Mathematical Physics-1
(Solved paper 2016)

Q. 1. (a) By calculating the Wronskian of the functions e^x, xe^x and e^{-x}, check whether the functions are linearly dependent or independent.

Ans.

$$y_1 = e^x$$
$$y_2 = xe^x$$
$$y_3 = e^{-x}$$
$$y'_1 = e^x$$
$$y'_2 = xe^x + e^x \cdot 1 = xe^x + e^x$$
$$y'_3 = -e^{-x}$$
$$y''_1 = e^x$$
$$y''_2 = x \cdot e^x + e^x + e^x = x \cdot e^x + 2e^x$$
$$y''_3 = e^{-x}$$

$$w = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix}$$

$$= \begin{vmatrix} e^x & xe^x & e^{-x} \\ e^x & xe^x + e^x & -e^{-x} \\ e^x & xe^x + 2e^x & e^{-x} \end{vmatrix}$$

$$= \begin{vmatrix} e^x & xe^x & e^{-x} \\ e^x & e^x(x+1) & -e^{-x} \\ e^x & e^x(x+2) & e^{-x} \end{vmatrix}$$

$$= e^x \cdot e^x \cdot e^{-x} \begin{vmatrix} 1 & x & 1 \\ 1 & x+1 & -1 \\ 1 & x+2 & 1 \end{vmatrix}$$

$$\therefore \quad W = e^x \begin{vmatrix} 1 & x & 1 \\ 1 & x+1 & -1 \\ 1 & x+2 & 1 \end{vmatrix}$$

$$C_1 \rightarrow C_1 - C_2$$

$$W = e^x \begin{vmatrix} 0 & x & 1 \\ 2 & x+1 & -1 \\ 0 & x+2 & 1 \end{vmatrix}$$

$$= e^x (-2) [x \cdot 1 - 1 \cdot (x+2)]$$
$$= -2 e^x (x - x - 2)$$
$$= 4e^x \neq 0$$

Hence the solutions are linearly independent.

(b) Solve the inexact equation
$(y^4 + 2y)\, dx + (xy^3 + 2y^4 - 4x)\, dy = 0$ **5**

Ans. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$ ... (i)

$$M = y^4 + 2y$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2$$

$$N = xy^3 + 2y^4 - 4x$$

$$\frac{\partial N}{\partial x} = y^3 - 4$$

The equation is not exact.

$$\frac{1}{M}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{3y^3 + 6}{y^4 + 2y} = \frac{3}{y}$$

This is a function of y

$$\text{I.F. } e^{-\int \frac{3}{y} dy} = e^{-3 \log y} = y^{-3}$$

Multiplying (i) by y^{-3}, we have

$$\left(y + \frac{2}{y^2}\right)dx + \left[x + 2y - \frac{4x}{y^3}\right]dy = 0 \quad \text{... (ii)}$$

Or $M_1\,dx + N_1\,dy = 0$

This (ii) is an exact equation.

$$\int M_1 dx = \int\left(y + \frac{2}{y^2}\right)dx$$

$$= \left(y + \frac{2}{y^2}\right)x$$

$$\int N_1 dy = \int 2y\, dy$$

$$= y^2$$

Therefore, the required solution is

$$\left(y + \frac{2}{y^2}\right)x + y^2 = C$$

(c) Solve the differential equation

$$\frac{d^2y}{dx^2} - y = e^x \cos x$$

Ans.
$$\frac{d^2y}{dx^2} - y = e^x \cos x$$

The auxilliary equation is $m^2 - 1 = 0$

$\Rightarrow$ $m^2 = 1$

$\Rightarrow$ $m = -1, 1$

$$\text{C.F.} = C_1 e^{-x} + C_2 e^x$$

Let $y_1 = e^{-x}$ and $y_2 = e^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix}$$

$$= 1 + 1 = 2 \neq 0$$

Let $\text{P.I.} = u_1 y_1 + u_2 y_2$

$$u_1 = -\int \frac{y_2 R}{w} dx = -\int \frac{e^x \cdot e^x \cos x \, dx}{2}$$

$$= -\frac{1}{2}\int e^{2x} \cos x \, dx$$

We have, $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$

Here $a = 2, b = 1$

$\therefore$
$$u_1 = -\frac{1}{2}\left[\frac{e^{2x}}{2^2 + 1^2}(2 \sin x - \cos x)\right]$$

