Free Study Material from

All Lab Experiments

Mathematical Physics - II

Chapter - 4, 5

4. Some Special Integral
5. Partial Differential Equations

Chapter - 4
Some Special Integrals

Some Special Integrals: Beta and Gamma Functions and Relation between them. Expression of Integrals in terms of Gamma Functions. **(4 Lectures)**

GAMMA FUNCTION:

The *gamma function*, denoted by $\Gamma(n)$, is defined by

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$

which is convergent for $n > 0$.

Q1: prove that

$$\Gamma 1 = 1$$

Sol: Using standard gamma integral

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$$

Putting n=1 ,we get

$$\Gamma 1 = \int_0^\infty e^{-x}\, dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = 1$$

Q2:Prove the following :

(a) $\Gamma(n+1) = n\Gamma(n), \ n > 0;$

(b) $\Gamma(n+1) = n!, \ n = 1, 2, 3, \ldots$

Sol:

$$(a)\ \Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx = \lim_{M\to\infty} \int_0^M x^n e^{-x}\, dx$$

$$= \lim_{M\to\infty} \left\{ (x^n)(-e^{-x}) \Big|_0^M - \int_0^M (-e^{-x})(nx^{n-1})\, dx \right\}$$

$$= \lim_{M\to\infty} \left\{ -M^n e^{-M} + n \int_0^M x^{n-1} e^{-x}\, dx \right\} = n\,\Gamma(n) \qquad \text{if } n > 0$$

(b) $\Gamma(1) = \int_0^\infty e^{-x}\,dx = \lim_{M\to\infty}\int_0^M e^{-x}\,dx = \lim_{M\to\infty}(1-e^{-M}) = 1$

Put $n = 1, 2, 3, \ldots$ in $\Gamma(n+1) = n\,\Gamma(n)$. Then

$\Gamma(2) = 1\,\Gamma(1) = 1$, $\Gamma(3) = 2\,\Gamma(2) = 2\cdot 1 = 2!$, $\Gamma(4) = 3\,\Gamma(3) = 3\cdot 2! = 3!$

In general, $\Gamma(n+1) = n!$ if n is a positive integer.

Q3:Evaluate

$$(a)\ \int_0^\infty x^3 e^{-x}\,dx, \quad (b)\ \int_0^\infty x^6 e^{-2x}\,dx.$$

Sol:

(a) $\int_0^\infty x^3 e^{-x}\,dx = \Gamma(4) = 3! = 6$

(b) Let $2x = y$. Then the integral becomes

$$\int_0^\infty \left(\frac{y}{2}\right)^6 e^{-y}\frac{dy}{2} = \frac{1}{2^7}\int_0^\infty y^6 e^{-y}\,dy = \frac{\Gamma(7)}{2^7} = \frac{6!}{2^7} = \frac{45}{8}$$

Q4: Prove that

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

Sol:

We have $\Gamma(\frac{1}{2}) = \int_0^\infty x^{-1/2}e^{-x}\,dx = 2\int_0^\infty e^{-u^2}\,du$, on letting $x = u^2$. It follows that

$$\{\Gamma(\tfrac{1}{2})\}^2 = \left\{2\int_0^\infty e^{-u^2}\,du\right\}\left\{2\int_0^\infty e^{-v^2}\,dv\right\} = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,du\,dv$$

Changing to polar coordinates (ρ, ϕ), where $u = \rho\cos\phi$, $v = \rho\sin\phi$, the last integral becomes

$$4\int_{\phi=0}^{\pi/2}\int_{\rho=0}^\infty e^{-\rho^2}\rho\,d\rho\,d\phi = 4\int_{\phi=0}^{\pi/2} -\tfrac{1}{2}e^{-\rho^2}\Big|_{\rho=0}^\infty\,d\phi = \pi$$

and so $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Q5: Evaluate:

$$(a)\ \Gamma(-1/2), \quad (b)\ \Gamma(-5/2).$$

Sol:

We use the generalization to negative values defined by $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$.

(a) Letting $n = -\frac{1}{2}$, $\Gamma(-1/2) = \dfrac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}$.

(b) Letting $n = -3/2$, $\Gamma(-3/2) = \dfrac{\Gamma(-1/2)}{-3/2} = \dfrac{-2\sqrt{\pi}}{-3/2} = \dfrac{4\sqrt{\pi}}{3}$, using (a).

Then $\Gamma(-5/2) = \dfrac{\Gamma(-3/2)}{-5/2} = -\dfrac{8}{15}\sqrt{\pi}$.

Q6: Evaluate

$$\int_0^\infty \frac{x^a}{a^x}\,dx.$$

Sol: Putting $a^x = e^t$ or $x \log a = t$,

$$x = \frac{t}{\log a}, \qquad dx = \frac{dt}{\log a}$$

$$I = \int_0^\infty \left(\frac{t}{\log a}\right)^a e^{-t}\frac{dt}{\log a} = \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t}\,t^a\,dt$$

$$= \frac{1}{(\log a)^{a+1}}\,\overline{|a+1} \qquad \textbf{Ans.}$$

BETA FUNCTION

The *beta function*, denoted by $B(m, n)$, is defined by

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

which is convergent for $m > 0$, $n > 0$.

