Free Study Material from

All Lab Experiments

All Lab Experiments

Mathematical Physics - I

Chapter - 4

4. Orthogonal Curvilinear Coordinates

Chapter-4
Orthogonal curvilinear coordinates

Orthogonal Curvilinear Coordinates:
Orthogonal Curvilinear Coordinates. Derivation of Gradient, Divergence, Curl and Laplacian in Cartesian, Spherical and Cylindrical Coordinate Systems. **(7 Lectures)**

ORTHOGONAL CURVILLINEAR COORDINATES

1. $\nabla\Phi = \text{grad }\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\mathbf{e}_3$

2. $\nabla\cdot\mathbf{A} = \text{div }\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$

3. $\nabla\times\mathbf{A} = \text{curl }\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$

4. $\nabla^2\Phi = \text{Laplacian of }\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]$

SPHERICAL COORDINATE SYSTEM

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$$

$$\text{where} \quad \rho \geqq 0, \quad 0 \leqq \phi < 2\pi, \quad -\infty < z < \infty$$

$$h_\rho = 1, \quad h_\phi = \rho, \quad h_z = 1$$

CYLINDRICAL COORDINATE SYSTEM

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$\text{where} \quad r \geqq 0, \quad 0 \leqq \phi < 2\pi, \quad 0 \leqq \theta \leqq \pi$$

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta$$

Q1:

If u, v, w are orthogonal curvilinear co-ordinates, show that $\frac{\partial\vec{r}}{\partial u}, \frac{\partial\vec{r}}{\partial v}, \frac{\partial\vec{r}}{\partial w}$ *and* $\nabla u, \nabla v, \nabla w$ *are reciprocal system of vectors.*

Sol: We know that

$$\nabla u = \frac{\overline{T}_u}{h_1},\ \nabla v = \frac{\overline{T}_v}{h_2},\ \nabla w = \frac{\overline{T}_w}{h_3}$$

$$(\nabla u,\ \nabla v,\ \nabla w) = (\nabla u \times \nabla v)\cdot \nabla w$$

$$= \left(\frac{\overline{T}_u}{h_1} \times \frac{\overline{T}_v}{h_r}\right)\cdot \frac{\overline{T}_w}{h_3}$$

$$= \frac{1}{h_1 h_2 h_3}\left[\overline{T}_w \cdot \overline{T}_w\right] = \frac{1}{h_1 h_2 h_3}$$

$$\frac{\partial \overline{r}}{\partial u} = h_1 T_u,\ \frac{\partial \overline{r}}{\partial v} = h_2 T_v,\ \frac{\partial \overline{r}}{\partial w} = h_3 T_w$$

Now,

$$\frac{\nabla u \times \nabla w}{[\nabla u \nabla v \nabla w]} = \frac{\left(\frac{1}{h_2}\right)\overline{T}_v \times \left(\frac{1}{h_3}\right)\overline{T}_w}{\frac{1}{h_1 h_2 h_3}} = h_1 \overline{T}_u \qquad [\overline{T}_v \times \overline{T}_w = \overline{T}_u]$$

$$= \frac{\partial \overline{r}}{\partial u}$$

Similarly,

$$\frac{\partial \overline{r}}{\partial v} = \frac{\nabla T_w \times \nabla T_u}{[\nabla u, \nabla v, \nabla w]},\ \frac{\partial \overline{r}}{\partial w} = \frac{\nabla u_1 \times \nabla u_2}{[\nabla u, \nabla v, \nabla w]}$$

This shows that $\frac{\partial \overline{r}}{\partial u}, \frac{\partial \overline{r}}{\partial v}, \frac{\partial \overline{r}}{\partial w}$ and $\nabla u, \nabla v, \nabla w$ form reciprocal system of vectors. **Proved.**

Q2: Derive an expression for curl of a vector field in orthogonal curvilinear coordinates. Express it in cylindrical coordinates.

