Mathematical Physics - I

Solved Paper – 2017

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1. Do any *five* of the following : $5\times3=15$

(*a*) Two sides of a triangle are formed by the vectors :

$$\vec{A} = 3\hat{i} + 6\hat{j} - 2\hat{k} \text{ and } \vec{B} = 4\hat{i} - \hat{j} + 3\hat{k}$$

Determine the angle between these two sides and length of the third side.

(b) Show that the area bounded by a simple closed curve C is given by :

$$\frac{1}{2}\oint_C (x\,dy - y\,dx).$$

(c) If $\vec{a}$ is a constant vector, then prove that :

$$\vec{\nabla} \times \left(\vec{a} \times \vec{r}\right) = 2\vec{a}.$$

(d) Solve :

$$\iint_R \sqrt{x^2 + y^2}\,dx\,dy$$

where, R is the region bounded by the circle, $x^2 + y^2 = 9$.

(e) Check whether the following functions are linearly independent or not :

$$e^x,\ x\,e^x.$$

(f) Solve the differential equation :

$$(b^2 + 2xy + y^2)dx + (x + y)^2 dy = 0.$$

(g) Form a differential equation whose solution is given by :

$$y = A\, e^{2x} + B\, e^{3x}.$$

(h) Solve :

(i) $\int_0^5 \delta(x - \pi) \cos 2x \, dx$

(ii) $\int_{-2}^{2} \left[x^2 + \log x\right] \delta(x - 1)\, dx.$

2. (a) Find the constants 'a' and 'b' so that the surface $ax^2 - byz = (a + 2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$. 4

(b) If $\vec{A} = r^n \vec{r}$, then find the value of n for which $\vec{A}$ is solenoidal. 5

(c) Prove that : 6

$$\vec{\nabla} \cdot \left[r\, \vec{\nabla}\left(\frac{1}{r^3}\right)\right] = \frac{3}{r^4}$$

where, $r = \sqrt{x^2 + y^2 + z^2}$.

3. (a) Prove that :

$$\vec{A} \times \left(\vec{\nabla} \times \vec{A}\right) = \frac{1}{2}\vec{\nabla} A^2 - \left(\vec{A}.\vec{\nabla}\right)\vec{A}.$$

(b) Evaluate $\iint_S \left(\vec{A}.\hat{n}\right) dS$, where :

$$\vec{A} = y\hat{i} + 2x\hat{j} - z\hat{k}$$

And,

S is the surface of the plane, $2x + y = 6$ in the first octant cut-off by the plane, $z = 4$. 9

4. (a) Prove that : 5

$$\oiint_S r^5 \hat{n}\, dS = \iiint_V 5r^3 \vec{r}\, dV$$

where, simple closed surface S encloses volume V.

(b) Write the mathematical form of Gauss's Divergence theorem and hence verify it for $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$, where S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. 1,9

5. (a) Evaluate :

$$\iiint_V (2x + y)\, dV,$$

where, V is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes, $x = 0$, $y = 0$, $y = 2$, $z = 0$. 6

(b) Derive an expression for curl of a vector field in orthogonal curvilinear coordinates. Express it in cylindrical coordinates. 7,2

6. Solve the differential equations :

(a) $(x^2y - 2xy^2)dx - (x^3 - 2x^2y)dy = 0$ 7

(b) $(D^2 + 1)y = \text{cosec } x \quad \left(D = \frac{d}{dx}\right).$ 8

7. (a) Solve the differential equation : 7

$$(D^2 - 6D + 8)y = (e^{2x} - 1)^2.$$

(b) Using method of variation of parameters, solve the differential equation : 8

$$(D^2 + 4)y = x \sin 2x.$$

8. (a) Solve the differential equation : 7

$$(D^2 - 4D + 3)y = xe^{2x}.$$

(b) Using method of undetermined coefficients, solve the differential equation : 8

