Mathematical physics-1
(Solved Paper - 2015)

Q. 1 (a). By calculating the Wronskian of the function $-1, \sin^2 x, \cos^2 x$, ɛck whether the function are linearly dependent or independent. 4

Ans.

$$y_1 = -1, y_2 = \sin^2 x, y_3 = \cos^2 x$$

$$y_1^1 = 0,$$

$$y_2^1 = 2 \sin x \cos x,$$

$$= \sin 2x$$

$$y_3^1 = 2 \cos x (-\sin x)$$

$$= -2 \sin x \cos x$$

$$= -\sin 2x$$

$$y_1^{11} = 0; \; y_2^{11} = 2 \cos 2x; \; y_3^{11} = -2 \cos 2x$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1^1 & y_2^1 & y_3^1 \\ y_1^{11} & y_2^{11} & y_3^{11} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & \sin^2 x & \cos^2 x \\ 0 & \sin 2x & -\sin 2x \\ 0 & 2\cos 2x & -2\cos 2x \end{vmatrix}$$

$$= (-1)[-2 \sin 2x \cos 2x + 2 \cos 2x \sin 2x$$

$$= 0$$

The system is linearly dependent.

All Lab Experiments

Q1(b): Solve

$$y\,(xy + 2x^2y^2)\,dx + x\,(xy - x^2y^2)\,dy = 0 \quad \text{............ (1)}$$

Sol: Dividing (1) by xy, we get

$$y\,(1 + 2xy)\,dx + x\,(1 - xy)\,dy = 0 \qquad \dots (2)$$

$$M = y\,f_1\,(xy), \quad N = x\,f_2\,(xy)$$

$$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{xy\,(1+2xy) - xy\,(1-xy)} = \frac{1}{3x^2y^2}$$

On multiplying (2) by $\frac{1}{3x^2y^2}$, we have an exact differential equation

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0 \quad \Rightarrow \quad \int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}\,dy = c$$

$$\Rightarrow \quad -\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c \quad \Rightarrow \quad -\frac{1}{xy} + 2\log x - \log y = b$$

Q1(c)

(c). Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x + x$$

Sol

Given equation is

$$\frac{\partial^2 y}{\partial x^2} - 2\frac{\partial y}{dx} + y = e^x + x.$$

Auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$m = \frac{2 \pm \sqrt{4-4}}{2} = 1, 1$$

$$\text{C.F.} = (c_1 + c_2 x)\,e^x$$

All Lab Experiments

particular solution, $P.I = \frac{e^x}{D^2-2D+1}$

$= \frac{e^x}{1-2+1}$ ∵ Here denominator = 0

Again, $P.I = \frac{xe^x}{2D-2}$ again denominator = 0

Again $P.I = \frac{x^2e^x}{2}$

$\frac{x}{D^2-2D+1} = \frac{x}{(1+D^2-2D)} = x(1+D^2-2D)^{-1}$

$\Rightarrow x(1-D^2+2D) = x - D^2x + 2Dx$

$\Rightarrow \frac{x}{D^2-2D+1} = x - 0 + 2 = x+2$.

∴ Complete solution is

$y = (C_1 + C_2x)e^x + \frac{x^2e^x}{2} + x + 2$.

All Lab Experiments

Q.2 (a). Solve the differential equation:

$$\frac{d^2y}{dx^2} - 4y = x \sin x$$

Ans.

$$\frac{d^2y}{dx^2} - 4y = x \sin x$$

The auxilliary equation is $m^2 - 4 = 0$

$$\Rightarrow \quad m^2 = 4 \Rightarrow m = \pm 2$$

$$\therefore \quad \text{C.F.} = C_1 e^{2x} + C_2 e^{-2x}$$

$$\text{P.I.} = I \frac{1}{(D^2 - 4)} x \sin x$$

$$= I \frac{1}{(D^2 - 4)} \cdot x e^{ix}$$

$$= I e^{ix} \frac{1}{\left[(D+i)^2 - 4\right]} \cdot x$$

$$= I e^{ix} \frac{1}{\left[D^2 + 2Di + i^2 - 4\right]} \cdot x$$

$$= I e^{ix} \frac{1}{\left[D^2 + 2Di - 5\right]} \cdot x$$

$$= \frac{1}{5} I e^{ix} \frac{1}{\left[\frac{D^2 + 2Di}{5} - 1\right]} \cdot x$$