$$= -\frac{1}{10}e^{2x}(2\sin x - \cos x)$$

$$u_1 y_1 = -\frac{1}{10}e^{2x}(2\sin x - \cos x)e^{-x}$$

$$= -\frac{1}{10}e^{x}(2\sin x - \cos x)$$

$$u_2 = \int \frac{y_1 R}{W} = \int \frac{e^{-x}\cdot e^{x}\cos x}{2}dx$$

$$= \frac{1}{2}\int \cos x \, dx$$

$$= \frac{1}{2}\sin x$$

$$u_2 y_2 = \frac{1}{2}\sin x \cdot e^{x} = \frac{1}{2}e^{x}\sin x$$

$$\text{P.I.} = u_1 y_1 + u_2 y_2$$

The complete solution is $y = \text{C.F.} + \text{P.I.}$

Q. 2. (a) Solve the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x} + \sin 3x$$ 8

Given equation is

$$y'' - 4y' + 4y = e^{2x} + \sin 3x$$

Auxillary equation is

$$m^2 - 4m + 4 = 0$$

$$m = \frac{4 \pm \sqrt{16 - 16}}{2} = 2, 2$$

$$\therefore \quad CF = (C_1 + C_2 x) e^{2x}.$$

$$PI = \frac{1}{D^2 - 4D + 4}\left(e^{2x} + \sin 3x\right)$$

Taking the first term of PI

$$\frac{e^{2x}}{D^2 - 4D + 4} = \frac{e^{2x}}{(D-2)^2} = \frac{x^2 e^{2x}}{2}.$$

Taking the second term, we get

$$\frac{\sin 3x}{D^2 - 4D + 4} = \frac{\sin 3x}{-9 - 4D + 4} = \frac{\sin 3x}{5 - 4D}.$$

multiplying num. & deno. by $5 + 4D$

$$\frac{\sin 3x\,(5 + 4D)}{(5)^2 - (4D)^2} = \frac{5 \sin 3x + 12 \cos 3x}{25 + (4 \times 9)}$$

$$\Rightarrow \quad PI = \frac{5 \sin 3x}{169} + \frac{12 \cos 3x}{169}.$$

∴ complete solution is

$$y = (C_1 + C_2 x)e^{2x} + \frac{x^2 e^{2x}}{2} + \frac{5\sin 3x}{169} + \frac{12\cos 3x}{169}.$$

(b) Solve the differental equation using method of undermined coefficients

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4 \qquad 7$$

Ans. $$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$$

$$y'' + y' = x^2 + 2x + 4$$

Aux. Equation : $(m^2 + m) = 0$

$\Rightarrow$ $m(m+1) = 0$

$\Rightarrow$ $m = 0, -1$

$$\text{C.F.} = C_1 + C_2 e^{-x}$$

Let $y = Ax^3 + Bx^2 + Cx$

$$y' = 3Ax^2 + 2Bx + C$$

$$y'' = 6Ax + 2B$$

Putting the value of y' and y'' in (i).

$$6Ax + 2B + 3Ax^2 + 2Bx + C = x^2 + 2x + 4$$

Comparing the co-efficients of same powers of x.

$$3A = 1$$

$\Rightarrow$ $$A = \frac{1}{3}$$

$$6A + 2B = 2$$

$\Rightarrow$ $$3A + B = 1$$

$$3\left(\frac{1}{3}\right) + B = 1$$

$$\Rightarrow \quad B = 0$$
$$C = 4$$
$$y = C_1 + C_2 e^{-x} + A x^3 + Bx^2 + Cx \quad \ldots (ii)$$
$$y' = -C_2 e^{-x} + 3Ax^2 + C$$
$$y'' = C_2 e^{-x} + 6Ax$$
$$y'' + y' = 3Ax^2 + 6Ax + C$$
$$= 3\left(\frac{1}{3}\right)x^2 + 6\left(\frac{1}{3}\right)x + 4$$
$$= x^2 + 2x + 4$$

Hence the solution is (ii). **Ans.**

Q. 3. (a) Solve the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 1 - 9x^2$$

given $y(0) = 0$ and $y'(0) = 1$.

