Free Study Material from

All Lab Experiments

Mathematical Physics - I
Chapter - 3
Vector Calculus

Chapter-3
Vector calculus

Vector Calculus:
Vector Differentiation: Directional derivatives and normal derivative. Gradient of a scalar field and its geometrical interpretation. Divergence and curl of a vector field. Del and Laplacian operators. Vector identities. **(10 Lectures)**

Vector Integration: Ordinary Integrals of Vectors. Multiple integrals, Jacobian. Notion of infinitesimal line, surface and volume elements. Line, surface and volume integrals of Vector fields. Flux of a vector field. Gauss' divergence theorem, Green's and Stokes Theorems and their verification (no rigorous proofs). **(14 Lectures)**

Q1: A particle moves along the curve

$$\bar{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k},$$

where t is the time. Find the magnitude of the tangential components of its acceleration at
t = 2.
Sol: We have,

$$\vec{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$$

$$\text{Velocity} = \frac{d\vec{r}}{dt} = (3t^2 - 4)\hat{i} + (2t + 4)\hat{j} + (16t - 9t^2)\hat{k}$$

$$t = 2, \quad \text{Velocity} = 8\hat{i} + 8\hat{j} - 4\hat{k}$$

$$\text{Acceleration} = \vec{a} = \frac{d^2\vec{r}}{dt^2} = 6t\hat{i} + 2\hat{j} + (16 - 18t)\hat{k}$$

At $t = 2$ $\quad \vec{a} = 12\hat{i} + 2\hat{j} - 20\hat{k}$

The direction of velocity is along tangent.
So the tangent vector is velocity.

$$\text{Unit tangent vector,} \quad \hat{T} = \frac{\vec{v}}{|v|} = \frac{8\hat{i} + 8\hat{j} - 4\hat{k}}{\sqrt{64 + 64 + 16}} = \frac{8\hat{i} + 8\hat{j} - 4\hat{k}}{12} = \frac{2\hat{i} + 2\hat{j} - \hat{k}}{3}$$

Tangential component of acceleration, $a_t = \bar{a}.\hat{T}$

$$= (12\hat{i} + 2\hat{j} - 20\hat{k}).\frac{2\hat{i} + 2\hat{j} - \hat{k}}{3} = \frac{24 + 4 + 20}{3} = \frac{48}{3} = 16 \text{ **Ans.**}$$

FORMULAE FOR VECTOR DIFFERENTION

(i) $\frac{d}{du}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du}$

(ii) $\frac{d}{du}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}$

(iii) $\frac{d}{du}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\times\mathbf{B}$

(iv) $\frac{d}{du}(\phi\mathbf{A}) = \phi\frac{d\mathbf{A}}{du} + \frac{d\phi}{du}\mathbf{A}$

(v) $\frac{d}{du}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}\times\frac{d\mathbf{C}}{du} + \mathbf{A}\cdot\frac{d\mathbf{B}}{du}\times\mathbf{C} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}\times\mathbf{C}$

(vi) $\frac{d}{du}\{\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\} = \mathbf{A}\times\left(\mathbf{B}\times\frac{d\mathbf{C}}{du}\right) + \mathbf{A}\times\left(\frac{d\mathbf{B}}{du}\times\mathbf{C}\right) + \frac{d\mathbf{A}}{du}\times(\mathbf{B}\times\mathbf{C})$

Q2:

If $\frac{d\vec{a}}{dt} = \vec{u}\times\vec{a}$ *and* $\frac{d\vec{b}}{dt} = \vec{u}\times\vec{b}$ *then prove that* $\frac{d}{dt}[\vec{a}\times\vec{b}] = \vec{u}\times(\vec{a}\times\vec{b})$

Sol: We have,

$$\frac{d}{dt}[\vec{a}\times\vec{b}] = \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b} = \vec{a}\times(\vec{u}\times\vec{b}) + (\vec{u}\times\vec{a})\times\vec{b}$$

$$= \vec{a}\times(\vec{u}\times\vec{b}) - \vec{b}\times(\vec{u}\times\vec{a})$$

$$= (\vec{a}.\vec{b})\,\vec{u} - (\vec{a}.\vec{u})\,\vec{b} - [(\vec{b}.\vec{a})\,\vec{u} - (\vec{b}.\vec{u})\,\vec{a}]$$

(Vector triple product)

$$= (\vec{a}.\vec{b})\,\vec{u} - (\vec{u}.\vec{a})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{u} + (\vec{u}.\vec{b})\,\vec{a}$$

$$= (\vec{u}.\vec{b})\,\vec{a} - (\vec{u}.\vec{a})\,\vec{b}$$

$$= \vec{u}\times(\vec{a}\times\vec{b})$$

Proved.

Q3: Find the angle between the surface $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at (2, –1, 2).

Sol: Here we have,

$$x^2 + y^2 + z^2 = 9 \quad \ldots(1)$$

$$z = x^2 + y^2 - 3 \quad \ldots(2)$$

Normal to (1) $\eta_1 = \nabla(x^2 + y^2 + z^2 - 9)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Normal to (1) at (2, – 1, 2), $\eta_1 = 4\hat{i} - 2\hat{j} + 4\hat{k}$ ...(3)

Normal to (2), $\eta_2 = \nabla(z - x^2 - y^2 + 3)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(z - x^2 - y^2 + 3) = -2x\hat{i} - 2y\hat{j} + \hat{k}$$

Normal to (2) at (2, – 1, 2), $\eta_2 = -4\hat{i} + 2\hat{j} + \hat{k}$ ...(4)

$$\eta_1 . \eta_2 = |\eta_1||\eta_2| \cos\theta$$

$$\cos\theta = \frac{\eta_1.\eta_2}{|\eta_1||\eta_2|} = \frac{(4\hat{i} - 2\hat{j} + 4\hat{k}).(-4\hat{i} + 2\hat{j} + \hat{k})}{|4\hat{i} - 2\hat{j} + 4\hat{k}||-4\hat{i} + 2\hat{j} + \hat{k}|} = \frac{-16 - 4 + 4}{\sqrt{16 + 4 + 16}\sqrt{16 + 4 + 1}}$$

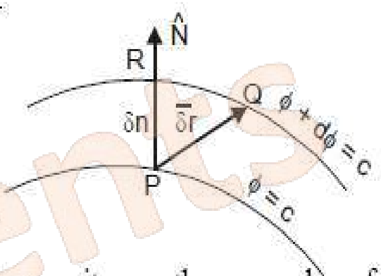
$$= \frac{-16}{6\sqrt{21}} = \frac{-8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{-8}{3\sqrt{21}}\right)$$

Hence the angle between (1) and (2) $\cos^{-1}\left(\frac{-8}{3\sqrt{21}}\right)$

Q4: what do you understand by point functions.

Sol:



Point function. A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a *point function*. There are two types of point functions.

(*i*) **Scalar point function.** If to each point *P* (*x*, *y*, *z*) of a region *R* in space there corresponds a unique scalar *f* (*P*), then *f* is called a scalar point function. *For example*, the temperature distribution in a heated body, density of a body and potential due to gravity are the examples of a scalar point function.

(*ii*) **Vector point function.** If to each point *P* (*x*, *y*, *z*) of a region *R* in space there corresponds a unique vector *f* (*P*), then *f* is called a *vector point function*. The velocity of a moving fluid, gravitational force are the examples of vector point function.

GRADIENT OF A SCALAR FIELD:

$$\text{grad } \phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Q5: give the geometrical interpretation of gradiant of a function.