All Lab Experiments

$$= -\frac{1}{5} I e^{ix} \frac{1}{\left[1 - \frac{D^2 + 2Di}{5}\right]} . x$$

$$= -\frac{I}{5} e^{ix} \left[1 - \frac{D^2 + 2Di}{5}\right]^{-1} . x$$

$$= -\frac{I}{5} e^{ix} \left[1 + \frac{D^2 + 2Di}{5}\right] . x$$

Using Binomial Theorem upto 2nd term

$$P.I. = -\frac{I}{5} e^{ix} \left[x + \frac{1}{5}\frac{\partial^2}{\partial x^2}(x) + \frac{2}{5}\frac{\partial}{\partial x}(x) i\right]$$

$$= -\frac{I}{5} e^{ix} \left[x + \frac{2}{5} i\right]$$

$$= -\frac{I}{5} (\cos x + i \sin x)\left(x + \frac{2}{5} i\right)$$

$$= -\frac{I}{5}\left[x \cos x + \frac{2}{5} i \cos x + \frac{2}{5} i^2 \sin x + i x \sin x\right]$$

$$= -\frac{I}{5}\left[\left(x \cos x - \frac{2}{5} \sin x\right) + i\left(\frac{2}{5} \cos x + x \sin x\right)\right]$$

$\therefore$ P.I. = Imaginary part

$$= -\frac{1}{5}\left(\frac{2}{5} \cos x + x \sin x\right)$$

Solution = C.F. + P.I.

(b). Solve the differential equation using method of undetermined coefficients :

$$\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + \cos 2x$$ 8

Ans. $$\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + \cos 2x$$

The auxiliary equation is $m^2 - 4m + 4 = 0$

or $(m-2)^2 = 0 \Rightarrow m = 2, 2$

$$\text{C.F.} = (C_1 + C_2x)e^{2x}$$

$$\text{P.I.} = \frac{x^2}{D^2 - 4D + 4} + \frac{\cos 2x}{D^2 - 4D + 4}$$

$$\frac{x^2}{D^2 - 4D + 4} = \frac{x^2}{4\left(\frac{1 + D^2 - 4D}{4}\right)}$$

$$= \frac{1}{4}\left(1 + \frac{D^2 - 4D}{4}\right)^{-1} . x^2$$

$$= \frac{1}{4}\left[1 - \left(\frac{D^2 - 4D}{4}\right) + \left(\frac{D^2 - 4D}{4}\right)^2\right]x^2$$

$$= \frac{1}{4}\left[1 - \frac{D^2}{4} + D + \frac{D^4}{16} + \frac{16D^2}{16} - \frac{8D^3}{16}\right]x^2$$

$$= \frac{1}{4}\left[x^2 - \frac{1}{4}D^2x^2 + Dx^2 + \frac{1}{16}D^4x^2 + D^2x^2 - \frac{1}{2}D^3x^2\right]$$

$$= \frac{1}{4}\left[x^2 - \frac{1}{4}.2 + 2x + 0 + 2x - 0\right]$$

$$= \frac{1}{4}\left[x^2 + 4x - \frac{1}{2}\right]$$

$$= \frac{1}{4}x^2 + x - \frac{1}{8}$$

$$\frac{\cos 2x}{D^2 - 4D + 4} = \frac{\cos 2x}{-4 - 4D + 4} = -\frac{\cos 2x}{4D}$$

$$= -\frac{1}{4}.\frac{D\cos 2x}{D.D} = -\frac{1}{4}.\frac{D\cos 2x}{(-4)} = \frac{1}{16}D\cos 2x$$

$$= \frac{1}{16}(-2\sin 2x) = -\frac{1}{8}\sin 2x$$

All Lab Experiments

$\therefore$ P.I. is $\frac{1}{4}x^2 + x - \frac{1}{8} - \frac{1}{8}\sin 2x$

The soln is C.F. + P.I.