The beta function is connected with the gamma function according to the relation

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Q7:Prove that :

(a) $B(m, n) = B(n, m)$,

(b) $B(m, n) = 2\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$.

Sol: (a) Using the transformation x = 1 – y, we have

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^1 (1-y)^{m-1}y^{n-1}\,dy$$
$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = B(n,m)$$

(b) using the transformation $x = \sin^2\theta$, we have

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$
$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\, d\theta$$

Q8: Prove that

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad m,n > 0.$$

Sol:

Letting $z = x^2$, we have $\Gamma(m) = \int_0^\infty z^{m-1}e^{-z}\,dz = 2\int_0^\infty x^{2m-1}e^{-x^2}\,dx.$

Similarly, $\Gamma(n) = 2\int_0^\infty y^{2n-1}e^{-y^2}\,dy.$ Then

$$\Gamma(m)\,\Gamma(n) = 4\left(\int_0^\infty x^{2m-1}e^{-x^2}\,dx\right)\left(\int_0^\infty y^{2n-1}e^{-y^2}\,dy\right)$$
$$= 4\int_0^\infty\int_0^\infty x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}\,dx\,dy$$

Transforming to polar coordinates, $x = \rho\cos\phi$, $y = \rho\sin\phi$,

$$\Gamma(m)\,\Gamma(n) = 4\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{\infty} \rho^{2(m+n)-1}e^{-\rho^2}\cos^{2m-1}\phi\,\sin^{2n-1}\phi\,d\rho\,d\phi$$
$$= 4\left(\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}\,d\rho\right)\left(\int_{\phi=0}^{\pi/2}\cos^{2m-1}\phi\,\sin^{2n-1}\phi\,d\phi\right)$$
$$= 2\,\Gamma(m+n)\int_0^{\pi/2}\cos^{2m-1}\phi\,\sin^{2n-1}\phi\,d\phi = \Gamma(m+n)\,B(n,m)$$
$$= \Gamma(m+n)\,B(m,n)$$

Using the results of previous question.
Hence the required result follows.

Q9: Evaluate the following

$$(a)\ \int_0^1 x^4(1-x)^3\,dx, \quad (b)\ \int_0^2 \frac{x^2\,dx}{\sqrt{2-x}}, \quad (c)\ \int_0^a y^4\sqrt{a^2-y^2}\,dy.$$

Sol:

$$(a)\ \int_0^1 x^4(1-x)^3\,dx = B(5,4) = \frac{\Gamma(5)\,\Gamma(4)}{\Gamma(9)} = \frac{4!\,3!}{8!} = \frac{1}{280}$$

(b) Letting $x = 2v$, the integral becomes

$$4\sqrt{2}\int_0^1 \frac{v^2}{\sqrt{1-v}}\,dv = 4\sqrt{2}\int_0^1 v^2(1-v)^{-1/2}\,dv = 4\sqrt{2}\,B(3,\tfrac{1}{2}) = \frac{4\sqrt{2}\,\Gamma(3)\,\Gamma(1/2)}{\Gamma(7/2)} = \frac{64\sqrt{2}}{15}$$

(c) Letting $y^2 = a^2x$ or $y = a\sqrt{x}$, the integral becomes

$$\frac{a^6}{2}\int_0^1 x^{3/2}(1-x)^{1/2}\,dx = \frac{a^6}{2}B(5/2, 3/2) = \frac{a^6\,\Gamma(5/2)\,\Gamma(3/2)}{2\,\Gamma(4)} = \frac{\pi a^6}{32}$$

Q10: Derive the relationship between Gamma and beta functions.

Sol: We know that

$$\Gamma(l) = \int_0^\infty e^{-x}\,x^{l-1}\,dx,$$

Putting zx =y

$$\frac{\Gamma(l)}{z^l} = \int_0^\infty e^{-zx}\,x^{l-1}\,dx$$

$$\Gamma(l) = \int_0^\infty z^l\,e^{-zx}\,x^{l-1}\,dx$$

Multiplying both sides by $e^{-z}z^{m-1}$, we have

$$\Gamma(l).\,e^{-z}.\,z^{m-1} = \int_0^\infty e^{-z}.z^{m-1}.z^l.e^{-zx}.x^{l-1}\,dx$$

$$\Gamma(l).\,e^{-z}.\,z^{m-1} = \int_0^\infty e^{-(1+x)z}.z^{l+m-1}.x^{l-1}\,dx$$