$$(D^2 - 1)y = e^x + 2x.$$

Que: 1(a)

$$\vec{A} = 3\hat{i} + 6\hat{j} - 2\hat{k}$$

and $$\vec{B} = 4\hat{i} - \hat{j} + 3\hat{k}$$

Angle between two vectors can be calculated as,

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$$

$$= \frac{(3\hat{i}+6\hat{j}-2\hat{k})\cdot(4\hat{i}-\hat{j}+3\hat{k})}{\sqrt{9+36+4}\cdot\sqrt{16+1+9}}$$

$$= \frac{12 - \cancel{6} + \cancel{6}}{7\sqrt{26}}$$

$$\cos\theta = \frac{12}{7\sqrt{26}} \Rightarrow \theta = \cos^{-1}\left(\frac{12}{7\sqrt{26}}\right)$$

The law of cosine states,

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where θ is angle between two sides of magnitude a, b respectively.

So length of third side can be calculated as,

$$c^2 = 49 + 26 - 2 \times \cancel{7} \times \cancel{\sqrt{26}}\ \frac{12}{\cancel{7}\cancel{\sqrt{26}}}$$

$$c^2 = 49 + 26 - 12$$

$$c^2 = 63$$

$$c = \sqrt{63}$$

Que : 1(b)

Sol^n By Green's theorem in the plane

$$\oint_C M\,dx + N\,dy = \int_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Let $M = -y$, $N = x$

So, $\int x\,dy - y\,dx = 2\int dx\,dy$

So, area $= \frac{1}{2}\oint (x\,dy - y\,dx)$

If we choose $M = 0$, $N = x$

then $\oint x\,dy = \int dx\,dy$

or area $= \oint x\,dy$

$$= \int_S \left\{ \frac{\partial(-2xy)}{\partial x} - \frac{\partial(x^2+y^2)}{\partial y} \right\}$$

$$= -4\int y\,dx\,dy$$

$$= -2y^2\Big|_{y=0}^{b} x\Big|_{-a}^{a} = -4ab^2$$

Que: 1 (c)

$\vec{a}$ is a constant vector

then assuming $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\,[a_2 z - a_3 y] - \hat{j}\,[a_1 z - a_3 x] + \hat{k}\,[a_1 y - a_2 x]$$

Now $\vec{\nabla} \times (\vec{a} \times \vec{r})$ is given by,

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (a_2 z - a_3 y) & (a_3 x - a_1 z) & (a_1 y - a_2 x) \end{vmatrix}$$

$$= \hat{i}\,[a_1 + a_1] - \hat{j}\,[-a_2 - a_2] + \hat{k}\,[a_3 + a_3]$$

$$= 2\vec{a}$$

Hence $\vec{\nabla} \times (\vec{a} \times \vec{r}) = 2\vec{a}$

Que: 1 (d) $\iint \sqrt{x^2+y^2}\, dx\, dy$

where R is the region bounded by the circle, $x^2+y^2=9$

in Polar co-ordinates,

$r = \sqrt{x^2+y^2}$, $x = r\cos\phi$, $y = r\sin\phi$

$dx\, dy = r\, dr\, d\phi$

So the given integral becomes,

$$\iint_R r^2\, dr\, d\phi$$

$$= \int_{r=0}^{3} \int_{\phi=0}^{2\pi} r^2\, dr\, d\phi$$

$$= 2\pi \left. \frac{r^3}{3} \right|_0^3$$

$$= 2\pi \times \frac{27}{3}$$

$$= 18\pi$$

Que: 1(e) e^x, $x e^x$

Solution: functions are linearly dependent when their Wronskian is ~~not~~ zero.

Let y_1 and y_2 be two differentiable functions. The Wronskian (y_1, y_2), associated to y_1 and y_2 is the function

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$= y_1(x)\, y_2'(x) - y_1'(x)\, y_2(x)$$

Now

$$W(e^x, x e^x)_{x=0} = \begin{vmatrix} e^x & x e^x \\ e^x & x e^x + e^x \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 1 - 0 = 1$$

So e^x and $x e^x$ are linearly independant functions.