Q. 3(a). Solve the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 10\cos x$$

given $y(0) = 1$ $(0) = -1$.

Solution

Given equation is

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 10\cos x.$$

Auxiliary equation is

$$(m^2 - m - 2) = 0$$

$$m = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

$$= 2, -1$$

$$\therefore y = C_1 e^{2x} + C_2 e^{-x}.$$

Particular Integral, $PI = \frac{10\cos x}{[D^2 - D - 2]}$

All Lab Experiments

$$\Rightarrow PI = \frac{10\cos x}{-1 - D - 2} = \frac{10\cos x}{-3 - D}$$

$$\Rightarrow PI = \frac{-10\cos x}{D+3}$$

Multiplying num. & denominator by $D-3$

$$PI = \frac{-10\cos x}{D+3}\frac{(D-3)}{(D-3)} = \frac{-10\cos x(D-3)}{D^2 - 9}$$

$$= \frac{+10\sin x + 30\cos x}{-1-9}$$

$$= -\sin x - 3\cos x$$

$\therefore$ complete solution is

$$y = C_1 e^{2x} + C_2 e^{-x} - \sin x - 3\cos x \quad \text{—(1)}$$

Given $y = 1$ $x = 0$.

Putting this in equation 1,

$$1 = C_1 + C_2 - 3 \Rightarrow \boxed{C_1 + C_2 = 4} \quad \text{—(2)}$$

Also, $y' = -1$ $x = 0$

$$\frac{dy}{dx} = 2C_1 e^{2x} + C_2 e^{-x}(-1) - \cos x + 3\sin x.$$

Putting $x = 0$ & $y' = -1$

$$-1 = 2C_1 - C_2 - 1$$

$$\Rightarrow \boxed{2C_1 - C_2 = 0} \quad \text{—(3)}$$

All Lab Experiments

on solving (2) and (3) we get

$$C_1 = \frac{4}{3} \quad \text{and} \quad C_2 = \frac{8}{3}$$

$\therefore$ complete solution is given by

$$y = \frac{4}{3}e^{2x} + \frac{8}{3}e^{-x} - \sin x - 3\cos x.$$

3(b)

(b). Solve the differential equation using method of variation of parameters

$$\frac{d^2y}{dx^2} + a^2y = \sec ax \qquad 7$$

Sol

Sol : Given equation is

$$\frac{d^2y}{dx^2} + a^2y = \sec ax.$$

Auxillary eqn is $m^2 + a^2 = 0 \Rightarrow m^2 = -a^2$

$$m = \pm ia.$$

complementary function is

$$Cf = A\cos ax + B\sin ax.$$

Here, $y_1 = \cos ax \qquad y_2 = \sin ax$

All Lab Experiments

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -\sin ax(a) & \cos ax\,(a) \end{vmatrix}$$

$$w = a\cos^2 ax + a\sin^2 ax = a$$

Now, $P.I = u\,y_1 + v\,y_2$.

$$u = -\int \frac{y_2 \sec ax\, dx}{w} = -\int \frac{\sin ax \sec ax\, dx}{w}$$

$$= -\frac{1}{a}\int \tan ax\, dx = -\frac{1}{a^2}\log(\sec x)$$

$$= \frac{\log(\cos x)}{a^2}$$

Now, $$V = \int \frac{y_1 \sec ax\, dx}{w} = \int \frac{\cos ax \sec ax\, dx}{a}$$

$$= \frac{x}{a}$$

$$\therefore \quad P.I. = \frac{\cos ax}{a^2}\log\cos ax + \frac{x}{a}\sin ax$$

Reqd. sol, $$y = A\cos ax + B\sin ax + \frac{\cos ax}{a^2}\log\cos ax + \frac{x}{a}\sin ax.$$

All Lab Experiments

Q. 4(a). Find the volume of a parallelepiped whose sides are given by $\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{B} = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{C} = -\hat{i} + 2\hat{j} + 2\hat{k}$. 3

Ans.
$$\vec{A} = 2i + 3j - k = A_1 i + A_2 j + A_3 k$$
$$\vec{B} = i - j - 2k = B_1 i + B_2 j + B_3 k$$
$$\vec{C} = -i + 2j + 2k = C_1 i + C_2 j + C_3 k$$