Integrating both sides w.r.t. 'z', we get

$$\int_0^\infty \Gamma(l)\,e^{-z}z^{m-1}\,dz = \int_0^\infty\int_0^\infty e^{-(1+x)z}z^{l+m-1}x^{l-1}\,dx\,dz$$

$$\Gamma(l)\,\Gamma(m) = \int_0^\infty x^{l-1}\,dx\int_0^\infty e^{-(1+x)z}z^{l+m-1}\,dz$$

$$= \int_0^\infty x^{l-1}dx\cdot\frac{\Gamma(l+m)}{(1+x)^{l+m}}$$

$$\Gamma l\,\Gamma m = \Gamma(l+m)\int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\,dx = \Gamma(l+m)\,.\,\beta(l,m)$$

$$\boxed{\beta(l,m) = \frac{\Gamma l\,\Gamma m}{\Gamma(l+m)}}$$

Q11:show that

$$\int_0^{\frac{\pi}{2}} \sin^p\theta\,\cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$$

Sol : we know that

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad \text{............ (1)}$$

Putting $x = \sin^2\theta,\ dx = 2\sin\theta\cos\theta\,d\theta$

and $1-x = 1-\sin^2\theta = \cos^2\theta$

Then (1) becomes

$$\beta(m,n) = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\,\cos^{2n-2}\theta\;2\sin\theta\cos\theta\,d\theta$$

$$\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

Putting $2m-1 = p,\quad i.e.,\quad m = \frac{p+1}{2}$

and $2n-1 = q,\quad i.e.,\quad n = \frac{q+1}{2}$

$$\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}} = 2\int_0^{\frac{\pi}{2}} \sin^p\theta\cdot\cos^q\theta\,d\theta$$

$$\boxed{\int_0^{\frac{\pi}{2}} \sin^p\theta\,\cos^q\theta\,d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}}$$

Chapter - 5
Partial Differential Equations

Partial Differential Equations: Solutions to partial differential equations, using separation of variables: Laplace's Equation in problems of rectangular geometry.

Solution of wave equation for vibrational modes of a stretched string, rectangular and circular membranes. **(15 Lectures)**

Q1: Applying the method of separation of variables techniques, find the solution of partial differential equation.

$$3u_x + 2u_y = 0 \ldots, \text{ where } u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}.$$

Sol:Here, we have

$$\frac{3\,\partial u}{\partial x} + \frac{2\,\partial u}{\partial y} = 0 \quad \ldots\ldots\ldots\ldots (1)$$

Let $$u = X(x)\, Y(y) \quad \ldots(2)$$

Where X is a function of x only and Y is a function of y only.

On differentiating (2) partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{\partial X}{\partial x}. Y \quad \ldots(3)$$

On differentiating (2) partially w.r.t. y, we get

$$\frac{\partial u}{\partial y} = X. \frac{\partial Y}{\partial Y} \quad \ldots(4)$$

Putting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ from (3) and (4) in (1), we get

$$3\frac{\partial X}{\partial x}. Y + 2\,X\frac{\partial Y}{\partial y} = 0 \quad \ldots(5)$$

Dividing (5) by XY, we get

$$\frac{3}{X}\frac{\partial X}{\partial x}+\frac{2}{Y}\frac{\partial Y}{\partial y}=0$$

[R.H.S is constant for L.H.S, So we take both equations are equal to k (constant)

$$\Rightarrow \quad \frac{3}{X}\frac{\partial X}{\partial x}=-\frac{2}{Y}\frac{\partial Y}{\partial y}=k \Rightarrow \frac{3}{X}\frac{\partial X}{\partial x}=k \text{ and } -\frac{2}{Y}\frac{\partial Y}{\partial y}=k$$

$$\Rightarrow \quad \frac{\partial X}{X}=\frac{k}{3}\partial x \text{ and } \frac{\partial Y}{Y}=-\frac{k}{2}\partial y \Rightarrow \log X=\frac{k}{3}x+c_1 \text{ and } \log Y=\frac{-k}{2}y+c_2$$

$$\Rightarrow \quad X=e^{\frac{k}{3}x+c_1} \text{ and } Y=e^{-\frac{k}{2}y+c_2}$$

Putting the values of X and Y in (2), we get

$$u=e^{\frac{k}{3}x+c_1}\,e^{\frac{-k}{2}y+c_2}=e^{k\left(\frac{x}{3}-\frac{y}{2}\right)+c_1+c_2}=e^{k\left(\frac{x}{3}-\frac{y}{2}\right)}.\,e^{c_1+c_2}$$

Hence $u=A\,e^{k\left(\frac{x}{3}-\frac{y}{2}\right)}$ [where $A=e^{c_1+c_2}$] **Ans.**

Q2: Using the method of separation of variables, solve

$$\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial t}+u$$, where

$u(x,0)=6\,e^{-3x}$

Solution. $$\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial t}+u \quad \ldots(1)$$

Let $$u=X(x).T(t) \quad \ldots(2)$$

where X is a function of x only and T is a function of t only.