$$y'' + 3y' = 1 - 9x^2$$

Aux. Equation is $m^2 + 3m = 0$

$$\Rightarrow \quad m(m+3) = 0$$
$$\Rightarrow \quad m = 0, -3$$
$$\text{C.F.} = C_1 + C_2 e^{-3x}$$

Let the trial solution be $y = Ax^3 + Bx^2 + Cx$

$$y' = 3Ax^2 + 2Bx + C$$
$$y'' = 6Ax + 2B$$

Putting these values in (i)

$$6Ax + 2B + 3(3Ax^2 + 2Bx + C) = 1 - 9x^2$$

or $\quad 6Ax + 2B + 9Ax^2 + 6Bx + 3C = 1 - 9x^2$

Comparing the co-efficients of various powers of x on both sides,

$$9A = -9$$
$$\Rightarrow \quad A = -1$$
$$6Ax + 6Bx = 0$$
$$\Rightarrow \quad A + B = 0$$

Since $\quad A = -1,$

$$\therefore \quad B = 1$$
$$\therefore \quad 2B + 3C = 1$$
$$\Rightarrow \quad 2(1) + 3C = 1$$

$$\Rightarrow \quad C = -\frac{1}{3}$$

$$Y = C_1 + C_2 e^{-3x} + Ax^3 + Bx^2 + Cx$$

$$Y' = -3C_2 e^{-3x} + 3Ax^2 + 2Bx + C$$

$$Y'' = 9C_2 e^{-3x} + 6Ax + 2B$$

$$Y(0) = C_1 - 3C_2 = 0 \Rightarrow C_1 = 3C_2$$

But

$$C_2 = -\frac{4}{9}$$

$\therefore$

$$C_1 = 3\left(-\frac{4}{9}\right) = -\frac{4}{3}$$

$$Y'(0) = -3C_2 + C = 1$$

$\therefore$

$$C = 1 + 3\left(-\frac{4}{9}\right) = 1 - \frac{4}{3} = -\frac{1}{3}$$

Hence the solution is :

$$Y = -\frac{1}{3} - \frac{4}{9}e^{-3x} - x^3 + x^2 - \frac{1}{3}Cx$$

(b) Solve the differential equation using method of variation of parameters

$$\frac{d^2y}{dx^2} + a^2y = \text{cosex } ax \qquad 7$$

Sol : Given equation is

$$\frac{d^2y}{dx^2} + a^2y = \sec ax.$$

Auxillary eq^n is $m^2 + a^2 = 0 \Rightarrow m^2 = -a^2$

$$m = \pm ia.$$

complementary function is

$$Cf = A\cos ax + B\sin ax.$$

Here, $y_1 = \cos ax$ $y_2 = \sin ax$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -\sin ax(a) & \cos ax\,(a) \end{vmatrix}$$

$$w = a\cos^2 ax + a\sin^2 ax = a$$

Now, $P.I = u\,y_1 + v\,y_2$

$$u = -\int \frac{y_2 \sec ax}{w}\,dx = -\int \frac{\sin ax \sec ax\,dx}{w}$$

$$= -\frac{1}{a}\int \tan ax\;dx = -\frac{1}{a^2}\log(\sec x)$$

$$= \frac{\log(\cos x)}{a^2}$$

Now, $V = \int \frac{y_1 \sec ax\,dx}{w} = \int \frac{\cos ax \sec ax\,dx}{a}$

$$= \frac{x}{a}$$

$\therefore$ $P.I. = \frac{\cos ax}{a^2}\log\cos ax + \frac{x}{a}\sin ax$

Reqd. sol, $y = A\cos ax + B\sin ax + \frac{\cos ax}{a^2}\log\cos ax + \frac{x}{a}\sin ax.$

Q. 4. (a) Find $\frac{d}{dt}\left(\vec{V}\cdot\frac{d\vec{V}}{dt}\times\frac{d^2\vec{V}}{dt^2}\right)$

where $\vec{V}$ is a function of t.