Sol: The cartesian orthogonal coordinate system is very intuitive and easy to handle. Once an origin has been fixed in space and three orthogonal scaled axis are anchored to this origin, any point in space is uniquely determined by three real numbers, its cartesian coordinates. A curvilinear coordinate system can be defined starting from the orthogonal cartesian one. If x, y, z are the cartesian coordinates, the curvilinear ones, u, v, w, can be expressed as smooth functions of x,y, z, according to:

$$\begin{aligned} u &= u(x, y, z) \\ v &= v(x, y, z) \\ w &= w(x, y, z) \end{aligned} \qquad (1)$$

These functions can be inverted to give x, y, z-dependency on u, v, w:

$$x = x(u, v, w)$$
$$y = y(u, v, w) \qquad (2)$$
$$z = z(u, v, w)$$

There are infinitely many curvilinear systems that can be defined using equations (1) and (2).

We are mostly interested in the so-called orthogonal curvilinear coordinate systems, defined as follows. Any point in space is determined by the intersection of three "warped" planes:

$$u = \text{const} \quad , \quad v = \text{const} \quad , \quad w = \text{const}$$

We could call these surfaces as coordinate surfaces. Three curves, called coordinate curves, are formed by the intersection of pairs of these surfaces. Accordingly, three straight lines can be calculated as tangent lines to each coordinate curve at the space point. In an orthogonal curved system these three tangents will be orthogonal for all points in space (see Figure 1). In order to express differential operators, like the gradient or the divergence, in curvilinear coordinates it is convenient to start from the infinitesimal increment in cartesian coordinates,

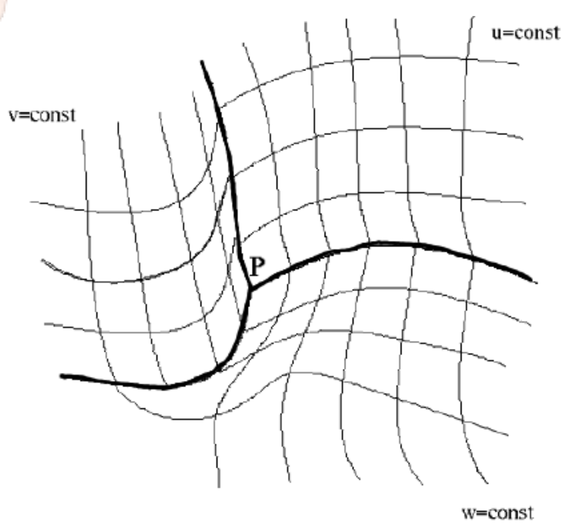


Figure 1: In this generic orthogonal curved coordinate system three coordinate surfaces meet at each point P in space. Their mutual intersection gives rise to three coordinate curves which are themselves perpendicular in P.

$\mathrm{d}\mathbf{r} \equiv (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$. By considering equations (2) and expanding the differential $\mathrm{d}\mathbf{r}$, the following equation can be obtained:

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{r}}{\partial v}\mathrm{d}v + \frac{\partial \mathbf{r}}{\partial w}\mathrm{d}w \quad (3)$$

$\partial\mathbf{r}/\partial u$, $\partial\mathbf{r}/\partial v$ and $\partial\mathbf{r}/\partial w$ are vectors tangent, respectively, to coordinate curves along u, v and w, in P. These vectors are mutually orthogonal, because we are working with orthogonal curvilinear coordinates. Let us call $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$, unit-length vectors along $\partial\mathbf{r}/\partial u$, $\partial\mathbf{r}/\partial v$ and $\partial\mathbf{r}/\partial w$, respectively. If we define by h_u, h_v and h_w as:

$$h_u \equiv \left|\frac{\partial \mathbf{r}}{\partial u}\right| \quad , \quad h_v \equiv \left|\frac{\partial \mathbf{r}}{\partial v}\right| \quad , \quad h_w \equiv \left|\frac{\partial \mathbf{r}}{\partial w}\right| \quad (4)$$

then the infinitesimal increment (3) can be re-written as:

$$\mathrm{d}\mathbf{r} = h_u \mathrm{d}u\mathbf{e}_u + h_v \mathrm{d}v\mathbf{e}_v + h_w \mathrm{d}w\mathbf{e}_w \quad (5)$$

Equation (5), and associated defnitions (4), are instrumental in the derivation of many fundamental quantities used in differential calculus, when passing from a cartesian to a curvilinear coordinate system. Let us consider, for example, polar coordinates, (r, θ), in the plane. x and y are functions of r and θ according to:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