Putting the value of u in (1), we get

$$\frac{\partial(X.T)}{\partial x}=2\frac{\partial}{\partial t}(X.T)+X.T$$

$$T\frac{dX}{dx}=2X\frac{dT}{dt}+X.T \Rightarrow TX'=2\,X.T'+X.T \Rightarrow T.\frac{X'}{X}=2\frac{T'}{T}+1=c\,(\text{say})$$

$$(a) \quad \frac{X'}{X} = c \Rightarrow \frac{1}{X}\frac{dX}{dx} = c \Rightarrow \frac{dX}{X} = c\,dx$$

On integration $\log X = c\,x + \log a. \Rightarrow \log \frac{X}{a} = cx \Rightarrow \frac{X}{a} = e^{cx} \Rightarrow X = ae^{cx}$

$$(b) \quad \frac{2T'}{T} + 1 = c \Rightarrow \frac{T'}{T} = \frac{1}{2}(c-1) \Rightarrow \frac{1}{T}\frac{dT}{dt} = \frac{1}{2}(c-1) \Rightarrow \frac{dT}{T} = \frac{1}{2}(c-1)dt$$

On integration $\log T = \frac{1}{2}(c-1)\,t + \log b \Rightarrow \log\frac{T}{b} = \frac{1}{2}(c-1)t$

$$\Rightarrow \quad \frac{T}{b} = e^{\frac{1}{2}(c-1)t} \quad \Rightarrow T = be^{\frac{1}{2}(c-1)t}$$

Putting the value of X and T in (2), we have

$$u = ae^{cx}.be^{\frac{1}{2}(c-1)t}$$

$$u = ab\ e^{cx+\frac{1}{2}(c-1)t} \qquad ...(3)$$

$$\Rightarrow \quad u(x,0) = ab\ e^{cx}$$

But $\quad u(x,0) = 6e^{-3x}$

i.e. $\quad ab\,e^{cx} = 6e^{-3x} \Rightarrow ab = 6$ and $c = -3$

Putting the value of ab and c in (3), we have

$$u = 6e^{-3x+\frac{1}{2}(-3-1)t}$$

https://alllabexperiments.com

$$u = 6e^{-3x-2t}$$

which is the required solution.

Q3: Use the method of separation of variables to solve the equation :

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$$

given that $v = 0$ *when* $t \to \infty$, *as well as* $v = 0$ *at* $x = 0$ *and* $x = l$.

Solution. $$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \quad ...(1)$$

Let us assume that $v = XT$ where X is a function of x only and T that of t only.

$$\frac{\partial v}{\partial t} = X\frac{dT}{dt} \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} = T\frac{d^2X}{dx^2}$$

Substituting these values in (1), we get

$$X\frac{dT}{dt} = T\frac{d^2X}{dx^2}$$

Let each side of (2) be equal to a constant ($-p^2$)

$$\Rightarrow \quad \frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -p^2 \quad ...(2)$$

$$\frac{1}{T}\frac{dT}{dt} = -p^2 \quad \Rightarrow \quad \frac{dT}{dt} + p^2T = 0 \quad ...(3)$$

and $$\frac{1}{X}\frac{d^2T}{dx^2} = -p^2 \quad \Rightarrow \quad \frac{d^2X}{dx^2} + p^2X = 0 \quad ...(4)$$

Solving (3) and (4), we have

$$T = C_1e^{-p^2t}$$

$$X = C_2\cos px + C_3\sin px \quad ...(5)$$

$$\therefore \quad v = C_1e^{-p^2t}(C_2\cos px + C_3\sin px)$$

Putting $x = 0$, $v = 0$ in (5), we get

$0 = C_1e^{-p^2t}C_2$ $\quad \therefore C_2 = 0$, since $C_1 \neq 0$

On putting the value of C_2 in (5), we get

$$v = C_1e^{-p^2t}C_3\sin px \quad ..(6)$$

Again putting $x = l$, $v = 0$ in (6), we get

$$0 = C_1e^{-p^2t}.C_3\sin pl$$

Since C_3 cannot be zero.

$\therefore \sin pl = 0 = \sin n\pi \ \therefore \quad p = \frac{n\pi}{l}$, n is any integer.

On putting the value of p in (6) it becomes

$$v = C_1C_3e^{-\frac{n^2\pi^2t}{l^2}}\sin\frac{n\pi x}{l}$$

Hence $$v = b_ne^{\frac{-n^2\pi^2t}{l^2}}\sin\frac{n\pi x}{l} \quad \text{where } b_n = C_1C_3$$

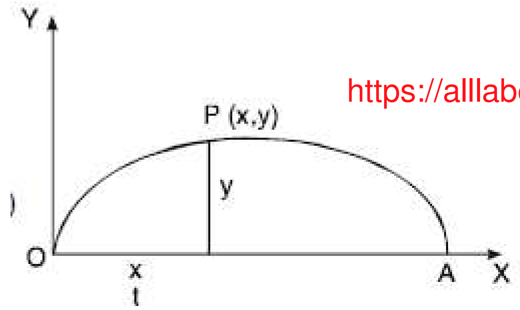
This equation satisfies the given condition for all integral values of n. Hence taking $n = 1, 2, 3, \ldots\ldots\ldots,$ *the most general solution is*

$$v = \sum_{n=1}^{\infty} b_ne^{-\frac{n^2\pi^2t}{l^2}}\frac{\sin n\pi x}{l}$$ **Ans.**

Q4: Obtain the solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

using the method of separation of variables.