Ans. $\frac{d}{dt}\left(V\cdot\frac{dV}{dt}\times\frac{d^2V}{dt^2}\right)=V\cdot\frac{dV}{dt}\times\frac{d^3V}{dt^3}+V\cdot\frac{d^2V}{dt^2}\times\frac{d^2V}{dt^2}+\frac{dV}{dt}\cdot\frac{dV}{dt}\times\frac{d^2V}{dt^2}$

$$=V\cdot\frac{dV}{dt}\times\frac{d^3V}{dt^3}+0+0=V\cdot\frac{dV}{dt}\times\frac{d^3V}{dt^3}$$

(b) Find the Jacobian $J\left(\frac{x,y,z}{u,v,w}\right)$ **of the transformation**

$u=x^2+y^2+z^2$, $v=x^2-y^2-z^2$ and $w=x^2+y^2-z^2$.

Given, $u = x^2+y^2+z^2$

$v = x^2-y^2-z^2$

$w = x^2+y^2-z^2$

$$J\left(\frac{u,v,w}{x,y,z}\right)=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

$$=\begin{vmatrix}2x & 2y & 2z\\ 2x & -2y & -2z\\ 2x & 2y & -2z\end{vmatrix}=32xyz$$

using the property of jacobian

$$J\left(\frac{x,y,z}{u,v,w}\right)=\frac{1}{J\left(\frac{u,v,w}{x,y,z}\right)}=\frac{1}{32xyz}$$

(c) If $\vec{v} = \vec{w} \times \vec{r}$, find the whether $\vec{v}$ is irrotational or not, wh $\vec{w}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Ans. $$\text{curl } v = \nabla \times v = \nabla \times (\omega \times r) = \nabla \times \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times [(\omega_2 z - \omega_3 y)\, i + (\omega_3 x - \omega_1 z)\, j + (\omega_1 y - \omega_2 x)\, k]$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix} = 2(\omega_1 i + \omega_2 j + \omega_3 k) = 2\omega$$

Since curl $\bar{v}$ is not zero, it is not irrotational

(d) Find $\vec{\nabla} \times \left(f(r)\, \vec{r} \right)$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Ans. $$\vec{\nabla}\times\left[f(r)\vec{r}\right] = \left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)\times\{xf(r)i+yf(r)j+zf(r)k\}$$

$[\because r = xi + yj + zk]$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$= i\left(z\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial z}\right)+j\left(y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial x}\right)+k\left(\frac{y\partial f}{\partial x}-x\frac{\partial f}{\partial y}\right) \quad \dots (1)$$

Again $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x}=\frac{\partial f}{\partial r}\cdot\frac{\partial}{\partial x}\sqrt{(x^2+y^2+z^2)}$

$= \frac{f'(r)z}{\sqrt{(x^2+y^2+z^2)}}=\frac{xf'}{r}$, Similarly

$\frac{\partial f}{\partial y}=\frac{\partial f'}{r}, \frac{\partial f}{\partial z}=\frac{\partial f'}{r}$ Thus by putting there values in (1)

We get curl $\{rf(r)\}$

$$= i\left(\frac{zyf'}{r}-\frac{yzf'}{r}\right)+j\left(\frac{zxf'}{r}-\frac{zxf'}{r}\right)+k\left(\frac{yxf'}{r}-\frac{xyf'}{r}\right)$$

or curl $\{r f(r) = 0$.

(e) Find the directional derivative of a scalar function $\phi = 2xz - y^2$ at the point (1, 3, 2) in the direction of $xz\hat{i}+yz\hat{j}+xy\hat{k}$. (3×5=15)

Sol : Given $\phi = 2xz - y^2$

Directional derivative, $\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2xz - y^2)$

$$= 2z\hat{i} - 2y\hat{j} + 2x\hat{k}$$

At point (1, 3, 2)

$$\nabla\phi = 2(2)\hat{i} - 2(3)\hat{j} + 2\hat{k}$$
$$= 4\hat{i} - 6\hat{j} + 2\hat{k}$$

unit vector in the direction of given ~~unit~~ vector at pt. (1,3,2)

$$xz\hat{i} + yz\hat{j} + xy\hat{k}$$
$$= (1)(2)\hat{i} + (3)(2)\hat{j} + (1)(3)\hat{k}$$
$$= 2\hat{i} + 6\hat{j} + 3\hat{k}$$

