x^{m+1}\,....]$

$-4\,[a_0x^m+a_1x^{m+1}+a_2x^{m+2}+....]=0$

The lowest power of x is $m-1$. So equating its coehicient to zero ⇒.

$$m\,(m-1)\,a_0+ma_0=0$$

$$m^2a_0-ma_0+ma_0=0$$

$$m^2a_0=0\Rightarrow m=0,0$$

∴ roots are same (repeated)

The coeff. of x^m equated to zero

$$-m\,(m-1)\,a_0+m(m+1)\,a_1-5ma_0+(m+1)\,a_1-4a_0=0$$

$$a_0\,[m^2+4m+4]=a_1\,[m^2+2m+2]$$

$$a_0\,(m+2)^2=a_1\,(m+1)^2$$

$$a_1=\frac{(m+2)^2}{(m+1)^2}a_0$$

The coeff of x^{m+1} equatied to zero,

$$-(m+1)\,ma_1+(m+2)\,(m+1)\,a_2+(m+2)\,a_2-5(m+1)\,a_1-4a_1=0$$

$$a_1\,[m^2+6m+a]=a_2\,[(m+2\,(m+2)]$$

$$a_1\,[(m+3)^2]=a_2\,[(m+2)^2]$$

$$a_2=\frac{(m+3)^2}{(m+1)^2}a_1=\frac{(m+3)^2}{(m+2)^2}\times\frac{(m+2)^2}{(m+1)^2}a_0$$

$$a2=\frac{(m+3)^2}{(m+1)^2}a_0$$

$$a_1 = 4a_0 \qquad ; \; a_0 = 9a_0$$

$$\therefore \quad y_1 = (y)_{m=0} = a_0 + 4a_0x^1 + 9a_0x^2 + \ldots$$

$$= a_0 + 4a_0x + 9a_0x^2 + \ldots$$

$$= a_0\,[1 + 4x + 9x^2 + \ldots]$$

$$\frac{\partial y}{\partial m} = mx^{m-1}\,[a_0 + a_1x^1 + a_2x^2 + \ldots]$$

$$\frac{\partial y}{\partial m}_{m=0} = 0$$

Hence the complete solution

$$y = c_1(y)_{m=m1} + c_2\frac{\partial y}{\partial m}_{m=m_1}$$

$$y = c_1a_0\,[1 + 4x + ax^2 + \ldots]$$

Q. 6. (*a*) Solve the differential equation in power series, $y'' + x\,y' + y = 0$.

Ans. $y'' + xy' + y = 0$

First, we have

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\Rightarrow \qquad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

so, from the given differential equation we have

$$y'' + xy' + y = 0$$

$$\Rightarrow \qquad \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \qquad \sum_{n=0}^{\infty} (n+2)(n+1)a_n + 2x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \qquad 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_n + 2x^n + \sum_{n=1}^{\infty} n a_n x^n + a_0 \sum_{n=1}^{\infty} a_n x^n = 0$$

$$\Rightarrow \quad 2a_2 + a_0 + \sum_{n=1}^{\infty}\left((n+2)(n+1)a_n + 2 + na_n + a_n\right)x^n = 0$$

$$\Rightarrow \quad 2a_2 + a_0 + \sum_{n=1}^{\infty}\left((n+2)(n+1)a_n + 2 + (n+1)a_n\right)x^n = 0$$

Since each coefficient of x^n must equal 0 for this equation to hold we have

$$2a_2 + a_0 = 0$$

$$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0 \Rightarrow a_{n+2} = \frac{-1}{n+2}a_n.$$

By induction we then have

$$a2n = \frac{(-1)^n}{2.4.....(2n)}$$

$$a2n+1 = \frac{(-1)^{n+1}}{1.3.....(2n-1)}$$

The coefficients a_0 and a_1 are arbitrary and we denote them by c_0 and c_1 respectively. Then we have

$$y = c_0\left(\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2.4...(2n)}\right) + c_1\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{1.3...(2n-1)}\right)$$

(*b*) Show that $\int_{-1}^{+1}[Pn(x)]dx = \frac{2}{2n+1}\delta_{mn}$.