Que: 1(f) $(b^2 + 2xy + y^2)\,dx + (x+y)^2\,dy = 0$

It can be written as

$$M\,dx + N\,dy = 0$$

Equation is said to be an exact differential equation when it satisfies,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Here $M = b^2 + 2xy + y^2$ and $N = 2xy + x^2 + y^2$

$$\frac{\partial M}{\partial y} = 2(x+y) \quad , \quad \frac{\partial N}{\partial x} = 2(x+y)$$

So given equation is an exact differential equation where solution can be written as

$$\int M\,dx + \int (\text{terms of } N \text{ is not containing } x)\,dy = C$$

$$\int (b^2 + 2xy + y^2)\,dx + \int y^2\,dy = C$$

$$b^2 + x^2 y + x y^2 + \frac{y^3}{3} = C$$

Que: 1(g)

$$y = A e^{2x} + B e^{3x} \qquad (1)$$

$$y_1 = 2A e^{2x} + 3B e^{3x} \qquad (2) \qquad \left[y_1 = \frac{dy}{dx}\right]$$

$$y_2 = 4A e^{2x} + 9B e^{3x} \qquad (3) \qquad \left[y_2 = \frac{d^2y}{dx^2}\right]$$

Eliminating A and B from above two equations we get,

$$\begin{vmatrix} e^{2x} & e^{3x} & -Y \\ 2e^{2x} & 3e^{3x} & -Y_1 \\ 4e^{2}x & 9e^{3x} & -Y_2 \end{vmatrix} = 0$$

So $$e^{2x}\, e^{3x} \begin{vmatrix} 1 & 1 & -Y \\ 2 & 3 & -Y_1 \\ 4 & 9 & -Y_2 \end{vmatrix} = 0$$

$$1(-3Y_2 + 9Y_1) - 1(-2Y_2 + 4Y_1) - Y(18 - 12) = 0$$

$$-3Y_2 + 9Y_1 + 2Y_2 - 4Y_1 - 6Y = 0$$

$$-Y_2 + 5Y_1 - 6Y = 0$$

$$-\frac{d^2Y}{dx^2} + 5\frac{dY}{dx} - 6Y = 0$$

Que :- 1(h) The integral of Dirac delta is given by

$$\int_{-\infty}^{\infty} f(t)\, \delta(t - T)\, dt = f(T)$$

(i) $$\int_0^5 \delta(x - \pi) \cos 2x \, dx$$

$\because \quad 0 < \pi < 5$, limits can be replaced by $-\infty$ to ∞

$$\int_{-\infty}^{\infty} \delta(x-\pi) \cos 2x \, dx$$

$$= \cos 2\pi$$

$$= 1$$

(ii) $$\int_{-2}^{2} [x^2 + \log x] \, \delta(x-1) \, dx$$

$-2 < 1 < 2$

$$= \int_{-\infty}^{\infty} [x^2 + \log x] \, \delta(x-1) \, dx$$

$$= (1)^2 + \log 1$$

$$= 1$$

Que:- 2(a)

Sol^n: Here we have,

$$ax^2 - byz = (a+2)x \quad (1)$$

$$4x^2y + z^3 = 4 \quad (2)$$

Normal to the surface (1)

$$= \nabla [ax^2 - byz - (a+2)x]$$

$$= \left[i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right] \left[ax^2 - byz - (a+2)x \right]$$

$$= \hat{i}(2ax - a - 2) + \hat{j}[-bz] + \hat{k}[-by]$$

Normal at $(1,-1,2) = \hat{i}\ (2a - a - 2) - \hat{j}\ (-2b) + \hat{k} b$ (3)