To derive the correct expression for $\mathrm{d}\mathbf{r} \equiv (\mathrm{d}x, \mathrm{d}y)$ we need first to compute h_r and h_θ. From (4) we get:

$$h_r = \left|\left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}\right)\right| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

$$h_\theta = \left|\left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}\right)\right| = \sqrt{[-r\sin(\theta)]^2 + [r\cos(\theta)]^2} = r$$

Thus, $\mathrm{d}\mathbf{r}$ is given by:

$$\mathrm{d}\mathbf{r} = \mathrm{d}r\mathbf{e}_r + r\mathrm{d}\theta\mathbf{e}_\theta$$

With this result we are able to derive the form of several quantities in polar coordinates. For example, the line element is given by:

$$d\ell \equiv \sqrt{d\mathbf{r} \cdot d\mathbf{r}} = \sqrt{(dr)^2 + r^2(d\theta)^2}$$

while the area element is:

$$dS = h_r h_\theta dr d\theta = r dr d\theta$$

For the general, 3D, case the line element is given by:

$$d\ell \equiv \sqrt{d\mathbf{r} \cdot d\mathbf{r}} = \sqrt{(h_u du)^2 + (h_v dv)^2 + (h_w dw)^2} \tag{6}$$

and the volume element is:

$$dV \equiv [(\mathbf{e}_u \cdot d\mathbf{r})\mathbf{e}_u] \cdot \{[(\mathbf{e}_v \cdot d\mathbf{r})\mathbf{e}_v] \times [(\mathbf{e}_w \cdot d\mathbf{r})\mathbf{e}_w]\} = h_u h_v h_w du dv dw \tag{7}$$

For the curl computation it is also important to have ready expressions for the surface elements perpendicular to each coordinate curve. These elements are simply given as:

$$dS_u = h_v h_w dv dw \quad , \quad dS_v = h_u h_w du dw \quad , \quad dS_w = h_u h_v du dv \tag{8}$$

Curl in curvilinear coordinates

The curl of a vector field is another vector field. Its component along an arbitrary vector $\mathbf{n}$ is given by the following expression:

$$[\nabla \times \mathbf{v}]_n \equiv \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_\gamma \mathbf{v} \cdot d\mathbf{r} \tag{17}$$

where γ is a curve encircling the small area element ΔS, and $\mathbf{n}$ is perpendicular to ΔS. Let us start with the w-component. We need to select a surface element perpendicular to $\mathbf{e}_w$. This is given in Figure 3. The contribution to the line integral coming from segments 1 and 3 are

$$v_u h_u \mathrm{d}u$$

computed at $v - \mathrm{d}v/2$, and

$$-v_u h_u \mathrm{d}u$$

computed at $v + \mathrm{d}v/2$. These, added together, gives:

$$-\frac{\partial (h_u v_u)}{\partial v}\mathrm{d}u\mathrm{d}v \tag{18}$$

The contribution from segments 2 and 4 gives, on the other hand,

$$v_v h_v \mathrm{d}v$$

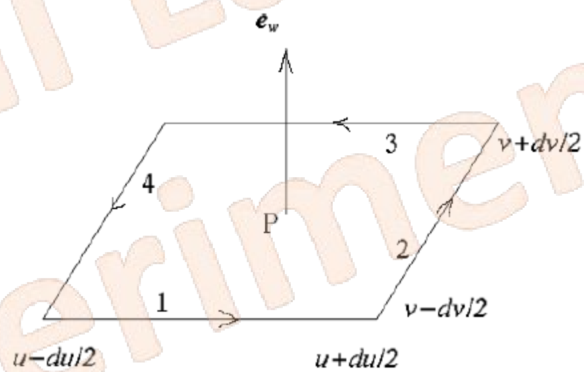


Figure 3: Surface element for the determination of curl's component along w, in curvilinear coordinates.

computed at $u + \mathrm{d}u/2$, and

$$-v_v h_v \mathrm{d}v$$

computed at $u - \mathrm{d}u/2$. Adding them together yields

$$\frac{\partial (h_v v_v)}{\partial u}\mathrm{d}u\mathrm{d}v \tag{19}$$

From the partial results (18) and (19) we obtain the contribution to the curl we were looking for:

$$[\nabla \times \mathbf{v}]_{\mathbf{e}_w} = \frac{1}{h_u h_v du dv}\left[\frac{\partial(h_v v_v)}{\partial u} - \frac{\partial(h_u v_u)}{\partial v}\right] du dv = \frac{1}{h_u h_v}\left[\frac{\partial(h_v v_v)}{\partial u} - \frac{\partial(h_u v_u)}{\partial v}\right]$$