Hence the required directional derivative is

$$\frac{(4\hat{i} - 6\hat{j} + 2\hat{k})(2\hat{i} + 6\hat{j} + 3\hat{k})}{\sqrt{4+36+9}}$$

$$= \frac{8 - 36 + 6}{7} = -\frac{22}{7}$$

Q. 5. (a) Prove that

$$\left(\vec{B}\times\vec{C}\right)\cdot\left(\vec{A}\times\vec{D}\right)+\left(\vec{C}\times\vec{A}\right)\cdot\left(\vec{B}\times\vec{D}\right)+\left(\vec{A}\times\vec{B}\right)\cdot\left(\vec{C}\times\vec{D}\right)=0$$ 5

Ans. $\left(\vec{B}\times\vec{C}\right)\cdot\left(\vec{A}\times\vec{D}\right)+\left(\vec{C}\times\vec{A}\right)\cdot\left(\vec{B}\times\vec{D}\right)+\left(\vec{A}\times\vec{B}\right)\cdot\left(\vec{C}\times\vec{D}\right)=0$

We have, $\left(\vec{A}\times\vec{B}\right)\cdot\left(\vec{C}\times\vec{D}\right)=\left(\vec{A}\cdot\vec{C}\right)\cdot\left(\vec{B}\cdot\vec{D}\right)-\left(\vec{A}\cdot\vec{D}\right)\cdot\left(\vec{B}\cdot\vec{C}\right)$ ...(i)

$$\left(\vec{B}\times\vec{C}\right)\cdot\left(\vec{A}\times\vec{D}\right)=\left(\vec{B}\cdot\vec{A}\right)\cdot\left(\vec{C}\cdot\vec{D}\right)-\left(\vec{B}\cdot\vec{D}\right)\cdot\left(\vec{C}\cdot\vec{A}\right) \quad ...(ii)$$

$$\left(\vec{C}\times\vec{A}\right)\cdot\left(\vec{B}\times\vec{D}\right)=\left(\vec{C}\cdot\vec{B}\right)\cdot\left(\vec{A}\cdot\vec{D}\right)-\left(\vec{C}\cdot\vec{D}\right)\cdot\left(\vec{A}\cdot\vec{B}\right) \quad ...(iii)$$

Add (i), (ii) and (iii),

L.H.S. of given expression

$$\left\{\left(\vec{A}\cdot\vec{C}\right)\cdot\left(\vec{B}\cdot\vec{D}\right)-\left(\vec{B}\cdot\vec{D}\right)\cdot\left(\vec{C}\cdot\vec{A}\right)\right\}+\left\{\left(\vec{B}\cdot\vec{A}\right)\cdot\left(\vec{C}\cdot\vec{D}\right)-\left(\vec{C}\cdot\vec{D}\right)\cdot\left(\vec{A}\cdot\vec{B}\right)\right\}$$

$$+\left\{\left(\vec{C}\cdot\vec{B}\right)\cdot\left(\vec{A}\cdot\vec{D}\right)-\left(\vec{A}\cdot\vec{D}\right)\cdot\left(\vec{B}\cdot\vec{C}\right)\right\}$$

$$= 0+0+0=0$$

$$\left[As\ \vec{A}\cdot\vec{C}=\vec{C}\cdot\vec{A},\ \vec{B}\cdot\vec{A}=\vec{A}\cdot\vec{B} \text{ and } \vec{C}\cdot\vec{B}=\vec{B}\cdot\vec{C}\right]$$

Hence Proved.

(b) Evaluate $\vec{\nabla}\left[r\vec{\nabla}\left(\frac{1}{r^3}\right)\right]$

where $r^2 = x^2+y^2+z^2.$

Ans. $\nabla\left(\frac{1}{r^3}\right) = \text{grad } r^{-3}$

$$= \frac{\partial}{\partial x}\left(r^{-3}\right)i+\frac{\partial}{\partial y}\left(r^{-3}\right)J+\frac{\partial}{\partial z}\left(r^{-3}\right)k$$

But $\frac{\partial}{\partial x}\left(r^{-3}\right) = -3r^{-4}\frac{\partial r}{\partial x}$

Also $r^2 = x^2+y^2+z^2$

$\Rightarrow$ $2r\frac{\partial r}{\partial x} = 2x$ $\quad\Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}$

$\Rightarrow$ $\frac{\partial}{\partial x}\left(r^{-3}\right) = -3r^{-4}\frac{x}{r}=-3r^{-5}x$