Sol:

If a surface $\phi(x, y, z) = c$ passes through a point *P*. The value of the function at each point on the surface is the same as at *P*. Then such a surface is called a *level surface* through *P*. *For example*, If $\phi(x, y, z)$ represents potential at the point *P*, then *equipotential surface* $\phi(x, y, z) = c$ is a *level surface*.

Two level surfaces can not intersect.

Let the level surface pass through the point P at which the value of the function is ϕ. Consider another level surface passing through Q, where the value of the function is $\phi + d\phi$.

Let $\bar{r}$ and $\bar{r} + \delta\bar{r}$ be the position vector of P and Q then $\overrightarrow{PQ} = \delta\bar{r}$

$$\nabla\phi . d\bar{r} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi \qquad ...(1)$$

If Q lies on the level surface of P, then $d\phi = 0$

Equation (1) becomes $\nabla\phi . dr = 0$. Then $\bar{\nabla}\phi$ is $\perp$ to $d\bar{r}$ (tangent).

Hence, $\nabla\phi$ is **normal** to the surface $\phi(x, y, z) = c$

Let $\nabla\phi = |\nabla\phi|\,\widehat{N}$, where $\widehat{N}$ is a unit normal vector. Let δn be the perpendicular distance between two level surfaces through P and R. Then the rate of change of ϕ in the direction of the normal to the surface through P is $\frac{\partial\phi}{\partial n}$.

$$\frac{d\phi}{dn} = \lim_{\delta n \to 0}\frac{\delta\phi}{\delta n} = \lim_{\delta n \to 0}\frac{\nabla\phi . d\vec{r}}{\delta n}$$

$$= \lim_{\delta n \to 0}\frac{|\nabla\phi|\,\widehat{N} . d\vec{r}}{\delta n}$$

$$\left\{\begin{aligned}\widehat{N}.\vec{\delta r} &= |\vec{N}|\,|\vec{\delta r}|\cos\theta \\ &= |\vec{\delta r}|\cos\theta = \delta n\end{aligned}\right\}$$

$$= \lim_{\delta n \to 0}\frac{|\nabla\phi|\,\delta n}{\delta n} = |\nabla\phi|$$

$$\therefore \qquad |\nabla\phi| = \frac{\partial\phi}{\partial n}$$

Hence, gradient ϕ is a vector normal to the surface $\phi = c$ and has a magnitude equal to the rate of change of ϕ along this normal.

Q6: If $\Phi = 3x^2y - y^3z^2$; find grad Φ at the point (1, –2, –1).

Sol:

$$\text{grad }\phi = \nabla\phi$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{i}(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$$\text{grad } \phi \text{ at } (1, -2, -1) = \hat{i}(6)(1)(-2) + \hat{j}[(3)(1) - 3(4)(1)] + \hat{k}(-2)(-8)(-1)$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

Q7: If u = x + y + z, v = $x^2 + y^2 + z^2$, w = yz + zx + xy prove that grad u, grad v and grad w are coplanar vectors.

Sol: We have

$$\text{grad } u = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x + y + z) = \hat{i} + \hat{j} + \hat{k}$$

$$\text{grad } v = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\text{grad } w = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(yz + zx + xy) = \hat{i}(z + y) + \hat{j}(z + x) + \hat{k}(y + x)$$

[For vectors to be coplanar, their scalar triple product is 0]

$$\text{Now, grad } u.(\text{grad } v \times \text{grad } w) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ z+y & z+x & y+x \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ z+y & z+x & y+x \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ z+y & z+x & y+x \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 + R_3]$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0$$

Since the scalar product of grad u, grad v and grad w are zero, hence these vectors are coplanar vectors.

Q8: Find the directional derivative of $x^2y^2z^2$ at the point (1, 1, –1) in the direction of the tangent to the curve x = e^t, y = sin 2t + 1, z = 1 – cos t at t = 0.

Sol: Let Φ = $x^2y^2z^2$

Directional Derivative of ϕ

$$= \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^2z^2)$$

$$\nabla\phi = 2xy^2z^2\,\hat{i} + 2yx^2z^2\,\hat{j} + 2zx^2y^2\,\hat{k}$$

Directional Derivative of ϕ at (1, 1, –1)

$$= 2(1)(1)^2(-1)^2\,\hat{i} + 2(1)(1)^2(-1)^2\,\hat{j} + 2(-1)(1)^2(1)^2\,\hat{k}$$

$$= 2\hat{i} + 2\hat{j} - 2\hat{k} \qquad \text{...(1)}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = e^t\,\hat{i} + (\sin 2t + 1)\,\hat{j} + (1 - \cos t)\,\hat{k}$$

Tangent vector, $$\vec{T} = \frac{d\vec{r}}{dt} = e^t\,\hat{i} + 2\cos 2t\,\hat{j} + \sin t\,\hat{k}$$

$$\text{Tangent(at } t = 0) = e^0\,\hat{i} + 2(\cos 0)\,\hat{j} + (\sin 0)\,\hat{k} = \hat{i} + 2\hat{j} \qquad \text{...(2)}$$

Required directional derivative along tangent $= (2\hat{i} + 2\hat{j} - 2\hat{k})\dfrac{(\hat{i} + 2\hat{j})}{\sqrt{1+4}}$

[From (1), (2)]

$$= \frac{2+4+0}{\sqrt{5}} = \frac{6}{\sqrt{5}} \qquad \textbf{Ans.}$$

Q9: Find the rate of change of phi = xyz in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point (1, 1, 1).

Solution. Rate of change of $\phi = \Delta\,\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x\,y\,z) = \hat{i}\,yz + \hat{j}\,xz + \hat{k}\,xy$$

Rate of change of ϕ at (1, 1, 1) $= (\hat{i} + \hat{j} + \hat{k})$

Normal to the surface $\Psi = x^2y + y^2x + yz^2 - 3$ is given as

$$\nabla\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y + y^2x + yz^2 - 3)$$

$$= \hat{i}(2xy + y^2) + \hat{j}(x^2 + 2xy + z^2) + \hat{k}\,2yz$$

$$(\nabla\psi)_{(1,\,1,\,1)} = 3\hat{i} + 4\hat{j} + 2\hat{k}$$

$$\text{Unit normal} = \frac{3\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{9+16+4}}$$

$$\text{Required rate of change of } \phi = (\hat{i} + \hat{j} + \hat{k}).\frac{(3\hat{i} + 4\hat{j} + 2\hat{k})}{\sqrt{9+16+4}} = \frac{3+4+2}{\sqrt{29}} = \frac{9}{\sqrt{29}}$$

Q10: show that

$$\nabla r^n = n r^{n-2} \mathbf{r}.$$

Sol:

$$\nabla r^n = \nabla(\sqrt{x^2+y^2+z^2})^n = \nabla(x^2+y^2+z^2)^{n/2}$$

$$= \mathbf{i}\frac{\partial}{\partial x}\{(x^2+y^2+z^2)^{n/2}\} + \mathbf{j}\frac{\partial}{\partial y}\{(x^2+y^2+z^2)^{n/2}\} + \mathbf{k}\frac{\partial}{\partial z}\{(x^2+y^2+z^2)^{n/2}\}$$

$$= \mathbf{i}\{\tfrac{n}{2}(x^2+y^2+z^2)^{n/2-1}\,2x\} + \mathbf{j}\{\tfrac{n}{2}(x^2+y^2+z^2)^{n/2-1}\,2y\} + \mathbf{k}\{\tfrac{n}{2}(x^2+y^2+z^2)^{n/2-1}\,2z\}$$

$$= n(x^2+y^2+z^2)^{n/2-1}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= n(r^2)^{n/2-1}\mathbf{r} = n r^{n-2}\mathbf{r}$$

Note that if $\mathbf{r} = r\mathbf{r}_1$ where $\mathbf{r}_1$ is a unit vector in the direction $\mathbf{r}$, then $\nabla r^n = n r^{n-1}\mathbf{r}_1$.