Ans. The legendre polynomials satisfies the condition of orthosonality as

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

Proof : When $m \neq n$

$\because$ Legendre eq. in

$$(1 - x^2)\, y'' - 2xy' + n(n+1)\, y = 0$$

Let $P_m(x)$ and $P_n(x)$ be legendre polynomials satisfying respectively legendre's eq.

$$(1 - x^2)\, P''_m - 2x\, P'_m + m(m+1)\, P_m = 0 \quad ...(1)$$

$$(1 - x^2)\, P''_n - 2x\, P'_n + n(n+1)\, P_n = 0 \quad ...(2)$$

[Pn × 1] — [Pm × 2] gives,

$$\Rightarrow \frac{d}{dx}[(1-x^2)(P_nP'_m - P_mP'_n)] + P_mP_n[m(m+1)-n(n+1)] = 0$$

$$\frac{d}{dx}(P_nP'_m - P_mP'_n) = P_nP''_m + P'_mP'_n - (P_mP''_n + P'_mP'_n)$$

$$\Rightarrow [m(m+1)-n(n+1)]\, P_mP_n = -\frac{d}{dx}[(1-x^2)(P_nP'_m - P_mP'_n)]$$

Integrating both sides w.r.t. x from -1 to 1, $[m(m+1)-n(n+1)]\int_{-1}^{1} P_m(x)P_n(x)dx$

$$= -\int_{-1}^{1}\frac{d}{dx}\left[(1-x^2)(P_nP'_m - P_mP'_n)\right]dx$$

$$\Rightarrow \qquad \int_{-1}^{1} P_m(x)P_n(x)dx = \frac{-1}{\left[m(m+1)-n(n+1)\right]}(1-x^2)(P_nP'_m - P_mP'_n)_{-1}^{1}$$

$$\Rightarrow \qquad \int_{-1}^{1} P_m(x)P_n(x)dx = 0 \text{ (if } m \neq n)$$

When $m = n$

$\because$ We know

$$(1-2xt)+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n \qquad \text{...(3)}$$

$$(1-2xt)+t^2)^{-1/2} = \sum_{m=0}^{\infty} P_m(x)t^m \qquad \text{...(4)}$$

(3) × (4) gives,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x)P_n(x)t^{m+n}dx = [1-2xt+t^2]^{-1}$$

Integrating both sides w.r.t. x from -1 to1.

$$\sum_{n=0}^{\infty}\int_{-1}^{1}\left[P_n(x)\right]^2 t^{2n}dx = \int_{-1}^{1}\frac{1}{(1-2xt+t^2)}dx$$

$$= \left|\frac{\log(1-2xt+t^2)}{-2t}\right|_{-1}^{1}$$

$$= \frac{-1}{2t}\left[\log(1-2t+t^2)-\log(1+2t+t^2)\right]$$

$$= \frac{-1}{2t}\left[\log(1-t)^2-\log(1+t)^2\right]$$

$$= \frac{1}{t}\left[\frac{1}{2}\log(1+t)^2-\frac{1}{2}\log(1-t)^2\right]$$

$$= \frac{1}{t}\left[\log(1+t)-\log(1-t)\right]$$

$$\sum_{n=0}^{\infty}\int_{-1}^{1}\left[P_n(x)\right]^2 t^{2n}dx = \frac{1}{t}\left[\log(1+t)-\log(1-t)\right]$$

$$\text{Log }(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\infty$$

$$\text{Log }(1-x) = -\left(x-\frac{x^2}{2}+\frac{x^3}{3}+\ldots\infty\right)$$

$$\sum_{n=0}^{\infty}\int_{-1}^{1}\left[P_n(x)\right]^2 t^{2n}dx = \frac{1}{t}\left[t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+t+\frac{t^2}{2}+\frac{t^3}{3}+\frac{t^4}{4}+\ldots\right]$$

$$= \frac{1}{t}\left[2t+\frac{2t^3}{3}+\frac{2t^5}{5}+\ldots\right]$$

$$= \frac{2t}{t}\left[1+\frac{t^2}{3}+\frac{t^4}{5}+\ldots.+\frac{t^{2n}}{2n+1}\right]$$

$$= 2\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}\right)t^{2n}$$

Equating the coeficient of t^{2n} on both sides,

$$\int_{-1}^{1}\left[P_n(x)\right]^2 dx = \frac{2}{2n+1} \qquad (\text{If } m=n)$$

Proved.