$= \hat{i}\ (a-2) + \hat{j}\ z(2b) + \hat{k}\ b$

Normal at the surface (2)

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(4x^2y + z^3 - 4\right)$$

$$= \hat{i}\ (8xy) + \hat{j}\ (4x^2) + \hat{k}\ (3z^2)$$

Normal at the point $(1,-1,2) = -8\hat{i} + 4\hat{j} + 12\hat{k}$ (4)

Since (3) and (4) are orthogonal so,

$$[\hat{i}(a-2) + \hat{j}\ z(2b) + \hat{k}b] \cdot [-8\hat{i} + 4\hat{j} + 12\hat{k}] = 0$$

$$-8\ (a-2) + 4\ (2b) + 12b = 0$$

$$\Rightarrow -8a + 16 + 8b + 12b = 0 \Rightarrow 4(-2a + 5b + 4) = 0$$

$$\Rightarrow 2a - 5b = 4 \quad - (5)$$

Point $(1,-1,2)$ will satisfy (1)

$$\therefore\ a(1)^2 - b(-1)(2) = (a+2)(1)$$

$$\Rightarrow a + 2b = a + 2$$

$$\Rightarrow b = 1$$

Putting $b = 1$ in (5), we get

$$2a - 5 = 4 \Rightarrow a = \frac{9}{2}$$

Hence $a = \frac{9}{2}$ and $b = 1$

Q 2(b)

Sol: Refer chapter 3 ques 16

Que: 2 (c) To prove:

$$\vec{\nabla} \cdot \left[r \vec{\nabla} \left(\frac{1}{r^3} \right) \right] = \frac{3}{r^4}$$

$$\vec{\nabla} \left(\frac{1}{r^3} \right) = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \left(x^2 + y^2 + z^2 \right)^{-3/2}$$

$$= \frac{-3}{2} \frac{2x}{(x^2+y^2+z^2)^{5/2}} \hat{i} - \frac{3}{2} \frac{2y}{(x^2+y^2+z^2)^{5/2}} - \frac{3}{2} \frac{2z}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{-3 \left(x\hat{i} + y\hat{j} + z\hat{k} \right)}{(x^2+y^2+z^2)^{5/2}}$$

$$= -\frac{3 \vec{r}}{r^5}$$

Now $r \vec{\nabla} \left(\frac{1}{r^3} \right) = \frac{-3 \vec{r}}{r^4}$

Finally $\vec{\nabla} \cdot \left(\frac{-3}{r^4} \vec{r} \right)$ is given by,

$$\left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \left(\frac{-3 \left(x\hat{i} + y\hat{j} + z\hat{k} \right)}{(x^2+y^2+z^2)^2} \right]$$

$$= -3 \left[\frac{(x^2+y^2+z^2)^2 - 2 (x^2+y^2+z^2] [x^2+y^2+z^2]}{(x^2+y^2+z^2]^4} \right]$$

$$= \frac{-3[-(x^2+y^2+z^2]^2}{[x^2+y^2+z^2]^4} = \frac{+3}{[x^2+y^2+z^2]^2} = \frac{3}{r^4}$$

Hence, $\vec{\nabla} \cdot \left[r \vec{\nabla} \left(\frac{1}{r^3} \right) \right] = \frac{3}{r^4}$

Que: 3(a) $\vec{A} \times (\vec{\nabla} \times \vec{A}) = \frac{1}{2}\vec{\nabla} A^2 - (\vec{A}\cdot\vec{\nabla})\vec{A}$