Q11: Prove that

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r).$$

Sol:

$$\nabla f(r) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) f(r)$$

$$\left[r^2 = x^2+y^2+z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}\right]$$

$$= i\,f'(r)\frac{\partial r}{\partial x} + j\,f'(r)\frac{\partial r}{\partial y} + k\,f'(r)\frac{\partial r}{\partial z} = f'(r)\left[i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r}\right]$$

$$= f'(r)\frac{xi + yj + zk}{r}$$

$$\nabla^2 f(r) = \nabla[\nabla f(r)] = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left[f'(r)\frac{xi + yj + zk}{r}\right]$$

$$= \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] + \frac{\partial}{\partial y}\left[f'(r)\frac{y}{r}\right] + \frac{\partial}{\partial z}\left[f'(r)\frac{z}{r}\right]$$

$$= \left(f''(r)\frac{\partial r}{\partial x}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r1 - x\frac{\partial r}{\partial x}}{r^2} + \left(f''(r)\frac{\partial r}{\partial y}\right)\left(\frac{y}{r}\right) + f'(r)\frac{r.1 - y\frac{\partial r}{\partial y}}{r^2} +$$

$$\left(f''(r)\frac{\partial r}{\partial z}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r.1 - z\frac{\partial z}{\partial r}}{r^2}$$

$$= \left(f''(r)\frac{x}{r}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r - \frac{x^2}{r}}{r^2} + \left(f''(r)\frac{y}{r}\right)\left(\frac{y}{r}\right) + f'(r)\frac{r - \frac{y^2}{r}}{r^2} + \left(f''(r)\frac{z}{r}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r - \frac{z^2}{r}}{r^2}$$

$$= \left(f''(r)\frac{x}{r}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r^2 - x^2}{r^3} + \left(f''(r)\frac{y}{r}\right) + f'(r)\frac{r^2 - y^2}{r^3} + \left(f''(r)\frac{z}{r}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r^2 - z^2}{r^3}$$

$$= f''(r)\frac{x^2}{r^2} + f'(r)\frac{y^2 + z^2}{r^3} + f''(r)\frac{y^2}{r^2} + f'(r)\frac{x^2 + z^2}{r^3} + f''(r)\frac{z^2}{r^2} + f'(r)\frac{x^2 + y^2}{r^3}$$

$$= f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\right] + f'(r)\left[\frac{y^2 + z^2}{r^3} + \frac{z^2 + x^2}{r^3} + \frac{x^2 + y^2}{r^3}\right]$$

$$= f''(r)\frac{x^2 + y^2 + x^2}{r^2} + f'(r)\frac{2(x^2 + y^2 + z^2)}{r^3} = f''(r)\frac{r^2}{r^2} + f'(r)\frac{2r^2}{r^3}$$

$$= f''(r) + f'(r)\frac{2}{r}$$

Ans.

DIVERGENCE OF VECTOR FUNCTION

$$\text{div}\,\vec{F} = \vec{\nabla}.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\hat{i}F_1 + \hat{j}F_2 + \hat{k}F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Q12:

If $\vec{v} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$, *find the value of div* $\bar{v}$.

Sol: We have

$$\vec{v} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\text{div}\,\vec{v} = \vec{\nabla}.\vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{1/2}}\right)$$

$$= \frac{\partial}{\partial x}\frac{x}{(x^2 + y^2 + z^2)^{1/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2 + y^2 + z^2)^{1/2}} + \frac{\partial}{\partial z}\frac{z}{(x^2 + y^2 + z^2)^{1/2}}$$

$$= \frac{\left[(x^2 + y^2 + z^2)^{1/2} - x.\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}.2x\right]}{(x^2 + y^2 + z^2)}$$

$$+\frac{\left[(x^2+y^2+z^2)^{\frac{1}{2}}-y.\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\times 2y\right]}{(x^2+y^2+z^2)}+\frac{\left[(x^2+y^2+z^2)^{1/2}-z.\frac{1}{2}(x^2+y^2+z^2)^{-1/2}.2z\right]}{(x^2+y^2+z^2)}$$

$$=\frac{(x^2+y^2+z^2)-x^2}{(x^2+y^2+z^2)^{3/2}}+\frac{(x^2+y^2+z^2)-y^2}{(x^2+y^2+z^2)^{3/2}}+\frac{(x^2+y^2+z^2)-z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$=\frac{y^2+z^2+x^2+z^2+x^2+y^2}{(x^2+y^2+z^2)^{3/2}}=\frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}=\frac{2}{\sqrt{(x^2+y^2+z^2)}}\quad \textbf{Ans.}$$

Q13:

If $u = x^2 + y^2 + z^2$, *and* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *then find div* $(u\vec{r})$ *in terms of u.*

Solution. $\text{div}\ (u\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).[(x^2+y^2+z^2)(x\hat{i}+y\hat{j}+z\hat{k})]$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).[(x^2+y^2+z^2)x\hat{i}+(x^2+y^2+z^2)y\hat{j}+(x^2+y^2+z^2)z\hat{k}]$$

$$=\frac{\partial}{\partial x}(x^3+xy^2+xz^2)+\frac{\partial}{\partial y}(x^2y+y^3+yz^2)+\frac{\partial}{\partial z}(x^2z+y^2z+z^3)$$

$$=(3x^2+y^2+z^2)+(x^2+3y^2+z^2)+(x^2+y^2+3z^2)=5(x^2+y^2+z^2)=5u \quad \textbf{Ans.}$$

Q14:

If $\mathbf{A} = x^2z\,\mathbf{i} - 2y^3z^2\,\mathbf{j} + xy^2z\,\mathbf{k}$, find $\nabla\cdot\mathbf{A}$ (or div $\mathbf{A}$) at the point $(1,-1,1)$.

Sol:

$$\nabla\cdot\mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot(x^2z\,\mathbf{i}-2y^3z^2\,\mathbf{j}+xy^2z\,\mathbf{k})$$

$$=\frac{\partial}{\partial x}(x^2z)+\frac{\partial}{\partial y}(-2y^3z^2)+\frac{\partial}{\partial z}(xy^2z)$$

$$=2xz-6y^2z^2+xy^2 = 2(1)(1)-6(-1)^2(1)^2+(1)(-1)^2 = -3 \quad \text{at } (1,-1,1).$$

LAPLACIAN OPERATOR:

$$\vec{\nabla}\cdot\left(\vec{\nabla}V\right)=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial V}{\partial x}\hat{x}+\frac{\partial V}{\partial y}\hat{y}+\frac{\partial V}{\partial z}\hat{z}\right)=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}.$$

Q15: Prove that

$$\nabla^2\left(\frac{1}{r}\right) = 0.$$

Sol:

$$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}\left[-x\,(x^2+y^2+z^2)^{-3/2}\right]$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly,

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}} \quad \text{and} \quad \frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$$

Then by addition, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = 0.$

Q16:

If $\vec{A} = r^n \vec{r}$, then find the value of n for which $\vec{A}$ is solenoidal. 5

Solution. Divergence $\vec{F} = \vec{\nabla}.\vec{F} = \vec{\nabla}.r^n\vec{r} = \nabla.(x^2+y^2+z^2)^{n/2}(x\hat{i}+y\hat{j}+z\hat{k})$

$$= \left[\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right].[(x^2+y^2+z^2)^{n/2}x\hat{i}+(x^2+y^2+z^2)^{n/2}y\hat{j}+(x^2+y^2+z^2)^{n/2}z\hat{k}]$$

$$= \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2x^2)+(x^2+y^2+z^2)^{n/2}+\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2y^2)$$

$$+(x^2+y^2+z^2)^{n/2}+\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2z^2)+(x^2+y^2+z^2)^{n/2}$$

$$= n(x^2+y^2+z^2)^{n/2-1}(x^2+y^2+z^2)+3(x^2+y^2+z^2)^{n/2}$$

$$= n(x^2+y^2+z^2)^{n/2}+3(x^2+y^2+z^2)^{n/2} = (n+3)(x^2+y^2+z^2)^{n/2}$$

If $r^n\vec{r}$ is solenoidal, then $(n+3)(x^2+y^2+z^2)^{n/2} = 0$

or $n+3=0$ or $n=-3$.