Volume of a parrellelopiped $= \vec{A}\cdot(\vec{B}\times\vec{C})$
$$= \vec{B}\cdot(\vec{C}\times\vec{A})$$
$$= \vec{C}\cdot(\vec{A}\times\vec{B})$$

Also,
$$V = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$
$$= \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix}$$
$$= 2\begin{vmatrix} -1 & -2 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ -1 & -2 \end{vmatrix}$$
$$= 2(-2+4) - 1(6+2) - 1(-6-1)$$
$$= 4 - 8 + 7$$
$$= 3 \text{ Cubic units.}$$

(b). Calculate the Jacobian $J\left(\frac{x, y, z}{u, v, w}\right)$ of the transformation $x = u^2 + w$, $y = u + v$ and $z = w^2 - u$. 3

$$J\left(\frac{x, y, z}{u, v, w}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x = u^2 + w \qquad y = u + v \qquad z = w^2 - u$

$$J\left(\frac{x, y, z}{u, v, w}\right) = \begin{vmatrix} 2u & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 2w \end{vmatrix}$$

$$= 2u\,(2w) + 1\,(1)$$

$$= 4uw + 1 \quad \text{Ans.}$$

All Lab Experiments

(c). If $\vec{v} = \vec{w} \times \vec{r}$, find whether $\vec{v}$ is solenoidal or not, where, $\vec{w}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. 3

Ans. Let $\vec{w} = A_1 i + A_2 j + A_3 k$

And $\vec{r} = xi + yj + zk$

$$\vec{w} \times \vec{r} = (A_1 i + A_2 j + A_3 k) \times (xi + yj + zk)$$

$$= \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$

$$= (A_2 z - A_3 y)i - (A_1 z - A_3 x)j + (A_1 y - A_2 x)k$$

$$\text{Div. } \vec{V} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left[(A_2 z - A_3 y)i - (A_1 z - A_3 x)j + (A_1 y - A_2 x)k\right]$$

$$= \frac{\partial}{\partial x}(A_2 z - A_3 y)i^2 - \frac{\partial}{\partial y}(A_1 z - A_3 x)j^2 + \frac{\partial}{\partial z}(A_1 y - A_2 x)k^2$$

$$= 0 - 0 + 0$$

$$= 0$$

Hence $\vec{V}$ is solenoidal.

All Lab Experiments

(d). Find, $\vec{\nabla}.(f(r)\vec{r})$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. 3

Ans. Pl. ref. your text.

(e). Find the directional derivative of a scalar function $\phi = (x^2 + y + z^2)^{-1/2}$ at the point $P(3, 1, 2)$ in the direction of the vector $yz\hat{i} + xz\hat{j} + xy\hat{k}$. 3

Ans.

$$\phi = (x^2 + y^2 + z^2)^{-1/2}$$

$$\vec{a} = yzi + zxj + xyk$$

Unit vector in the direction of given vector

$$\hat{\alpha} = \frac{yzi + zxj + xyk}{\sqrt{y^2z^2 + z^2x^2 + x^2y^2}}$$

$$\frac{\partial\phi}{\partial x} = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}(2x)$$

$$= -x\left(x^2 + y^2 + z^2\right)^{-3/2}$$

Similarly,

$$\frac{\partial\phi}{\partial y} = -y\left(x^2 + y^2 + z^2\right)^{-3/2}$$

And $$\frac{\partial\phi}{\partial z} = -z\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$\text{grade } \phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= -x\left(x^2 + y^2 + z^2\right)^{-3/2}(xi + yj + zk)$$

$\therefore$ $$\frac{\partial\phi}{\partial S} = \hat{\alpha} \cdot \text{grade } \phi$$

$$= \frac{yzi + zxj + xyk}{\sqrt{y^2z^2 + z^2x^2 + x^2y^2}} \cdot \left[-\left(x^2 + y^2 + z^2\right)^{-3/2}(xi + yj + zk)\right]$$

$$= \frac{-3xy\left(x^2 + y^2 + z^2\right)^{-3/2}}{\sqrt{y^2z^2 + z^2x^2 + x^2y^2}}$$