LHS $\vec{A} \times (\vec{\nabla} \times \vec{A})$ can be evaluated part by part, let $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{A} = \hat{i}\underbrace{\left[\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right]}_{B_1} - \hat{j}\underbrace{\left[\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right]}_{B_2} + \hat{k}\underbrace{\left[\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right]}_{B_3}$$

$$\vec{A} \times (\vec{\nabla} \times \vec{A}) = \hat{i}\,[A_y B_3 - A_z B_2] - \hat{j}\,[A_x B_3 - A_z B_1] + \hat{k}\,[A_x B_2 - A_y B_1]$$

$$[\vec{A} \times (\vec{\nabla} \times \vec{A})]_x = A_y\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) - A_z\left[\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right]$$

$$[\vec{A} \times (\vec{\nabla} \times \vec{A})]_y = -A_x\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) + A_z\left[\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right]$$

$$[\vec{A} \times (\vec{\nabla} \times \vec{A})]_z = A_x\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right) - A_y\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right)$$

$$[\vec{A} \times (\vec{\nabla} \times \hat{A})]_x = \frac{1}{2}\frac{\partial}{\partial x}(A_y^2 + A_z^2) - A_y\frac{\partial A_x}{\partial y} - A_z\frac{\partial A_x}{\partial z}$$

$$= \frac{1}{2}\frac{\partial}{\partial x}(A_y^2 + A_z^2 + A_x^2) - \frac{1}{2}\frac{\partial}{\partial x}A_x^2 - A_y\frac{\partial}{\partial y}A_x - A_z\frac{\partial A_x}{\partial z}$$

$$= \left[\frac{1}{2}\vec{\nabla} A^2\right]_x - |(\vec{A}\cdot\vec{\nabla})\vec{A}|_x$$

Similarly we can write for y and z components.

Hence $\vec{A} \times (\vec{\nabla} \times \vec{A}) = \frac{1}{2}\vec{\nabla} A^2 - (\vec{A}\cdot\vec{\nabla})\vec{A}$.

Que: 3(b) $\iint_S (\vec{A}\cdot\hat{n})\, ds$

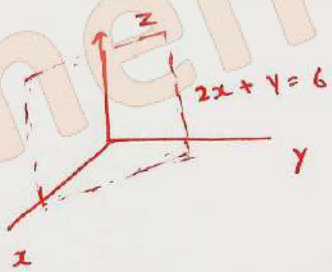
where $\vec{A} = y\hat{i} + 2x\hat{j} - z\hat{k}$

and S is the surface of the plane, $2x+y=6$ in the first octant cut-off by the plane, $z=4$

$$\iint \vec{A}\cdot\hat{n}\, ds = \iint_R \vec{A}\cdot\hat{n}\, \frac{dx\, dz}{|\hat{n}\cdot\hat{j}|}$$

A normal to $2x+y=6$ is $\nabla(2x+y-6) = 2\hat{i}+\hat{j}$

Then the unit normal to S as shown in the adjoining figure is



$$\hat{n} = \frac{2\hat{i}+\hat{j}}{\sqrt{4+1}} = \frac{2\hat{i}+\hat{j}}{\sqrt{5}}$$

$$\vec{A}\cdot\hat{n} = (y\hat{i}+2x\hat{j}-z\hat{k})\cdot\frac{(2\hat{i}+\hat{j})}{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}}(2y+2x)$$

$$n\cdot\hat{j} = \frac{(2\hat{i}+\hat{j})}{\sqrt{5}}\cdot\hat{j} = \frac{1}{\sqrt{5}}$$

$$\iint \vec{A}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\hat{j}|} = \iint (2y+2x)\, dx\, dz\, \frac{\cancel{\sqrt{5}}}{\cancel{\sqrt{5}}}$$

$$= 2\int_{z=0}^{4}\int_{x=0}^{3} (y+x)\, dx\, dz = 4\times 2\int_{x=0}^{3} (6-2x+x)\, dx$$

$$= 8\int_0^3 (6-x)\, dx = 8\left[6x - \frac{x^2}{2}\right]_0^3$$

$$= 81$$

Que: 4(a) $\oiint_S r^5 \, \hat{n} \, dS$

Using Gauss Divergence Theorem,

$$\oiint_S r^5 \, \hat{n} \, dS = \iiint_V \vec{\nabla} \cdot r^5 \, dV$$

$$\vec{\nabla} \cdot r^5 = \left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right] \cdot \left[x^2+y^2+z^2\right]^{5/2}$$