The other two components can be derived from the previous expression with the cyclic permutation $u \to v \to w \to u$. To extract all three components the following compressed determinantal form can be used:

$$\nabla \times \mathbf{v} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u \mathbf{e}_u & h_v \mathbf{e}_v & h_w \mathbf{e}_w \\ \partial/\partial u & \partial/\partial v & \partial/\partial w \\ h_u v_u & h_v v_v & h_w v_w \end{vmatrix} \tag{20}$$

Curl of a vector A in spherical cordinates

$$\nabla \times \mathbf{A} = \frac{1}{(1)(r)(r\sin\theta)}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\left\{\frac{\partial}{\partial\theta}(r\sin\theta\, A_\phi) - \frac{\partial}{\partial\phi}(rA_\theta)\right\}\mathbf{e}_r + \left\{\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r\sin\theta\, A_\phi)\right\} r\,\mathbf{e}_\theta + \left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right\} r\sin\theta\,\mathbf{e}_\phi\right]$$

Q3:

Express $z\,\hat{i} - 2x\,\hat{j} + y\,\hat{k}$ *in cylindrical co-ordinates.*

Sol:

$x = r\cos\phi,\ y = r\sin\phi,\ z = z$

$\bar{R} = x\hat{i} + y\hat{j} + z\hat{k}$

$\bar{R} = \rho\cos\phi\ \hat{i} + \rho\sin\phi\ \hat{j} + z\hat{k}$

If T_ρ, T_ϕ, T_z be the unit vectors at P in the directions of the tangents to the ρ, ϕ, z curves respectively, then

$$\overline{T}_\rho = \frac{\frac{\partial \overline{R}}{\partial \rho}}{\left|\frac{\partial \overline{R}}{\partial \rho}\right|} = \frac{\cos\phi\hat{i} + \sin\phi\hat{j}}{\sqrt{\cos^2\phi + \sin^2\phi}} = \cos\phi\hat{i} + \sin\phi\hat{j}$$

$$\overline{T}_\phi = \frac{\frac{\partial \overline{R}}{\partial \phi}}{\left|\frac{\partial \overline{R}}{\partial \phi}\right|} = \frac{-\rho\sin\phi\hat{i} + \rho\cos\phi\hat{j}}{\sqrt{(-\rho\sin\phi)^2 + (\rho\cos\phi)^2}} = -\sin\phi\hat{i} + \sin\phi\hat{j}$$

$$\overline{T}_z = \frac{\frac{\partial \overline{R}}{\partial z}}{\left|\frac{\partial \overline{R}}{\partial z}\right|} = \hat{k}$$

$$\overline{f} = z\hat{i} - 2x\hat{j} + y\hat{k}$$

$$\overline{f} = z\hat{i} - 2\rho\cos\phi\hat{j} + \rho\sin\phi\hat{k}$$

$$f_1 = \overline{f}\cdot T_\rho = (z\hat{i} - 2\rho\cos\phi\hat{j} + \rho\sin\phi\hat{k})\cdot(\cos\phi\hat{i} + \sin\phi\hat{j})$$
$$= z\cos\phi - 2\rho\sin\phi\cos\phi$$

$$f_2 = \overline{f}\cdot\overline{T}_\phi = (z\hat{i} - 2\rho\cos\phi\hat{j} + \rho\sin\phi\hat{k})\cdot(-\sin\phi\hat{i} + \cos\phi\hat{j})$$
$$= -z\sin\phi - 2\rho\cos^2\phi$$

$$f_3 = \overline{f}\cdot\overline{T}_z = (z\hat{i} - 2\rho\cos\phi\hat{j} + \rho\sin\phi\hat{k})\cdot\hat{k}$$
$$= \rho\sin\phi$$

$$\overline{f} = f_1\overline{T}_\rho + f_2\overline{T}_\phi + f_3\overline{T}_z, \text{ where } f_1 = z\cos\phi - 2\rho\sin\phi\cos\phi,$$
$$f_2 = -z\sin\phi - 2\rho\cos^2\phi,\quad f_3 = \rho\sin\phi$$