$\frac{\partial\phi}{\partial S}$ at P(3, 1, 2) is

$$= \frac{-3(6)(9 + 1 + 4)^{-3/2}}{\sqrt{4 + 36 + 9}}$$

$$= \frac{-18(14)^{-3/2}}{7}$$

$$= \frac{-18}{7 \times 14\sqrt{14}}$$

All Lab Experiments

$$= \frac{-9}{49\sqrt{14}}$$

$$\left|\frac{\partial\phi}{\partial S}\right| = \frac{9}{49\sqrt{14}}$$

Q. 5(a). Prove that,

$$(\vec{A}\times\vec{B}).(\vec{C}\times\vec{D}) = (\vec{A}\times\vec{C}).(\vec{B}\times\vec{D})-(\vec{A}\times\vec{D}).(\vec{B}\times\vec{C})$$ 3

Ans. $(\vec{A}\times\vec{B}).(\vec{C}\times\vec{D}) = (\vec{A}\times\vec{C}).(\vec{B}\times\vec{D})-(\vec{A}\times\vec{D}).(\vec{B}\times\vec{C})$

Put $\vec{C}\times\vec{D} = \vec{r}$

L.H.S. $= (\vec{A}\times\vec{B}).\vec{r}$

$= \vec{A}.(\vec{B}\times\vec{r})$ ..... Dot and cross can be exchanged.

$= \vec{A}.[\vec{B}\times\vec{C}\times\vec{D}]$

$= \vec{A}[(\vec{B}.\vec{D})\vec{C}-(\vec{B}.\vec{C})\vec{D}]$

$= (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$

= R.H.S.

Hence proved.

All Lab Experiments

(b). Evaluate

$$\vec{\nabla}\left[\vec{\nabla}.\left(\frac{\vec{r}}{r^2}\right)\right].$$

where $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$ 6

Sol:

$$\vec{\nabla}\left[\vec{\nabla}\cdot\left(\frac{\bar{r}}{r^2}\right)\right]$$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\bar{r}}{r^2} = \frac{x}{x^2+y^2+z^2}\hat{i} + \frac{y}{x^2+y^2+z^2}\hat{j} + \frac{z}{x^2+y^2+z^2}\hat{k}$$

$$= A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{\nabla}\cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2} + \frac{x^2-y^2+z^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\bar{A}) = \frac{\partial}{\partial x}(\nabla\cdot A)\hat{i} + \frac{\partial}{\partial y}(\nabla\cdot A)\hat{j} + \frac{\partial}{\partial z}(\nabla\cdot A)\hat{k}$$

All Lab Experiments

$$= \frac{-2x}{x^2+y^2+z^2}\hat{i} - \frac{2y}{x^2+y^2+z^2}\hat{j} - \frac{2z\hat{k}}{x^2+y^2+z^2}$$

$$= \frac{-2(x\hat{i}+y\hat{j}+z\hat{k})}{x^2+y^2+z^2} = \frac{-2\vec{r}}{r^2}$$

(c). Evaluate

$$1 = \int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

where C is the boundary f the region defined by $y^2 = x$ and $y = x^2$.

Ans. See Book portion for solution

Q. 6(a). Verify Stoke's theorem for

$$\vec{A} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$$

where S is the upper half surface of the sphere x2 + y2 + z2 = 1 and C is its boundary. **10**

Ans.

$$\vec{A} = (2x-y)i - yz^2j - y^2zk$$

$$\int_C \vec{A}.dr = \oint_C (A_x dx + Aydy + Azdz)$$

$$= \int_C \{(2x - y)dx - yz^2dy - y^2zdz\}$$

But the boundary C of S is a circle in the xy–plane of radius unity and centre at (0, 0, 0), Hence the parametric equations of C are :

$$x = \cos\theta$$
$$y = \sin\theta$$
$$z = 0, \text{ where } \theta \text{ varies}$$