Q17: Prove that

(a) $\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B}$

(b) $\nabla\cdot(\phi\mathbf{A}) = (\nabla\phi)\cdot\mathbf{A} + \phi(\nabla\cdot\mathbf{A})$.

(a) Let $\mathbf{A} = A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i}+B_2\mathbf{j}+B_3\mathbf{k}$.

Then $\nabla\cdot(\mathbf{A}+\mathbf{B}) = (\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k})\cdot[(A_1+B_1)\mathbf{i}+(A_2+B_2)\mathbf{j}+(A_3+B_3)\mathbf{k}]$

$$= \frac{\partial}{\partial x}(A_1+B_1)+\frac{\partial}{\partial y}(A_2+B_2)+\frac{\partial}{\partial z}(A_3+B_3)$$

$$= \frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}+\frac{\partial B_1}{\partial x}+\frac{\partial B_2}{\partial y}+\frac{\partial B_3}{\partial z}$$

$$= (\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k})\cdot(A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k})$$

$$+ (\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k})\cdot(B_1\mathbf{i}+B_2\mathbf{j}+B_3\mathbf{k})$$

$$= \nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B}$$

(b) $\nabla\cdot(\phi\mathbf{A}) = \nabla\cdot(\phi A_1\mathbf{i} + \phi A_2\mathbf{j} + \phi A_3\mathbf{k})$

$$= \frac{\partial}{\partial x}(\phi A_1) + \frac{\partial}{\partial y}(\phi A_2) + \frac{\partial}{\partial z}(\phi A_3)$$

$$= \frac{\partial\phi}{\partial x}A_1 + \phi\frac{\partial A_1}{\partial x} + \frac{\partial\phi}{\partial y}A_2 + \phi\frac{\partial A_2}{\partial y} + \frac{\partial\phi}{\partial z}A_3 + \phi\frac{\partial A_3}{\partial z}$$

$$= \frac{\partial\phi}{\partial x}A_1 + \frac{\partial\phi}{\partial y}A_2 + \frac{\partial\phi}{\partial z}A_3 + \phi(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z})$$

$$= (\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k})\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) + \phi(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k})\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$

$$= (\nabla\phi)\cdot\mathbf{A} + \phi(\nabla\cdot\mathbf{A})$$

CURL OF A VECTOR

$$\text{curl } \vec{F} = \vec{\nabla}\times\vec{F} \qquad (\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Q18: find the divergence and curl of V at point (2,-1,1)

$$\vec{v} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$$

Sol: here, we have

$$\vec{v} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$$

$$\text{Div. } \vec{v} = \nabla\phi$$

$$\text{Div } \vec{v} = \frac{\partial}{\partial x}(x\,y\,z) + \frac{\partial}{\partial y}(3x^2 y) + \frac{\partial}{\partial z}(xz^2 - y^2 z)$$

$$= yz + 3x^2 + 2x\,z - y^2 \qquad = -1 + 12 + 4 - 1 = 14 \text{ at } (2, -1, 1)$$

$$\text{Curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2 y & xz^2 - y^2 z \end{vmatrix} = -2yz\,\hat{i} - (z^2 - xy)\,\hat{j} + (6xy - xz)\,\hat{k}$$

$$= -2yz\,\hat{i} + (xy - z^2)\,\hat{j} + (6xy - xz)\,\hat{k}$$

Curl at $(2, -1, 1)$

$$= -2(-1)(1)\,\hat{i} + \{(2)(-1) - 1\}\,\hat{j} + \{6(2)(-1) - 2(1)\}\,\hat{k}$$

$$= 2\hat{i} - 3\,\hat{j} - 14\hat{k}$$

Q19: Prove that

$$(y^2 - z^2 + 3yz - 2x)\,\hat{i} + (3xz + 2xy)\,\hat{j} + (3xy - 2xz + 2z)\,\hat{k}$$

Is both solenoidal and irrotational

Sol: Let

$$\vec{F} = (y^2 - z^2 + 3yz - 2x)\,\hat{i} + (3xz + 2xy)\,\hat{j} + (3xy - 2xz + 2z)\,\hat{k}$$

For solenoidal, we have to prove $\vec{\nabla}.\vec{F} = 0$.

$$\text{Now, } \vec{\nabla}.\vec{F} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] . \left[(y^2 - z^2 + 3yz - 2x)\,\hat{i} + (3xz + 2xy)\,\hat{j} + (3xy - 2xz + 2z)\,\hat{k}\right]$$

$$= -2 + 2x - 2x + 2 = 0$$

Thus, $\vec{F}$ is solenoidal. For irrotational, we have to prove Curl $\overline{F} = 0$.

$$\text{Now, } \quad \text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= (3z + 2y - 2y + 3z)\,\hat{i} - (-2z + 3y - 3y + 2z)\,\hat{j} + (3z + 2y - 2y - 3z)\,\hat{k}$$

$$= 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} = 0$$

Thus, $\vec{F}$ is irrotational.

Hence, $\vec{F}$ is both solenoidal and irrotational. **Proved.**

Q20: Evalute

$$\nabla\cdot(\mathbf{A}\times\mathbf{r}) \text{ if } \nabla\times\mathbf{A}=\mathbf{0}.$$

Sol:

Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Then
$$\mathbf{A}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$

$$= (zA_2 - yA_3)\mathbf{i} + (xA_3 - zA_1)\mathbf{j} + (yA_1 - xA_2)\mathbf{k}$$

and
$$\nabla\cdot(\mathbf{A}\times\mathbf{r}) = \frac{\partial}{\partial x}(zA_2 - yA_3) + \frac{\partial}{\partial y}(xA_3 - zA_1) + \frac{\partial}{\partial z}(yA_1 - xA_2)$$

$$= z\frac{\partial A_2}{\partial x} - y\frac{\partial A_3}{\partial x} + x\frac{\partial A_3}{\partial y} - z\frac{\partial A_1}{\partial y} + y\frac{\partial A_1}{\partial z} - x\frac{\partial A_2}{\partial z}$$

$$= x\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + y\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + z\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= [x\mathbf{i} + y\mathbf{j} + z\mathbf{k}]\cdot\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}\right]$$

$$= \mathbf{r}\cdot(\nabla\times\mathbf{A}) = \mathbf{r}\cdot\text{curl }\mathbf{A}.$$ If $\nabla\times\mathbf{A}=\mathbf{0}$ this reduces to zero.