Sol: here ,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Let $y = XT$ where X is a function of x only and T is a function of t only.

$$\frac{\partial y}{\partial t} = X\frac{dT}{dt} \quad \text{and} \quad \frac{\partial y}{\partial x} = T\frac{dX}{dx}$$

Since T and X are functions of a single variable only.

$$\frac{\partial^2 y}{\partial t^2} = X\frac{d^2T}{dt^2} \quad and \quad \frac{\partial^2 y}{\partial x^2} = T\frac{d^2X}{dx^2}$$

Substituting these values in the given equation, we get

$$X\frac{d^2T}{dt^2} = c^2T\frac{d^2X}{dx^2}$$

By separating the variables, we get

$$\frac{\frac{d^2T}{dt^2}}{c^2T} = \frac{\frac{d^2X}{dx^2}}{X} = k \qquad \text{(say).}$$

(Each side is constant, since the variables x and y are independent).

$$\therefore \qquad \frac{d^2T}{dt^2} - k\,c^2T = 0 \qquad \text{and} \qquad \frac{d^2X}{dx^2} - kX = 0$$

Auxiliary equations are

$$m^2 - kc^2 = 0 \;\Rightarrow\; m = \pm c\sqrt{k} \quad \textit{and} \quad m^2 - k = 0 \;\Rightarrow\; m = \pm\sqrt{k}$$

Case 1. If $k > 0$.

$$T = C_1 e^{c\sqrt{k}t} + C_2 e^{-c\sqrt{k}t}$$

$$X = C_3 e^{c\sqrt{k}x} + C_4 e^{-\sqrt{k}x}$$

Case 2. If $k < 0$.

$$T = C_5 \cos c\sqrt{k}t + C_6 \sin c\sqrt{k}t$$

$$X = C_7 \cos\sqrt{k}x + C_8 \sin\sqrt{k}x$$

Case 3. If $k = 0$.

$$T = C_9 t + C_{10}$$

$$X = C_{11}x + C_{12}$$

These are the three cases depending upon the particular problems. Here we are dealing with wave motion ($k < 0$).

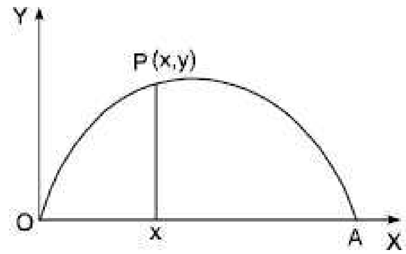
$$y = TX$$

$$y = \left(C_5 \cos c\sqrt{k}t + C_6 \sin c\sqrt{k}t\right) \times \left(C_7 \cos\sqrt{k}x + C_8 \sin\sqrt{k}x\right) \qquad \textbf{Ans.}$$

Example 5. *A string is stretched and fastened to two points l apart Motion is started by displacing the string in the form $y = a \sin \frac{\pi x}{l}$ from which it is released at a time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by*

$$y(x,t) = a \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi c t}{l}\right)$$

Solution. The vibration of the string is given by:



$$\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2} \quad ...(1)$$

As the end points of the string are fixed, for all time,

$$y(0,t) = 0 \quad ...(2)$$

and $$y(l,t) = 0 \quad ...(3)$$

Since the initial transverse velocity of any point of the string is zero, therefore,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad ...(4)$$

Also $$y(x,0) = a \sin \frac{\pi x}{l} \quad ...(5)$$

Now we have to solve (1), subject to the above boundary conditions. Since the vibration of the string is periodic, therefore, the solution of (1) is of the form

$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos C\,pt + C_4 \sin C\,pt) \quad ...(6)$$

By (2) $$y(0,t) = C_1 (C_3 \cos C\,pt + C_4 \sin C\,pt) = 0$$

For this to be true for all time, $C_1 = 0$.

Hence $$y(x,t) = C_2 \sin px (C_3 \cos Cpt + C_4 \sin Cpt) \quad ...(7)$$

and $$\frac{\partial y}{\partial t} = C_2 \sin px \left[C_3 (-Cp \sin Cpt) + C_4 (Cp \cos Cpt) \right]$$

By (4) $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px (C_4 Cp) = 0$$

Whence $$C_2 C_4 Cp = 0$$

If $C_2 = 0$, (7) will lead to the trivial solution $y(x, t) = 0$.

$\therefore$ the only possibility is that $C_4 = 0$

Thus (7) becomes

$$y(x, t) = C_2 C_3 \sin px \cos Cpt \quad ...(8)$$

If $x = l$ then $y = 0, 0 = C_2 C_3 \sin p\,l \cos Cpt$, for all t.