$$= \frac{5}{\cancel{2}} \, \cancel{2} \, (x\hat{i} + y\hat{j} + z\hat{k}) \, (x^2+y^2+z^2)^{3/2}$$

$$= 5 \, \vec{r} \, r^3$$

Hence $\iiint_V \vec{\nabla} \cdot r^5 \, dV = \iiint_V 5 r^3 \, \vec{r} \, dV$

where, simple closed surface S encloses volume V.

Q 4 (b) Ans

Refer ques 40 and 41 of chapter 3

Que : 5 (a)

$$\iiint_V (2x+y)\, dV$$

where V is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes, $x = 0,\ y = 0,\ y = 2,\ z = 0$

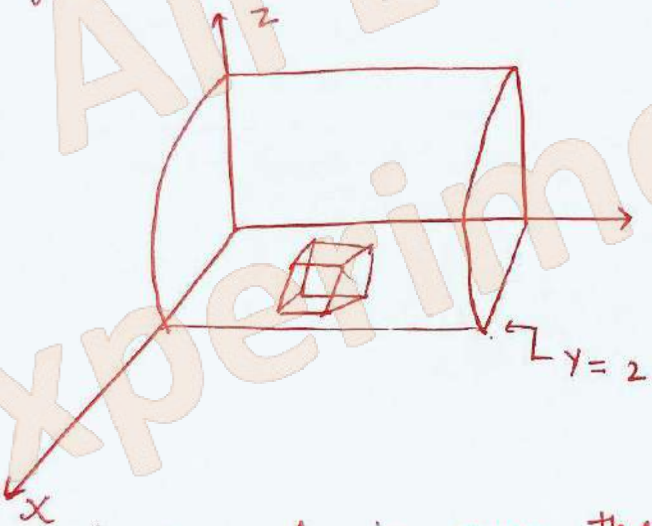
Sol^n

$$\int (2x+y)\, dx\, dy\, dz$$

$$= \int (2x+y)\, dx\, dy \int_{z=0}^{z=4-x^2} dz$$

$$= \int (2x+y)\, dx\, dy\, (4-x^2)$$

$$= \int (2x+y)(4-x^2)\, dx\, dy$$



The surface integral is over the area bounded by $x = 0,\ y = 0,\ x = 2,\ y = 2$

$$\int (2x+y)\, dV = \int 2x(4-x^2)\, dx\, dy + \int (4-x^2)\, y\, dx\, dy$$

$$= 2\int (8x - 2x^3)\, dx + \frac{2^2}{2}\int (4-x^2)\, dx$$

$$= 2\left(4x^2 - \frac{x^4}{2}\right)\Bigg|_{x=0}^{2} + 2\left(4x - \frac{x^3}{3}\right)\Bigg|_{x=0}^{x=2}$$

$$= 2(16-8) + 2\left(8 - \frac{8}{3}\right)$$

$$= 16 + 16 \times \frac{2}{3} = 16\left(\frac{2}{3}+1\right) = \frac{80}{3}$$

Que 5(b) Refer Q2 chapter 4

Ques 6

Que: 6(a) $(x^2y - 2xy^2)\,dx - (x^3 - 2x^2y)\,dy = 0$

$$\frac{dy}{dx} = \frac{x^2y - 2xy^2}{x^3 - 2x^2y} \quad -(i)$$

All terms in the given equation are of same degree. Putting $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{vx^3 - 2x^3v^2}{x^3 - 2x^3v}$$

$$v + x\frac{dv}{dx} = \frac{x^3(v - 2v^2)}{x^3(1-v)}$$

$$v + x\frac{dv}{dx} = \frac{v(1-2v)}{(1-v)}$$

$$x\frac{dv}{dx} = \frac{v(1-2v)}{1-v} - v = \frac{\cancel{v} - 2v^2 - \cancel{v} + v^2}{(1-v)}$$