VECTOR IDENTITES

1. $\nabla(\phi+\psi) = \nabla\phi + \nabla\psi$ or $\text{grad}(\phi+\psi) = \text{grad}\,\phi + \text{grad}\,\psi$
2. $\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B}$ or $\text{div}(\mathbf{A}+\mathbf{B}) = \text{div}\,\mathbf{A} + \text{div}\,\mathbf{B}$
3. $\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$ or $\text{curl}(\mathbf{A}+\mathbf{B}) = \text{curl}\,\mathbf{A} + \text{curl}\,\mathbf{B}$
4. $\nabla\cdot(\phi\mathbf{A}) = (\nabla\phi)\cdot\mathbf{A} + \phi(\nabla\cdot\mathbf{A})$
5. $\nabla\times(\phi\mathbf{A}) = (\nabla\phi)\times\mathbf{A} + \phi(\nabla\times\mathbf{A})$
6. $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$
7. $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - \mathbf{B}(\nabla\cdot\mathbf{A}) - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B})$
8. $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B})$
9. $\nabla\cdot(\nabla\phi) \equiv \nabla^2\phi \equiv \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$

   where $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called the *Laplacian operator*.
10. $\nabla\times(\nabla\phi) = \mathbf{0}$. The curl of the gradient of ϕ is zero.
11. $\nabla\cdot(\nabla\times\mathbf{A}) = 0$. The divergence of the curl of $\mathbf{A}$ is zero.
12. $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

VECTOR INTEGRATION

Q21:

If $\mathbf{R}(u) = (u-u^2)\mathbf{i} + 2u^3\,\mathbf{j} - 3\mathbf{k}$, find (*a*) $\int \mathbf{R}(u)\,du$ and (*b*) $\int_1^2 \mathbf{R}(u)\,du$.

Sol:

$$(a) \int \mathbf{R}(u)\,du = \int [(u-u^2)\mathbf{i} + 2u^3\,\mathbf{j} - 3\mathbf{k}]\,du$$

$$= \mathbf{i}\int (u-u^2)\,du + \mathbf{j}\int 2u^3\,du + \mathbf{k}\int -3\,du$$

$$= \mathbf{i}(\frac{u^2}{2} - \frac{u^3}{3} + c_1) + \mathbf{j}(\frac{u^4}{2} + c_2) + \mathbf{k}(-3u + c_3)$$

$$= (\frac{u^2}{2} - \frac{u^3}{3})\mathbf{i} + \frac{u^4}{2}\mathbf{j} - 3u\,\mathbf{k} + c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$$

$$= (\frac{u^2}{2} - \frac{u^3}{3})\mathbf{i} + \frac{u^4}{2}\mathbf{j} - 3u\,\mathbf{k} + \mathbf{c}$$

where $\mathbf{c}$ is the constant vector $c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$.

$$\int_1^2 \mathbf{R}(u)\,du = \mathbf{i}\int_1^2 (u-u^2)du + \mathbf{j}\int_1^2 2u^3\,du + \mathbf{k}\int_1^2 -3\,du$$

$$= \mathbf{i}(\frac{u^2}{2} - \frac{u^3}{3})\Big|_1^2 + \mathbf{j}(\frac{u^4}{2})\Big|_1^2 + \mathbf{k}(-3u)\Big|_1^2 = -\frac{5}{6}\mathbf{i} + \frac{15}{2}\mathbf{j} - 3\mathbf{k}$$

Q22: Evaluate

$$\int \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2}\,dt\,.$$

Sol:

$$\frac{d}{dt}(\mathbf{A} \times \frac{d\mathbf{A}}{dt}) = \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2} + \frac{d\mathbf{A}}{dt} \times \frac{d\mathbf{A}}{dt} = \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2}$$

$$\int \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2}\,dt = \int \frac{d}{dt}(\mathbf{A} \times \frac{d\mathbf{A}}{dt})\,dt = \mathbf{A} \times \frac{d\mathbf{A}}{dt} + \mathbf{c}$$

Q23:

The equation of motion of a particle P of mass m is given by

$$m\frac{d^2\mathbf{r}}{dt^2} = f(r)\,\mathbf{r}_1$$

where $\mathbf{r}$ is the position vector of P measured from an origin O, $\mathbf{r}_1$ is a unit vector in the direction $\mathbf{r}$, and $f(r)$ is a function of the distance of P from O.

(a) Show that $\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \mathbf{c}$ where $\mathbf{c}$ is a constant vector.

(b) Interpret physically the cases $f(r) < 0$ and $f(r) > 0$.

(c) Interpret the result in (a) geometrically.

Sol:

(a) Multiply both sides of $m\frac{d^2\mathbf{r}}{dt^2} = f(r)\,\mathbf{r}_1$ by $\mathbf{r}\times$. Then

$$m\,\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} = f(r)\,\mathbf{r}\times\mathbf{r}_1 = \mathbf{0}$$

since $\mathbf{r}$ and $\mathbf{r}_1$ are collinear and so $\mathbf{r}\times\mathbf{r}_1 = \mathbf{0}$. Thus

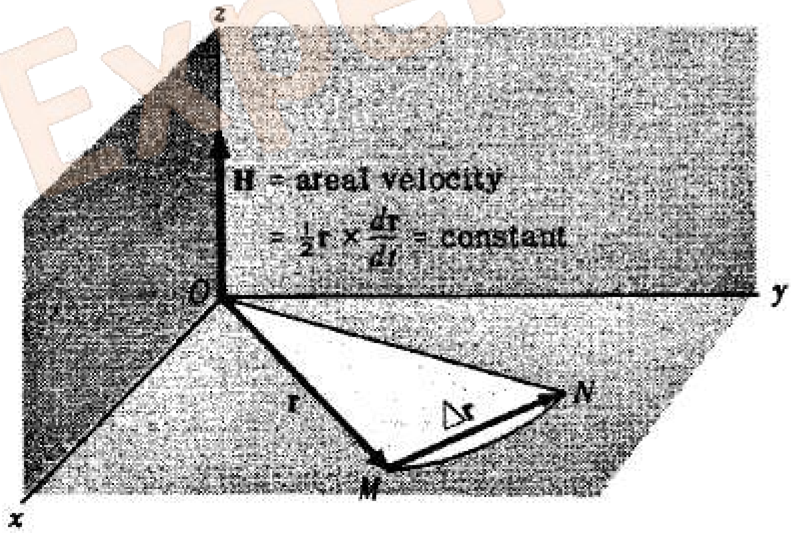
$$\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} = \mathbf{0} \quad \text{and} \quad \frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right) = \mathbf{0}$$

Integrating, $\mathbf{r}\times\frac{d\mathbf{r}}{dt} = \mathbf{c}$, where $\mathbf{c}$ is a constant vector.

b) If $f(r) < 0$ the acceleration $\frac{d^2\mathbf{r}}{dt^2}$ has direction opposite to $\mathbf{r}_1$; hence the force is directed toward O and the particle is always *attracted* toward O.

If $f(r) > 0$ the force is directed away from O and the particle is under the influence of a *repulsive* force at O.

A force directed toward or away from a fixed point O and having magnitude depending only on the distance r from O is called a *central force*.