Since C_2 and $C_3 \neq 0$, we have $\sin p\,l = 0 \;\therefore\; p\,l = n\pi$

i.e. $p = \frac{n\pi}{l}$,where n is an integer.

Hence (8) reduces to

$$y(x,t) = C_2 C_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi Ct}{l} \qquad ...(9)$$

Finally imposing the last condition (5), we have

$$y(x,0) = C_2 C_3 \sin\frac{n\pi x}{l} = a\sin\frac{\pi x}{l}$$

which will be satisfied by taking $C_2 C_3 = a$ and $n = 1$

Hence the required solution is

$$y(x,t) = a\sin\frac{\pi x}{l} = \cos\frac{\pi C t}{l}$$

Proved.

Q6: The vibrations of an elastic string is governed by the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

The length of the string is and the ends are fixed .The initial velocity is zero and the initial deflection is u (x, 0) = 2(sin x +sin 3x).Find the deflection u (x, t)of the vibrating string for t 0 .

Solution. $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

$$\Rightarrow \quad u = (c_1\cos pt + c_2\sin pt)(c_3\cos px + c_4\sin px) \qquad ...(1)$$

On putting $x = 0, u = 0$ in (1),we get

$$0 = (c_1\cos pt + c_2\sin pt)c_3 \Rightarrow c_3 = 0$$

On putting $c_3 = 0$ in (1),it reduces

$$u = (c_1\cos pt + c_2\sin pt)c_4\sin px \qquad ...(2)$$

On putting $x = \pi$ and $u = 0$ in (2), we have

$$0 = (c_1\cos pt + c_2\sin pt)c_4\sin p\pi$$

$\sin p\pi = 0 = \sin n\pi \qquad n = 1, 2, 3, 4...$

$\therefore \quad p\pi = n\pi \quad$ or $\quad p = n$

On substituting the value of p in (2), we get

$$u = (c_1\cos nt + c_2\sin nt)c_4\sin nx \qquad ...(3)$$

On differentiating (3) w.r.t. "t", we get

$$\frac{du}{dt} = (-c_1 n \sin nt + c_2 n \cos nt) c_4 \sin nx \quad ...(4)$$

On putting $\frac{du}{dt} = 0,\ t = 0$ in (4) we have

$$0 = (c_2 n)(c_4 \sin nx) \Rightarrow c_2 = 0$$

On putting $c_2 = 0$, (3) becomes

$$u = (c_1 \cos nt)(c_4 \sin nx)$$

$$u = c_1 c_4 \cos nt \sin nx \quad ...(5)$$

given $u(x,0) = 2(\sin x + \sin 3x)$

On putting $t = 0$ in (5), we have

$$u = (x,0) = c_1 c_4 \sin nx$$

$$2(\sin x + \sin 3x) = c_1 c_4 \sin nx$$

$$4 \sin 2x \cos x = c_1 c_4 \sin nx$$

$$c_1 c_4 = 4 \cos x \qquad 2 = n$$

On substituting the value of $c_1 c_4$ and $n = 2$,(5) becomes

$$u(x,t) = 4 \cos x \cos 2t \sin 2x \quad \textbf{Ans}$$

Q7: Solve the wave equation by D'Alemert's method

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Sol: let the given equation be (1)

Now Let us introduce the two new independent variables u = x + ct, v = x – ct

So that y becomes a function of u and v

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u}(1) + \frac{\partial y}{\partial v}(1) = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \quad ...(2)$$

$$\frac{\partial y}{\partial t}=\frac{\partial y}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial y}{\partial v}\frac{\partial v}{\partial t}=\frac{\partial y}{\partial u}c+\frac{\partial y}{\partial v}(-c)=c\left(\frac{\partial y}{\partial u}-\frac{\partial y}{\partial v}\right) \qquad \left[\because \frac{\partial u}{\partial t}=c,\frac{\partial v}{\partial t}=-c\right]$$

$$\frac{\partial}{\partial t}=c\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial v}\right)$$

$$\frac{\partial^2 y}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)=c\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial v}\right)c\left(\frac{\partial y}{\partial u}-\frac{\partial y}{\partial v}\right)$$

$$=c^2\left(\frac{\partial^2 y}{\partial u^2}-2\frac{\partial^2 y}{\partial u\partial v}+\frac{\partial^2 y}{\partial v^2}\right) \qquad \ldots(3)$$

Substituting the values of $\frac{\partial^2 y}{\partial x^2}$ and $\frac{\partial^2 y}{\partial t^2}$ from (2) and (3) in (1), we get

$$c^2\left(\frac{\partial^2 y}{\partial u^2}-2\frac{\partial^2 y}{\partial u\partial v}+\frac{\partial^2 y}{\partial v^2}\right)=c^2\left(\frac{\partial^2 y}{\partial u^2}+2\frac{\partial^2 y}{\partial u\partial v}+\frac{\partial^2 y}{\partial v^2}\right) \text{ or } \frac{\partial^2 y}{\partial u\partial v}=0 \qquad \ldots(4)$$