Q4: Given

(i) $\vec{A} = 2xy\hat{x} + z\hat{y} + yz^2\hat{z}$, find $\vec{\nabla}\cdot\vec{A}$ at $(2,-1,3)$

(ii) $\vec{A} = 2r\cos^2\phi\hat{r} + 3r^2\sin z\hat{\phi} + 4z\sin^2\phi\hat{z}$, find $\vec{\nabla}\cdot\vec{A}$

(iii) $\vec{A} = 10\hat{r} + 5\sin\theta\hat{\theta}$, Find $\vec{\nabla}\cdot\vec{A}$

Sol:

(i) In Cartesian coordinates $\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

$A_x = 2xy, A_y = z, A_z = yz^2 \Rightarrow \vec{\nabla}\cdot\vec{A} = 2y + 0 + 2yz$, At $(2,-1,3)$, $\vec{\nabla}\cdot\vec{A} = -2-6=-8$

(ii) In cylindrical coordinates $\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$

$$A_r = 2r\cos^2\phi, A_\phi = 3r^2\sin z, A_z = 4z\sin^2\phi$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r}4r\cos^2\phi + 0 + 4\sin^2\phi = 4(\cos^2\phi + \sin^2\phi) = 4$$

(iii) In spherical coordinates, $\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

$$A_r = 10, A_\theta = 5\sin\theta, A_\phi = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}20r + \frac{1}{r\sin\theta}10\sin\theta\cos\theta = (2+\cos\theta)(10/r)$$

Q5:

Express $x\hat{i} + 2y\hat{j} + yz\hat{k}$ in spherical polar co-ordinates.

Sol:

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

$\bar{R} = x\hat{i} + y\hat{j} + z\hat{k}$

$\bar{R} = r\sin\theta\cos\phi\hat{i} + r\sin\theta\sin\phi\hat{j} + r\cos\theta\hat{k}$

In spherical co-ordinates, T_r, T_θ, T_ϕ be the unit vectors along the tangents to r, θ, ϕ curves respectively, then

$$\bar{T}_r = \frac{\partial\bar{R}/\partial r}{\left|\frac{\partial\bar{R}}{\partial r}\right|} = \frac{\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}}{\sqrt{(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta}}$$

$$= \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

$$\bar{T}_\theta = \frac{\frac{\partial\bar{R}}{\partial\theta}}{\left|\frac{\partial\bar{R}}{\partial\theta}\right|} = \frac{r\cos\theta\cos\phi\hat{i} + r\cos\theta\sin\phi\hat{j} - r\sin\theta\hat{k}}{\sqrt{(r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2}}$$

$$= \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}$$

$$\overline{T}_\phi = \frac{\frac{\partial \overline{R}}{\partial \phi}}{\left|\frac{\partial \overline{R}}{\partial \phi}\right|} = \frac{-r\sin\theta\sin\phi\hat{i} + r\sin\theta\cos\phi\,\hat{j}}{\sqrt{(-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2}}$$

$$= -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\overline{f} = x\hat{i} + 2y\hat{j} + yz\hat{k}$$

$$= r\sin\theta\cos\phi\hat{i} + 2r\sin\theta\sin\phi\hat{j} + r^2\sin\theta\sin\phi\cos\theta\hat{k}$$

$$f_1 = \overline{f}\cdot\overline{T}_r = [r\sin\theta\cos\phi\hat{i} + 2r\sin\theta\sin\phi\hat{j} + r^2\sin\theta\sin\phi\cos\theta\hat{k}]$$

$$[\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\,\hat{j} - \cos\theta\hat{k}]$$

$$= r\sin^2\theta\cos^2\phi + 2r\sin^2\theta\sin^2\phi + r^2\sin\theta\sin\phi\cos^2\theta$$

$$= r\sin^2\theta(\cos^2\phi + \sin^2\phi) - r\sin^2\theta\sin^2\phi + r^2\sin\theta\sin\phi\cos^2\theta$$

$$= r\sin^2\theta + r\sin^2\theta\sin^2\phi + r^2\sin\theta\sin\phi\cos^2\theta$$