(c) In time Δt the particle moves from M to N (see adjoining figure). The area swept out by the position vector in this time is approximately half the area of a parallelogram with sides $\mathbf{r}$ and $\Delta\mathbf{r}$, or $\frac{1}{2}\mathbf{r}\times\Delta\mathbf{r}$. Then the approximate area swept out by the radius vector per unit time is $\frac{1}{2}\mathbf{r}\times\frac{\Delta\mathbf{r}}{\Delta t}$; hence the instantaneous time rate of change in area is

$$\lim_{\Delta t\to 0}\frac{1}{2}\mathbf{r}\times\frac{\Delta\mathbf{r}}{\Delta t} = \frac{1}{2}\mathbf{r}\times\frac{d\mathbf{r}}{dt} = \frac{1}{2}\mathbf{r}\times\mathbf{v}$$

where $\mathbf{v}$ is the instantaneous velocity of the particle. The quantity $\mathbf{H} = \frac{1}{2}\mathbf{r}\times\frac{d\mathbf{r}}{dt} = \frac{1}{2}\mathbf{r}\times\mathbf{v}$ is called the *areal velocity*. From part (a),

$$\text{Areal Velocity} = \mathbf{H} = \frac{1}{2}\mathbf{r}\times\frac{d\mathbf{r}}{dt} = \text{constant}$$

Since $\mathbf{r}\cdot\mathbf{H} = 0$, the motion takes place in a plane, which we take as the xy plane in the figure above.

Q24:

Find the total work done in moving a particle in a force field given by $\mathbf{F} = 3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=1$ to $t=2$.

Sol:

$$\text{Total work} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C 3xy\,dx - 5z\,dy + 10x\,dz$$

$$= \int_{t=1}^{2} 3(t^2+1)(2t^2)\,d(t^2+1) - 5(t^3)\,d(2t^2) + 10(t^2+1)\,d(t^3)$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt = 303$$

Q25: A vector field is given by

$\vec{F} = (2y+3)\,\hat{i} + xz\hat{j} + (yz - x)\,\hat{k}$. *Evaluate* $\int_C \vec{F}\cdot\vec{dr}$ *along the path c is* $x = 2t$, $y = t$, $z = t^3$ *from* $t = 0$ *to* $t = 1$. *(Nagpur University, Winter 2003)*

Sol:

$$\int_C \vec{F}.\vec{dr} = \int_C (2y+3)\,dx + (xz)\,dy + (yz-x)\,dz$$

$$\begin{bmatrix} \text{Since } x = 2t & y = t & z = t^3 \\ \therefore \dfrac{dx}{dt} = 2 & \dfrac{dy}{dt} = 1 & \dfrac{dz}{dt} = 3t^2 \end{bmatrix}$$

$$= \int_0^1 (2t+3)\,(2\,dt) + (2t)\,(t^3)\,dt + (t^4 - 2t)\,(3t^2\,dt) = \int_0^1 (4t + 6 + 2t^4 + 3t^6 - 6t^3)\,dt$$

$$= \left[4\frac{t^2}{2} + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{6}{4}t^4\right]_0^1 = \left[2t^2 + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{3}{2}t^4\right]_0^1$$

$$= 2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2} = 7.32857.$$ **Ans.**

Q26: The acceleration of a particle at time t is given by

$$\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}.$$

If the velocity $\vec{v}$ *and displacement* $\vec{r}$ *be zero at t = 0, find* $\vec{v}$ *and* $\vec{r}$ *at any point t.*

Sol:Here

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}.$$

On integrating, we have

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\int 18\cos 3t\,dt + \hat{j}\int -8\sin 2t\,dt + \hat{k}\int 6t\,dt$$

$$\Rightarrow \quad \vec{v} = 6\sin 3t\,\hat{i} + 4\cos 2t\,\hat{j} + 3t^2\,\hat{k} + \vec{c} \qquad \text{....... (1)}$$

At $t = 0,\ \vec{v} = \vec{0}$

Putting $t = 0$ and $\vec{v} = 0$ in (1), we get

$$\vec{0} = 4\hat{j} + \vec{c} \quad \Rightarrow \quad \vec{c} = -4\hat{j}$$

$$\therefore \quad \vec{v} = \frac{d\vec{r}}{dt} = 6\sin 3t\,\hat{i} + 4(\cos 2t - 1)\,\hat{j} + 3t^2\hat{k}$$

Again integrating, we have

$$\vec{r} = \hat{i}\int 6\sin 3t\,dt + \hat{j}\int 4(\cos 2t - 1)\,dt + \hat{k}\int 3t^2\,dt$$

$$\Rightarrow \quad \vec{r} = -2\cos 3t\,\hat{i} + (2\sin 2t - 4t)\hat{j} + t^3\,\hat{k} + \vec{c_1} \qquad \text{......... (2)}$$

At, $\quad t = 0, \ \vec{r} = 0$

Putting $t = 0$ and $\vec{r} = 0$ in (2), we get

$$\therefore \quad \vec{0} = -2\hat{i} + \vec{C_1} \quad \Rightarrow \quad \vec{C_1} = 2\hat{i}$$

Hence, $\quad \vec{r} = 2(1 - \cos 3t)\,\hat{i} + 2(\sin 2t - 2t)\,\hat{j} + t^3\hat{k}$

Q27:

If $\vec{A} = (3x^2 + 6y)\,\hat{i} - 14yz\hat{j} + 20xz^2\,\hat{k}$, *evaluate the line integral* $\oint \vec{A}.\,d\vec{r}$ *from* (0, 0, 0) to (1, 1, 1) *along the curve C.*

$x = t,\ y = t^2,\ z = t^3.$

Sol: We have,

$$\int_C \vec{A}.\,d\vec{r} = \int_C [\,(3x^2 + 6y)\,\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\,].[\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz]$$

$$= \int_C [\,(3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz]$$

If $x = t,\ y = t^2,\ z = t^3$, then points (0, 0, 0) and (1, 1, 1) correspond to $t = 0$ and $t = 1$ respectively.

$$\text{Now, } \int_C \vec{A}.\,d\vec{r} = \int_{t=0}^{t=1} [\,(3t^2 + 6t^2)\,d(t) - 14t^2\,t^3\,d(t^2) + 20t\,(t^3)^2\,d(t^3)]$$

$$= \int_{t=0}^{t=1} [\,9t^2\,dt - 14t^5\,.2t\,dt + 20\,t^7\,.3\,t^2\,dt] = \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$$

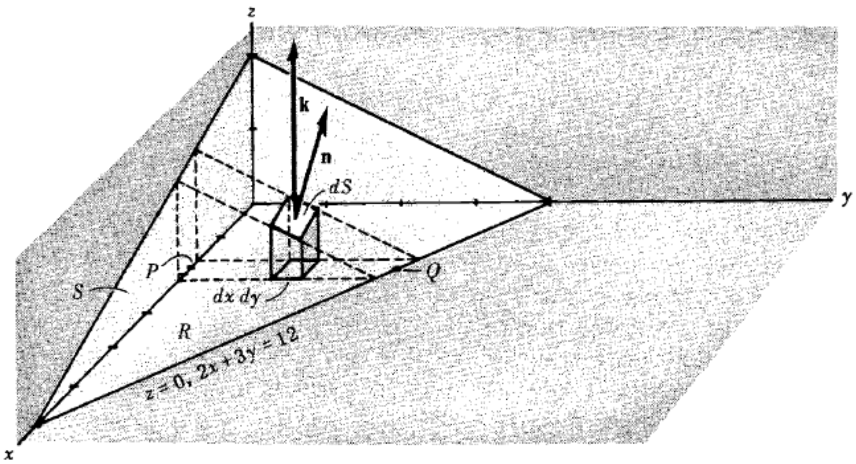
$$= \left[9\left(\frac{t^3}{3}\right) - 28\left(\frac{t^7}{7}\right) + 60\left(\frac{t^{10}}{10}\right)\right]_0^1 \qquad = 3 - 4 + 6 = 5$$

Q28:

Evaluate $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$, where $\mathbf{A} = 18z\,\mathbf{i} - 12\,\mathbf{j} + 3y\,\mathbf{k}$ and S is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant.