$$x\frac{dv}{dx} = -\frac{v^2}{(1-v)}$$

$$\frac{dx}{x} = -\frac{(1-v)}{v^2}\,dv$$

$$\ln x = -\frac{dv}{v^2} + \frac{1}{v}\,dv$$

$$\ln x = \frac{1}{v} + \ln v + C$$

$$\ln \frac{x}{v} = \frac{1}{v} + c$$

Putting $v = \frac{y}{x}$

$$\ln \frac{x^2}{y} = \frac{x}{y} + C$$

$$\ln \frac{x^2}{y} - \frac{x}{y} + K = 0$$

where K is some constant.

Q 6 (b) Ans.
Refer Q: 34 chapter 1

Que: 7(a) $(D^2 - 6D + 8)y = (e^{2x} - 1)^2$

The auxiliary equation is

$$m^2 - 6m + 8 = 0$$

$$m^2 - 4m - 2m + 8 = 0$$

$$m(m-4) - 2(m-4) = 0$$

$$\Rightarrow \quad m = 4, \quad m = 2$$

$$CF = c_1 e^{4x} + c_2 e^{2x}$$

$$PI = \frac{1}{D^2 - 6D + 8}(e^{2x} - 1)^2 = \frac{1}{D^2 - 6D + 8} e^{4x} + 1 - 2e^{2x}$$

$$= \frac{1}{D^2 - 6D + 8} e^{4x} + \frac{1}{D^2 - 6D + 8} 1 - \frac{2}{D^2 - 6D + 8} e^{2x}$$

$$= \frac{1}{(4)^2 - 6(4) + 8} e^{4x} + \frac{1}{0^2 - 6(0) + 8} 1 - \frac{2}{(2)^2 - 6(2) + 8} e^{2x}$$

$\hookrightarrow f(a) = 0$ $\qquad$ $f(a) = 0$

So, $\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}$

$$PI = \frac{x}{2D - 6} e^{4x} + \frac{1}{8} - \frac{2x}{2D - 6} e^{2x}$$

$$= \frac{x}{2(4) - 6} e^{4x} + \frac{1}{8} - \frac{2x}{2(2) - 6} e^{2x}$$

$$= \frac{x}{2} e^{4x} + \frac{1}{8} - \frac{2x}{-2} e^{2x}$$

$$= \frac{x}{2} e^{4x} + \frac{1}{8} + x e^{2x}$$

So complete solution is, $y = c_1 e^{4x} + c_2 e^{2x} + \frac{x}{2} e^{4x} + x e^{2x} + \frac{1}{8}$

Que: 7(b) $(D^2+4)y = x\sin 2x$

$$m^2+4=0$$
$$m=\pm 2i$$

$CF = A\cos 2x + B\sin 2x$

Now using variation of parameters method,

$$y_1 = \cos 2x \quad \text{and} \quad y_2 = \sin 2x$$

$$X = x\sin 2x$$

$$u = \int \frac{-y_2 X}{y_1 y_2' - y_1' y_2}\,dx \quad \text{and} \quad v = \int \frac{y_1 X}{y_1 y_2' - y_1' y_2}\,dx$$

$$u = \int \frac{-\sin 2x \; x \sin 2x}{2\cos 2x \cos 2x + 2\sin 2x \sin 2x}\,dx$$

$$u = \frac{1}{2}\int -\sin^2 2x \; x \,dx$$

$$u = -\frac{1}{2}\int x\frac{(1-\cos 4x)}{2}\,dx$$

$$u = -\frac{1}{4}\int x\,dx + \int \frac{1}{4} x \cos 4x\,dx$$

$$= -\frac{x^2}{8} + \left[\frac{x\sin 4x}{16}\right] - \int \frac{\sin 4x}{16}\,dx$$