$$= r\sin^2\theta(1 + \sin^2\phi) + r^2\sin\theta\cos^2\theta\sin\phi$$

$$f_2 = \overline{f}\cdot\overline{T}_\theta = [r\sin\theta\cos\phi\hat{i} + 2r\sin\theta\sin\phi\,\hat{j} + r^2\sin\theta\sin\phi\cos\theta\hat{k}]$$

$$[\cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}]$$

$$= r\sin\theta\cos\theta\cos^2\phi + 2r\sin\theta\cos\theta\sin^2\phi - r^2\sin^2\theta\sin\phi\cos\theta$$

$$= r\sin\theta\cos\theta(1 + \sin^2\phi) - r^2\sin^2\theta\cos\theta\sin\phi$$

$$f_3 = \overline{f}\cdot\overline{T}_\phi = [r\sin\theta\cos\phi\hat{i} + 2r\sin\theta\sin\phi\hat{j} + r^2\sin\theta\sin\phi\cos\theta\hat{k}]\ [-\sin\phi\hat{i} + \cos\phi\hat{j}]$$

$$= -r\sin\theta\sin\phi\cos\phi + 2r\sin\theta\sin\phi\cos\phi$$

$$= r\sin\theta\sin\phi\cos\phi$$

$$\overline{f} = f_1\overline{T}_r + f_2\overline{T}_\theta + f_3\overline{T}_\phi$$

where

$$f_1 = r\sin^2\theta(1 + \sin^2\phi) + r^2\sin\theta\cos^2\theta\sin\phi$$

$$f_2 = r\sin\theta\cos\theta(1 + \sin^2\phi) - r^2\sin^2\theta\cos\theta\sin\phi$$

$$f_3 = r\sin\theta\sin\phi\cos\phi$$

Q6: Find the curl of the vector

$$\overline{A} = \left(e^{-r}/r\right)\hat{\theta}$$

Sol:

$$\vec{A} = \left(e^{-r}/r\right)\hat{\theta} \Rightarrow A_r = 0,\ A_\theta = \left(e^{-r}/r\right),\ A_\phi = 0$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} = -\frac{e^{-r}}{r}\hat{\phi}$$

Q7: Find the nature of the following fields by determining divergence and curl.

(i) $\vec{F}_1 = 30\hat{x} + 2xy\hat{y} + 5xz^2\hat{z}$

(ii) $\vec{F}_2 = \left(\frac{150}{r^2}\right)\hat{r} + 10\hat{\phi}$ (Cylindrical coordinates)

Sol:

(i) $\vec{F}_1 = 30\hat{x} + 2xy\hat{y} + 5xz^2\hat{z} \Rightarrow \vec{\nabla}\cdot\vec{F}_1 = \frac{\partial F_{1x}}{\partial x} + \frac{\partial F_{1y}}{\partial y} + \frac{\partial F_{1z}}{\partial z} = 2x(1+5z)$

Divergence exists, so the field is non-solenoidal.

$$\vec{\nabla} \times \vec{F}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 30 & 2xy & 5xz^2 \end{vmatrix} = -5z^2\hat{y} + 2y\hat{z}$$

The field has a curl so it is rotational.

(ii) In cylindrical coordinates,

$$\text{Divergence } \vec{\nabla}\cdot\vec{F}_2 = \frac{1}{r}\frac{\partial}{\partial r}(rF_{2r}) + \frac{1}{r}\frac{\partial F_{2\phi}}{\partial \phi} + \frac{\partial F_{2z}}{\partial z} = \frac{-150}{r^3}$$

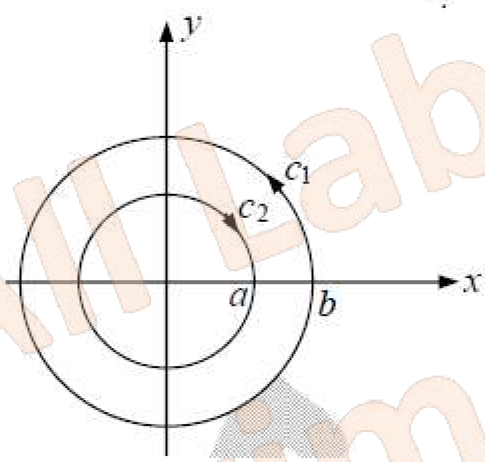
The field is non-solenoid.

$$\vec{\nabla}\times\vec{F}_2=\frac{1}{r}\begin{vmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ \left(\frac{150}{r^2}\right) & 10r & 0\end{vmatrix}=\frac{10}{r}\hat{z}$$

It has non-zero curl so it is rotational.