Integrating (4) *w.r.t v*, we get $\frac{\partial y}{\partial u}=f(u)$ ... (5)

where $f(u)$ is constant in respect of v. Again integrating (5) *w. r.t 'u,'* we get

$$y=\int f(u)du+\psi(v)$$

where $\psi(v)$ is constant in respect of u

$$y=\phi(u)+\psi(v) \text{ where } \phi(u)=\int f(u)du$$

$$\Rightarrow \qquad y(x,t)=\phi(x+ct)+\psi(x-ct) \qquad \ldots(6)$$

This is D'Almberts solution of wave equations (1)

To determine ϕ,ψ let us apply initial conditions, $y(x,0)=f(x)$ and $\frac{\partial y}{\partial t}=0$ when $t=0$.

Differentiating (6) w. r.t. "t", we get

$$\frac{\partial y}{\partial t}=c\phi'(x+ct)-c\psi'(x-ct) \qquad \ldots(7)$$

Putting $\frac{\partial y}{\partial t}=0$, and $t=0$ in (7) we get $\qquad 0=c\phi'(x)-c\psi'(x)$

$$\Rightarrow \qquad \phi'(x)=\psi'(x) \quad \Rightarrow \quad \phi(x)=\psi(x)+b$$

Again substituting $y=f(x)$ and $t=0$ in (6) we get

$$f(x)=\phi(x)+\psi(x) \quad \Rightarrow \quad f(x)=[\psi(x)+b]+\psi(x)$$

$$\Rightarrow \qquad f(x)=2\psi(x)+b$$

Again substituting $y = f(x)$ and $t = 0$ in (6) we get

$$f(x) = \phi(x) + \psi(x) \quad \Rightarrow \quad f(x) = [\psi(x) + b] + \psi(x)$$

$$\Rightarrow \qquad f(x) = 2\psi(x) + b$$

so that $\psi(x) = \frac{1}{2}[f(x) - b]$ and $\phi(x) = \frac{1}{2}[f(x) + b]$

On putting the values of $\phi(x+ct)$ and $\psi(x-ct)$ in (6), we get

$$y(x,t) = \frac{1}{2}[f(x+ct) + b] + \frac{1}{2}[f(x-ct) - b]$$

$$\Rightarrow \qquad y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] \qquad \textbf{Ans.}$$

Q8: A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along the short edge y = 0 is given by

u (x, 0) = 20x, 0 < x 5

= 20 (10 – x), 5 < x < 10

while the two long edges x = 0 and x = 10 as well as the other short edges are kept at 0°C.

Find the steady state temperature at any point (x, y) of the plate.

Sol: In the steady state, the temperature u (x, y) at any point p (x, y) satisfy the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{.......................} \ (1)$$

The boundary conditions are

$$u(0, y) = 0 \text{ for all values of } y \qquad \text{... (2)}$$

$$u(10, y) = 0 \text{ for all values of } y \qquad \text{... (3)}$$

$$u(x, \infty) = 0 \text{ for all values of } x \qquad \text{... (4)}$$

$$u(x, 0) = 20x \qquad 0 < x \leq 5$$

$$= 20(10 - x) \qquad 5 < x < 10 \qquad \text{...(5)}$$

Now three possible solutions of (1) are

$$u = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \qquad \text{...(6)}$$

$$u = (C_5 \cos px + C_6 \sin px)(C_7 e^{px} + C_8 e^{-py}) \qquad \text{...(7)}$$

$$u = (C_9 x + C_{10})(C_{11} y + C_{12}) \qquad \text{...(8)}$$

Of these, we have to choose that solution which is consistent with the physical nature of the problem. The solution (6) and (8) cannot satisfy the condition (2), (3)and (4). Thus, only possible solution is (7) *i.e.*, of the form.

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad ...(9)$$

By (2) $\quad u(0, y) = C_1(C_3 e^{py} + C_4 e^{-py}) = 0 \quad$ for all values of y

$$C_1 = 0$$

(9) reduces to $u(x, y) = C_2 \sin px\,(C_3 e^{py} + C_4 e^{-py}) \quad ...(10)$

By (3) $u(10, y) = C_2 \sin 10p\,(C_3 e^{py} + C_4 e^{-py}) = 0 \qquad C_2 \neq 0$

$\sin 10p = 0 \quad \Rightarrow \quad 10p = n\pi \quad \Rightarrow \quad p = \dfrac{n\pi}{10}$

Also to satisfy the condition (4) *i.e.*, $u = 0$ as $y \to \infty$

$$C_3 = 0$$

Hence (10) takes the form $u(x, y) = C_2 C_4 \sin px.e^{-py}$

$\Rightarrow \quad u(x, y) = b_n \sin px.e^{-py} \qquad$ where $b_n = C_2 C_4$

The most general solution that satisfies (2), (3) & (4) is of the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin px e^{-py} \quad ...(5)$$