Similarly, $\frac{\partial}{\partial y}\left(r^{-3}\right) = -3r^{-5}y$

and $\frac{\partial}{\partial z}\left(r^{-3}\right) = -3r^{-5}z$

$\therefore$ $\nabla\left(\frac{1}{r^3}\right) = -3r^{-5}\left(xi+yj+zk\right)$

$$\Rightarrow \quad r\nabla\left(\frac{1}{r^3}\right) = -3r^{-4}(xi + yj + zk)$$

$$\Rightarrow \quad \nabla \bullet \left(r\nabla\frac{1}{r^3}\right) = \frac{\partial}{\partial x}\left(-3r^{-4}x\right) + \frac{\partial}{\partial y}\left(-3r^{-4}y\right) + \frac{\partial}{\partial z}\left(-3r^{-4}z\right)$$

Again $$\frac{\partial}{\partial x}\left(-3r^{-4}x\right) = 12r^{-5}\frac{\partial r}{\partial x}x - 3r^{-4}$$

$$= 12r^{-6}x^2 - 3r^{-4}$$

Similarly, $$\frac{\partial}{\partial y}\left(-3r^{-4}y\right) = 12r^{-6}y^2 - 3r^{-4}$$

And $$\frac{\partial}{\partial z}\left(-3r^{-4}z\right) = 12r^{-6}z^2 - 3r^{-4}$$

$$\Rightarrow \quad \left(r\nabla\left(\frac{1}{r^3}\right)\right) = 12r^{-6}\left(x^2 + y^2 + z^2\right) - 9r^{-4}$$

$$= 12r^{-6}, r^2 - 9r^{-4}$$

$$= 3r^{-4}$$

Q. 6. (a) Verify Stoke's theorem when $\vec{F} = \left(2xy - x^2\right)\hat{i} - \left(x^2 - y^2\right)\hat{j}$ where C is the boundary of the region enclosed by $y^2 = x$ and $x^2 = y$ 10

Ans. $\vec{F} = (2xy - x^2)\hat{i} - (x^2 - y^2)\hat{j}$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - x^2 & x^2 - y^2 & 0 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x^2 - y^2)\right] - j\left[\frac{\partial}{\partial z}(2xy - x^2) - \frac{\partial}{\partial x}(0)\right]$$

$$+k\left[\frac{\partial}{\partial x}(x^2 - y^2) - \frac{\partial}{\partial y}(2xy - x^2)\right]$$

$$= 0$$

and $\vec{F}\cdot dr = \left[(2xy - x^2)i - (x^2 - y^2)j\right]\cdot[idx + jdy + kdz]$

$$= dx\,(2xy) - dy\,(x^2 - y^2)$$

Since both(zero), the Stokes theorem is verified.

(b) Using Gauss Divergence theorem, prove that

$$\iiint_V \vec{\nabla} \times \vec{F}\, dV = \iint_S d\vec{S} \times \vec{F}$$

where V is the volume enclosed by surface S.

Sol: refer Q6 (b) in solved paper 2015

Q. 7. (a)Derive an expression of curl of a vector field in orthogonal curvilinear coordinates. Express it in spherical coordinates. **6**

Sol: Refer Q2 chapter 4

(b) Evaluate $\iint_V (y^2 + z^2)\, dV$, where V is the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $x = 0$ and $z = h$. **6**

(c) Define the Dirac Delta function and establish

$$\int_{-\infty}^{+\infty} f(x)\delta'(x)dx = -f'(0)$$ 3

Ans.

$$\int_{-\infty}^{+\infty} f(x)\delta'(x)dx = -f'(0)$$

$$\int_{-\infty}^{+\infty} f(x)\delta'(x)dx = \left[\delta(x)f(x)\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \delta(x)f'(x)dx$$

If $\lim_{x\to\pm\infty}$ is finite, then the first term in the R.H.S. of above equation is zero, and we get

$$\int_{-\infty}^{+\infty} \delta'(x)f(x)dx = -\int_{-\infty}^{+\infty} \delta(x)f'(x)dv$$

or

$$\int_{-\infty}^{+\infty} \delta'(x)f(x)dx = -f'(0)$$

Hence Proved.