$$= \frac{-x^2}{8} + \frac{x\sin 4x}{16} + \frac{\cos 4x}{64}$$

$$V = \frac{\int x \cos 2x \sin 2x \, dx}{2\cos 2x \cos 2x + 2\sin 2x \sin 2x}$$

$$V = \frac{1}{2} \int x \sin 2x \cos 2x \, dx$$

$$V = \frac{1}{2 \times 2} \int x \sin 4x \, dx$$

$$V = \frac{1}{4} \int x \sin 4x \, dx$$

$$V = \frac{1}{4}\left[\left[-x \frac{\cos 4x}{4}\right] + \int \frac{\cos 4x}{4} dx\right]$$

$$V = -\frac{x}{16} \cos 4x - \frac{\sin 4x}{64}$$

$$PI = u y_1 + v y_2$$

$$= \left(-\frac{x^2}{8} + \frac{x \sin 4x}{16} + \frac{\cos 4x}{64}\right) \cos 2x +$$

$$\left(-\frac{x}{16} \cos 4x - \frac{\sin 4x}{64}\right) \sin 2x$$

$$Y = CF + PI$$

$$= A \cos 2x + B \sin 2x + \cos 2x \left(-\frac{x^2}{8} + \frac{x \sin 4x}{16} + \frac{\cos 4}{64}\right.$$

$$\left. + \sin 2x \left(-\frac{x}{16} \cos 4x - \frac{\sin 4x}{64}\right)\right)$$

Que:- 8 (a) $(D^2 - 4D + 3)y = xe^{2x}$

Sol^n. The auxiliary equation is

$$m^2 - 4m + 3 = 0$$

$$m^2 - 3m - m + 3 = 0$$

$$\Rightarrow (m-3)(m-1) = 0$$

$$\Rightarrow m = 3, \quad m = 1$$

$$C.F. = C_1 e^{x} + C_2 e^{3x}$$

$$PI = \frac{1}{D^2 - 4D + 3} x e^{2x}$$

$$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 3} x$$

$$= e^{2x} \frac{1}{D^2 + 4 + \cancel{4D} - \cancel{4D} - 8 + 3} x$$

$$= e^{2x} \frac{1}{D^2 - 1} x$$

$$= e^{2x} (D+1)^{-1} (D-1)^{-1} x$$

$$= -e^{2x} (1+D)^{-1} (1-D)^{-1} x$$

$$= -e^{2x} (1 - D + D^2 + \cdots)(1 + D + D^2 + \cdots) x$$

$$= -e^{2x} (1 + D - D) x$$

$$= -e^{2x} x$$

The complete solution is

$$y = c_1 e^{x} + c_2 e^{x} - e^{2x} x$$

Que: 8 (b) $(D^2-1)y = e^x + 2x$ -(1)

Auxiliary equation is,

$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

$$CF = C_1 e^x + C_2 e^{-x}$$

$$y(x) = CF + Y$$

Let $Y = Axe^x + Bx + C$

Hint: We guess / seek solution of the same form as the source term and will determine the coefficients

$$Y' = Axe^x + Ae^x + B$$

$$Y'' = Axe^x + Ae^x + Ae^x$$

$$Y'' = Axe^x + 2Ae^x \quad -(2)$$

Putting eq^n (2) in eq^n (1)

$$Axe^x + 2Ae^x - Axe^x - Bx - C = e^x + 2x$$

Equating co-efficients on both sides,

$$\Rightarrow 2Ae^x = e^x \quad \text{and} \quad -Bx - C = 2x$$

$$\Rightarrow A = \frac{1}{2}, \quad B = -2, \quad C = 0$$

So the general solution can be written as,

$$y(x) = C_1 e^x + C_2 e^{-x} + \frac{x}{2} e^x - 2x$$

C_1 and C_2 can be determined from initial conditions