From 0 to 2π, Thus

$$\int_C \bar{A}.dr = \int_{\theta=0}^{2\pi} \{(2\cos\theta - \sin\theta)(-\sin\theta d\theta) - 0 - 0\}$$

$$= -\int_0^{2\pi} (2\cos\theta - \sin\theta)\sin\theta d\theta$$

$$= -\int_0^{2\pi} \left(\sin 2\theta - \sin^2\theta\right) d\theta$$

$$= -\int_0^{2\pi} \left(\sin 2\theta - \frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= -\left[\frac{\cos 2\theta}{2} - \frac{\theta}{2} + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \pi$$

Now,
$$\nabla \times \bar{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & -yz^2 & -y^2 z \end{vmatrix}$$

$$= k$$

All Lab Experiments

Hence, $$\iint_S (\nabla \times \bar{A}).ds = \iint_S k.ds$$

$$= \iint_R dxdy,$$

where R is the projection of S on xy–plane and $k.ds = dxdy =$ projection of ds on xy–plane.

Thus, R is $x^2 + y^2 = 1$

$$\therefore \quad \iint_R dxdy = 4\int_0^1\int_0^1 \sqrt{1-x^2}\,dxdy$$

$$= 4\int_0^1 \sqrt{1-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= 4 \cdot \frac{\pi}{4}$$

$$= \pi$$

Hence, $$\int_C \bar{A}.dr = \iint_S \left(\nabla \times \bar{A}\right).ds$$

This verifies stoke's theorem.

(b) Using Gauss Divergence theorem, prove that

$$\iiint_V \vec{\nabla}\phi\, dV = \iint_S \phi\, d\vec{S}$$

where V is the volume enclosed by the surface S.

All Lab Experiments

In the divergence theorem, let $\mathbf{A} = \phi\mathbf{C}$ where $\mathbf{C}$ is a constant vector. Then

$$\iiint_V \nabla\cdot(\phi\mathbf{C})\,dV = \iint_S \phi\mathbf{C}\cdot\mathbf{n}\,dS$$

Since $\nabla\cdot(\phi\mathbf{C}) = (\nabla\phi)\cdot\mathbf{C} = \mathbf{C}\cdot\nabla\phi$ and $\phi\mathbf{C}\cdot\mathbf{n} = \mathbf{C}\cdot(\phi\mathbf{n})$,

$$\iiint_V \mathbf{C}\cdot\nabla\phi\,dV = \iint_S \mathbf{C}\cdot(\phi\mathbf{n})\,dS$$

Taking $\mathbf{C}$ outside the integrals,

$$\mathbf{C}\cdot\iiint_V \nabla\phi\,dV = \mathbf{C}\cdot\iint_S \phi\mathbf{n}\,dS$$

and since $\mathbf{C}$ is an arbitrary constant vector,

$$\iiint_V \nabla\phi\,dV = \iint_S \phi\mathbf{n}\,dS$$

All Lab Experiments

Q. 7(a) Derive an expression of divergence of a vector field in orthogonal curvilinear coordinates. Express it in cylindrical coordinates. **6**

Ans. Let f (u, v, w) be a vector point function of orthogonal curvilinear co-ordinates u, v, w.

Divergence $\bar{f} = \nabla \cdot \bar{f}$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(h_2 h_3 f_1) + \frac{\partial}{\partial v}(h_3 h_1 f_2) + \frac{\partial}{\partial w}(h_1 h_2 f_3)\right]$$

Proof. Let $\bar{f} = f_1 \bar{T}_u + f_2 \bar{T}_v + f_3 \bar{T}_w$

$$= f_1 \bar{T}_v \times \bar{T}_w + f_2 \bar{T}_w \times \bar{T}_u + f_3 \bar{T}_u \times \bar{T}_v$$

$$= f_1 h_2 h_3 \nabla v \times \nabla w + f_2 h_3 h_1 \nabla w \times \nabla u + f_3 h_1 h_2 \nabla u \times \nabla v$$