Putting $y = 0$, $\quad u(x, 0) = \sum_{n=1}^{\infty} b_n \sin px \quad$ where $p = \dfrac{n\pi}{10}$

This requires the expansion of u in Fourier series in the interval $x = 0$ and $x = 5$ and from $x = 5$ to $x = 10$.

$$b_n = \frac{2}{10}\int_0^5 20x \sin px dx + \frac{2}{10}\int_5^{10} 20\,(10 - x) \sin px dx$$

$$b_n = 4\int_0^5 x \sin px dx + 4\int_5^{10} (10 - x) \sin px dx$$

$$= 4\left[x\left(\frac{-\cos px}{p}\right) - (1)\left(\frac{-\sin px}{p^2}\right)\right]_0^5 + 4\left[(10 - x)\left(\frac{-\cos px}{p}\right) - (-1)\left(\frac{-\sin px}{p^2}\right)\right]_5^{10}$$

$$= 4\left[\frac{-5\cos 5p}{p} + \frac{\sin 5p}{p^2}\right] + 4\left[0 - \frac{\sin 10p}{p^2} + \frac{5\cos 5p}{p} + \frac{\sin 5p}{p^2}\right]$$

$$= 4\left[\frac{2\sin 5p}{p^2} - \frac{\sin 10p}{p^2}\right] \qquad \left(p = \frac{n\pi}{10}\right)$$

$$= 4\left[\frac{2\sin 5.\frac{n\pi}{10}}{\frac{n^2\pi^2}{100}} - \frac{\sin 10.\frac{n\pi}{10}}{\frac{n^2\pi^2}{100}}\right] = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{400}{n^2\pi^2}\sin n\pi$$

$$= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2} = 0 \text{ if n is even.} = \pm\frac{800}{n^2\pi^2} \text{ if n is odd.} \quad \text{or} \quad b_n = \frac{(-1)^{n+1}800}{(2n-1)^2\pi^2}$$

On putting the value of b_n in (5) the temperature at any point (x, y) is given by

$$u(x,y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{10}e^{-\frac{(2n-1)\pi y}{10}} \quad \text{Ans.}$$

Q9: The diameter of a semicircular plate of radius a is kept at 0°C and the temperature at the semicircular boundary is T° C. Find the steady state temperature in the plate.

Sol: Let the centre O of the semicircular plate be the pole and the bounding diameter be as the initial line. Let u (r,) be the steady state temperature at any point p (r,) and u satisfies the equation

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

https://alllabexperiments.com

The boundary conditions are

(i) $u(r, 0) = 0 \qquad 0 \le r \le a$

(ii) $u(r, \pi) = 0 \qquad 0 \le r \le a$

(iii) $u(a, \theta) = T.$

Support by Donating

From conditions (ii) and (iii), we have $u \to 0$ as $r \to 0$. Hence the appropriate solution of (i) is as solved in example 15.

$$u = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \qquad \ldots(2)$$

Putting $u\ (r, 0) = 0$ in (2), we get

$$0 = (c_1 r^p + c_2 r^{-p}) c_3 \to c_3 = 0$$

(2) becomes

$$u = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\theta \qquad \ldots(3)$$

Putting $u(r, \pi) = 0$ in (3), we get

$$0 = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\pi \Rightarrow \sin p\pi = 0 = \sin n\pi$$

$$\Rightarrow \qquad p\pi = n\pi \quad \Rightarrow \quad p = n$$

(3) becomes, on putting $p = n$

$$u = (c_1 r^n + c_2 r^{-n}) c_4 \sin n\theta \qquad \ldots(4)$$

Since, $u = 0$ when $r = 0$

$$0 = c_2$$

(4) becomes, $u = c_1 c_4 r^n \sin n\theta$

The most general solution of (1) is

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \qquad \ldots(5)$$

Putting $r = a$ and $u = T$ in (5), we have

$$T = \sum_{n=1}^{\infty} b_n a^n \sin n\theta$$

By Fourier half range series, we get

Support by Donating

$$b_n\ a^n = \frac{2}{\pi}\int_0^{\pi} T \sin n\theta\, d\theta = \frac{2}{\pi} T \left(\frac{-\cos n\theta}{n}\right)_0^{\pi} = \frac{2T}{n\pi}[-(-1)^n + 1]$$

$$b_n a^n = 0, \qquad \text{When } n \text{ is even.}$$

$$b_n a^n = \frac{4T}{n\pi}, \qquad \text{When } n \text{ is odd.}$$

$$\Rightarrow \qquad b_n = \frac{4T}{n\pi a^n}$$

https://alllabexperiments.com

Hence, (5) becomes

$$u(r, \theta) = \frac{4T}{\pi}\left[\frac{r/a}{1}\sin\theta + \frac{(r/a)^3}{3}\sin 3\theta + \frac{(r/a)^5}{5}\sin 5\theta + \ldots\right] \qquad \textbf{Ans.}$$