Sol:

The surface S and its projection R on the xy plane are shown in the figure below.



We know

$$\iint_S \mathbf{A}\cdot\mathbf{n}\, dS = \iint_R \mathbf{A}\cdot\mathbf{n} \frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$$

$$\mathbf{n} = \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$$

Thus $\mathbf{n}\cdot\mathbf{k} = (\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k})\cdot\mathbf{k} = \frac{6}{7}$ and so $\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|} = \frac{7}{6}\,dx\,dy$.

Also $\mathbf{A}\cdot\mathbf{n} = (18z\,\mathbf{i} - 12\mathbf{j} + 3y\,\mathbf{k})\cdot(\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}) = \frac{36z - 36 + 18y}{7} = \frac{36 - 12x}{7}$,

using the fact that $z = \frac{12 - 2x - 3y}{6}$ from the equation of S. Then

$$\iint_S \mathbf{A}\cdot\mathbf{n}\, dS = \iint_R \mathbf{A}\cdot\mathbf{n} \frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|} = \iint_R \left(\frac{36-12x}{7}\right)\frac{7}{6}\,dx\,dy = \iint_R (6 - 2x)\,dx\,dy$$

To evaluate this double integral over R, keep x fixed and integrate with respect to y from $y = 0$ (P in the figure above) to $y = \frac{12-2x}{3}$ (Q in the figure above); then integrate with respect to x from $x = 0$ to $x = 6$. In this manner R is completely covered. The integral becomes

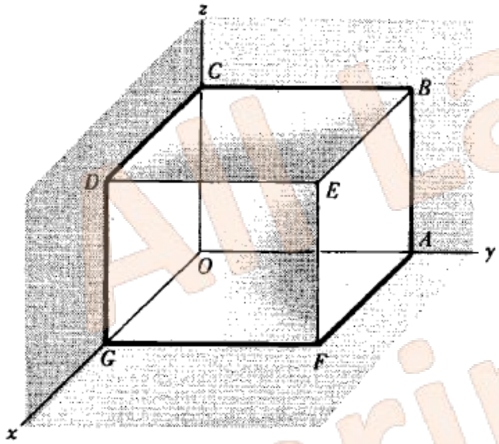
$$\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3} (6 - 2x)\,dy\,dx = \int_{x=0}^{6} \left(24 - 12x + \frac{4x^2}{3}\right) dx = 24$$

If we had chosen the positive unit normal $\mathbf{n}$ opposite to that in the figure above, we would have obtained the result -24.

Q29:

If $\mathbf{F} = 4xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$, evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$

where S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.



Sol:

Face DEFG: $\mathbf{n}=\mathbf{i}$, $x=1$. Then

$$\iint_{DEFG} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4z\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k})\cdot\mathbf{i}\,dy\,dz$$

$$= \int_0^1\int_0^1 4z\,dy\,dz = 2$$

Face ABCO: $\mathbf{n}=-\mathbf{i}$, $x=0$. Then

$$\iint_{ABCO} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (-y^2\,\mathbf{j} + yz\,\mathbf{k})\cdot(-\mathbf{i})\,dy\,dz = 0$$

Face ABEF: $\mathbf{n}=\mathbf{j}$, $y=1$. Then

$$\iint_{ABEF} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4xz\,\mathbf{i} - \mathbf{j} + z\,\mathbf{k})\cdot\mathbf{j}\,dx\,dz = \int_0^1\int_0^1 -dx\,dz = -1$$

Face OGDC: $\mathbf{n} = -\mathbf{j},\ y = 0$. Then

$$\iint_{OGDC} \mathbf{F}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 (4xz\,\mathbf{i})\cdot(-\mathbf{j})\, dx\, dz = 0$$

Face BCDE: $\mathbf{n} = \mathbf{k},\ z = 1$. Then

$$\iint_{BCDE} \mathbf{F}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 (4x\,\mathbf{i} - y^2\,\mathbf{j} + y\,\mathbf{k})\cdot\mathbf{k}\, dx\, dy = \int_0^1\int_0^1 y\, dx\, dy = \tfrac{1}{2}$$

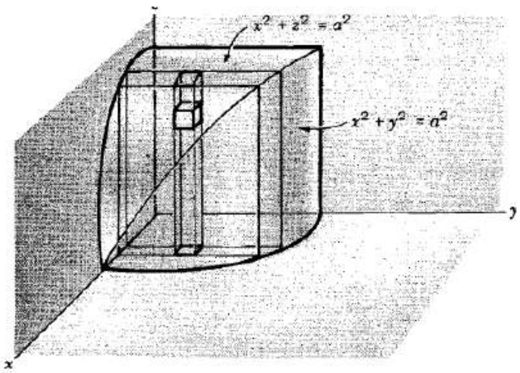
Face AFGO: $\mathbf{n} = -\mathbf{k},\ z = 0$. Then

$$\iint_{AFGO} \mathbf{F}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 (-y^2\,\mathbf{j})\cdot(-\mathbf{k})\, dx\, dy = 0$$

Adding, $$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = 2 + 0 + (-1) + 0 + \tfrac{1}{2} + 0 = \frac{3}{2}$$

Q30: Find the volume of the region common to the intersecting cylinders

$x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$



Sol: Required volume = 8 times the region shown in above figure

$$= 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2}} dz\, dy\, dx$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\, dy\, dx = 8 \int_{x=0}^{a} (a^2-x^2)\, dx = \frac{16a^3}{3}$$

Q31:

If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, *evaluate* $\iiint_V \vec{F}\, dv$ *where, v is the region bounded by the surfaces*

$$x = 0,\ y = 0,\ x = 2,\ y = 4,\ z = x^2,\ z = 2.$$

Sol:

$$\iiint_V \vec{F}\, dv = \iiint (2z\hat{i} - x\hat{j} + y\hat{k})\, dx\, dy\, dz$$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\, dz = \int_0^2 dx \int_0^4 dy\, [z^2\hat{i} - xz\hat{j} + yz\hat{k}]_{x^2}^2$$

$$= \int_0^2 dx \int_0^4 dy\, [4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2 y\hat{k}]$$

$$= \int_0^2 dx \left[4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4 y\hat{i} + x^3 y\hat{j} - \frac{x^2y^2}{2}\hat{k}\right]_0^4$$

$$= \int_0^2 (16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k})\, dx$$

$$= \left[16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} + x^4\hat{j} - \frac{8x^3}{3}\hat{k}\right]_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k} = \frac{32\hat{i}}{5} + \frac{32\hat{k}}{3} = \frac{32}{15}(3\hat{i} + 5\hat{k})$$

Jacobian

If u and v are functions of the two independent variables x and y, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the jacobian of u, v with respect to x, y and is written as

$$\frac{\partial(u, v)}{\partial(x, y)} \text{ or } J\left(\frac{u, v}{x, y}\right)$$

Q32:

If $x = r\cos\theta,\ y = r\sin\theta$; *evaluate* $\frac{\partial(x, y)}{\partial(r, \theta)}$, *and* $\frac{\partial(r, \theta)}{\partial(x, y)}$

Sol:We have

$$x = r\cos\theta,$$
$$\frac{\partial x}{\partial r} = \cos\theta,$$
$$\frac{\partial x}{\partial \theta} = -r\sin\theta,$$
$$y = r\sin\theta$$
$$\frac{\partial y}{\partial r} = \sin\theta$$
$$\frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$$