Divergence $f = \nabla \cdot \bar{f}$

$$= \nabla \cdot [f_1 h_2 h_3 \nabla v \times \nabla w + f_2 h_3 h_1 \nabla w \times \nabla u + f_3 h_1 h_2 \nabla u \times \nabla v]$$

$$= \nabla \cdot (f_1 h_2 h_3 \nabla v \times \nabla w) + \nabla \cdot (f_2 h_3 h_1 \nabla w \times \nabla u) + \nabla \cdot (f_3 h_1 h_2 \nabla u \times \nabla v)$$

$$\nabla \cdot (f_1 h_2 h_3 \nabla v \times \nabla w) = f_1 h_2 h_3 \nabla \cdot (\nabla v \times \nabla w) + (\nabla v \times \nabla w).\nabla (f_1 h_2 h_3)$$

$$= f_1 h_2 h_3 [\text{curl } \nabla v \cdot \nabla w - \text{curl } \nabla w \cdot \nabla v] + [\nabla v \times \nabla w].\nabla f_1 h_2 h_3$$

$$= f_1 h_2 h_3 [\text{curl grade } v \cdot \nabla w - \text{curl grade } w \cdot \nabla v] + [\nabla v \times \nabla w].\nabla (f_1 h_2 h_3)$$

$$= (\nabla v \times \nabla w).\nabla (f_1 h_2 h_3)$$

$$= (\nabla v \times \nabla w).\left[\frac{\partial}{\partial u}(f_1 h_2 h_3)\nabla u + \frac{\partial}{\partial v}(f_1 h_2 h_3)\nabla u + \frac{\partial}{\partial w}(f_1 h_2 h_3)\nabla w_3\right]$$

$$= (\nabla v \times \nabla w).\nabla u \frac{\partial}{\partial u}(f_1 h_2 h_3) \qquad \begin{bmatrix}\nabla v \times \nabla w \cdot \nabla v = 0 \\ \nabla v \times \nabla w \cdot \nabla w = 0\end{bmatrix}$$

$$= \frac{1}{h_1 h_2 h_3}(T_v \times T_w).T_u \frac{\partial}{\partial u}(f_1 h_2 h_3)$$

$$= \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u}(f_1\, h_2\, h_3) \quad [(T_v \times T_w)\cdot T_u = T_u \cdot T_u = 1]$$

By symmetry

$$\nabla \cdot (f_2 h_3 h_1 \nabla w \times \nabla u) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial v}(f_2\, h_3\, h_1)$$

and $\nabla \cdot (f_3 h_1 h_2 \nabla u \times \nabla v) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial w}(f_3\, h_1\, h_2)$

On substituting the values

$$Div.\bar{f} = \nabla \cdot \bar{f}$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(f_1 h_2 h_3) + \frac{\partial}{\partial v}(f_2 h_3 h_1) + \frac{\partial}{\partial w}(f_3 h_1 h_2)\right]$$

In cylindrical co-ordinates.

$$div.\bar{F} = \nabla \cdot \bar{F}$$

$$= \frac{1}{\rho}\partial\frac{(\partial F_\rho)}{\partial \rho} + \frac{1}{\rho}\partial\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

(b) Evaluate

$$\iiint_V (x^2 + y^2 + z^2)\,dV$$

where V is the volume of the sphere $x^2 + y^2 + z^2 = a^2$. **6**

Ans. $\iiint_V (x^2 + y^2 + z^2)\,dV$ ... (i)

$$x^2 + y^2 + z^2 = a^2$$

$$\iiint_V (x^2 + y^2 + z^2)dv = \iiint_V (x^2 + y^2 + z^2)dx\,dy\,dz$$

Converting the given intergral into spherical polar co-ordinates.

Put,

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\iiint (x^3 + y^2 + z^2)dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2(r^2\sin\theta\,d\theta\,d\phi\,dr)$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{a} r^4 dr$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \left|\frac{r^5}{5}\right|_0^a$$

$$= \frac{a^5}{5}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta$$

$$= \frac{a^5}{5}\int_0^{2\pi} d\phi(-\cos\theta)_0^{\pi}$$

$$= \frac{2a^5}{5}\int_0^{2\pi} d\phi$$

$$= \frac{2}{5}a^5\cdot(2\pi)$$

$$= \frac{4\pi}{5}a^5$$

All Lab Experiments

(c) Define the Dirac Delta function and establish

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = -f'(a)$$

Ans.

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = -f'(a)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = \left[f(x)\delta(x-a)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\delta(x-a)dx$$

$$= 0 - 0 - f'(a)$$

$$= -f'(a)$$

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All Lab Experiments