Now, $r^2 = x^2 + y^2$, $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2} = -\frac{y}{r^2}$$
$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{x}{r^2}$$

$$\frac{\partial(r, \theta)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{x^2 + y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

Q33: Prove that

$$\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1$$

Sol:

Let $u = f(x, y)$ ...(1)

$v = \varphi(x, y)$ ...(2)

$$\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}\times\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

On interchanging the rows and columns of second determinant

$$=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}\times\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v}\\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v}\end{vmatrix} \quad ...(3)$$

On differentiating (1) and (2) w. r. t. u and v, we get

$$\left.\begin{aligned}\frac{\partial u}{\partial u}&=1=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u}\\ \frac{\partial u}{\partial v}&=0=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v}\\ \frac{\partial v}{\partial v}&=1=\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v}\\ \frac{\partial v}{\partial u}&=0=\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u}\end{aligned}\right\} \quad ...(4)$$

On making substitutions from (4) in (3), we get

$$\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=1$$

Proved.

Q34:

$$If\ x=uv,\ y=\frac{u+v}{u-v},\ find\ \frac{\partial(u,v)}{\partial(x,y)}.$$

Sol:

Here it is easy to find $\frac{\partial x}{\partial u},\frac{\partial x}{\partial v},\frac{\partial y}{\partial u},\frac{\partial y}{\partial v}$. But to find $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}$ is comparatively difficult. So we first find $\frac{\partial(x,y)}{\partial(u,v)}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{-2v} & \frac{u}{2u} \\ (u-v)^2 & (u-v)^2 \end{vmatrix} = \frac{uv}{(u-v)^2}\begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} = \frac{uv}{(u-v)^2}(2+2) = \frac{4uv}{(u-v)^2}$$

But $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1 \Rightarrow \frac{\partial(u,v)}{\partial(x,y)} \times \frac{4uv}{(u-v)^2} = 1 \Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \frac{(u-v)^2}{4uv}$ **Ans.**

Q35: Find the value of the Jacobian

$\frac{\partial(u,v)}{\partial(r,\theta)}$, where $u = x^2 - y^2$, $v = 2xy$ and

$x = r\cos\theta$, $y = r\sin\theta$.

Sol: $u = x^2 - y^2$, $v = 2xy$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2) = 4r^2$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} = 4r^2 \cdot r = 4r^3$$ **Ans.**

Q36: Verify whether the following functions are functionally dependent, and if so, find the relation between them.

$$u = \frac{x+y}{1-xy}, \qquad v = \tan^{-1}x + \tan^{-1}y$$

Sol:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Hence u, v are functionally related.

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

$$v = \tan^{-1}u$$

$$u = \tan v.$$

Q37:

If $x^2 + y^2 + u^2 - v^2 = 0$ *and* $uv + xy = 0$, *prove that* $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{x^2 - y^2}{u^2 + v^2}$

Sol: Let

$$f_1 = x^2 + y^2 + u^2 - v^2, \quad f_2 = uv + xy$$

$$\frac{\partial(f_1, f_2)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} = 2(x^2 - y^2)$$

$$\frac{\partial(f_1, f_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2(u^2 + v^2)$$

But
$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\dfrac{\partial(f_1, f_2)}{\partial(x, y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}} = \frac{2(x^2 - y^2)}{2(u^2 + v^2)} = \frac{x^2 - y^2}{u^2 + v^2}$$

Q38: State Green's theorem

Sol: If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

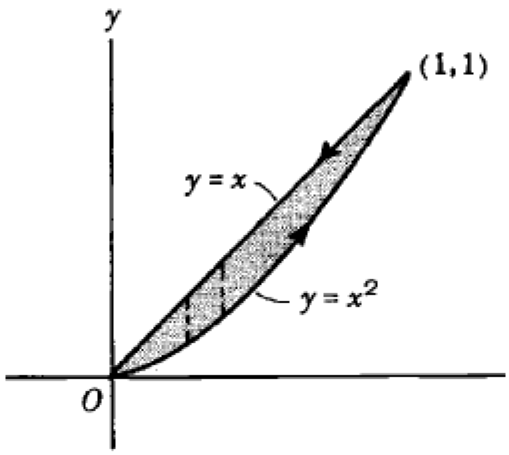
where C is traversed in the positive (counterclockwise direction

Q39: Verify Green's theorem in the plane for

$$\oint_C (xy + y^2)\,dx + x^2\,dy$$

where C is the closed curve of the region bounded by y =x and y =x^2

Sol: y=x and y =x^2 intersect at (0,0) and (1,1) . The positive direction in traversing C is as shown in the adjacent diagram.



Along $y = x^2$, the line integral equals

$$\int_0^1 \left((x)(x^2) + x^4\right) dx + (x^2)(2x)\,dx = \int_0^1 (3x^3 + x^4)\,dx = \frac{19}{20}$$

Along $y = x$ from (1,1) to (0,0) the line integral equals

$$\int_1^0 \left((x)(x) + x^2\right) dx + x^2\,dx = \int_1^0 3x^2\,dx = -1$$

Then the required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\,dy$$

$$= \iint_R (x - 2y)\,dx\,dy = \int_{x=0}^1 \int_{y=x^2}^x (x - 2y)\,dy\,dx$$

$$= \int_0^1 \left[\int_{x^2}^x (x - 2y)\,dy\right] dx = \int_0^1 (xy - y^2)\Big|_{x^2}^x dx$$

$$= \int_0^1 (x^4 - x^3)\,dx = -\frac{1}{20}$$

so that the theorem is verified.

Q40: State divergence theorem of Gauss.

Sol:It states that if V is the volume bounded by a closed surface S and A is a vector function of position with continuous derivatives, then

$$\iiint_V \nabla\cdot\mathbf{A}\,dV = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \oiint_S \mathbf{A}\cdot d\mathbf{S}$$

where n is the positive (outward drawn) normal to S.

Q41:Evaluate

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS, \quad \text{where} \quad \mathbf{F} = 4xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$$

and S is the surface of the cube bounded by x =0, x=1, y=0, y=1, z=0, z=1.

Sol: By the divergence theorem, the required integral is equal to

$$\iiint_V \nabla\cdot\mathbf{F}\,dV = \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_V (4z - y)\,dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z-y)\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} 2z^2 - yz \Big|_{z=0}^{1} dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1} (2-y)\,dy\,dx = \frac{3}{2}$$

Q42: State Stoke's theorem

Sol: It states that if S is an open, two-sided surface bounded by a closed, non-intersecting curve C (simple closed curve) then if A has continuous derivatives

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = \iint_S (\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

where C is traversed in the positive direction

Q43: Verify Stokes' theorem for A = $(2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Sol: The boundary C of S is a circle in the xy plane of radius one and center at the origin. Let x = cost,
y = sin t, z = 0, 0 < t < 277 be parametric equations of C. Then

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \oint_C (2x-y)\,dx - yz^2\,dy - y^2z\,dz$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt = \pi$$

$$\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \mathbf{k}$$

Then $$\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = \iint_S \mathbf{k}\cdot\mathbf{n}\,dS = \iint_R dx\,dy$$

since $\mathbf{n}\cdot\mathbf{k}\,dS = dx\,dy$ and R is the projection of S on the xy plane. This last integral equals

$$\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1\int_0^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1 \sqrt{1-x^2}\,dx = \pi$$

and Stokes' theorem is verified.