Q8:

Given $\vec{A}=2r\cos\phi\,\hat{r}+r\hat{\phi}$ in cylindrical coordinates. Find $\oint_{C_1}\vec{A}\cdot d\vec{l}+\oint_{C_2}\vec{A}\cdot d\vec{l}$ where c_1 and c_2 are contours shown in figure.



Sol: In cylindrical coordinate system

$$d\vec{l}=dr\hat{r}+rd\phi\hat{\phi}+dz\hat{z}\ ,\ \vec{A}=2r\cos\phi\,\hat{r}+r\hat{\phi}$$

$$\vec{A}\cdot d\vec{l}=2r\cos\phi dr+r^2d\phi$$

In figure on curve c1,Φ varies from 0 to 2π , r = b and dr = 0

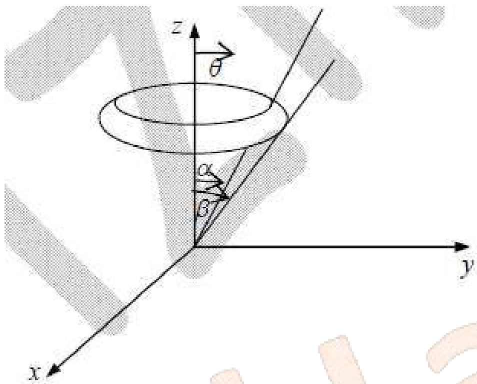
$$\oint_{c_1}\vec{A}\cdot d\vec{l}=\int_{\substack{\phi=0\\ r=b}}^{2\pi}r^2d\phi=2\pi b^2$$

In figure on curve c2,Φ varies from 0 to -2π , r = a and dr = 0

$$\oint_{c_2}\vec{A}\cdot d\vec{l}=\int_{\substack{\phi=0\\ r=a}}^{-2\pi}r^2d\phi=-2\pi a^2$$

$$\oint_{c_1} \vec{A}\cdot d\vec{l} + \oint_{c_2} \vec{A}\cdot d\vec{l} = 2\pi\left(b^2 - a^2\right)$$

Q9: Use spherical coordinate system to find the area of the strip α ≤ ϴ ≤ β on the spherical shell of radius 'a'. Calculate the area when α = 0 and β =π.



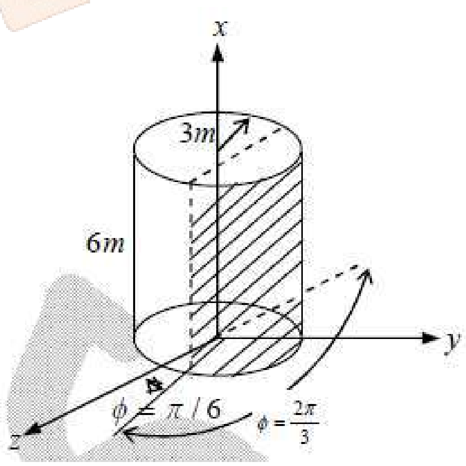
Sol: Sphere has radius 'a' and ϴ varies between α and β. For fixed radius the elemental surface is

$$da = (r\sin\theta\, d\phi)(r\, d\theta) = r^2 \sin\theta\, d\theta\, d\phi$$

$$\text{Area } A = \int_{\theta=\alpha}^{\beta}\int_{\phi=0}^{2\pi} r^2 \sin\theta\, d\theta\, d\phi = 2\pi a^2 \int_{\theta=\alpha}^{\beta} \sin\theta\, d\theta = 2\pi a^2(\cos\alpha - \cos\beta)$$

For $\alpha = 0, \beta = \pi$, Area $= 2\pi a^2(1+1) = 4\pi a^2$, is surface area of the sphere.

Q10: Use the cylindrical coordinate system to find the area of a curved surface on the right circular cylinder having radius = 3 m and height = 6 m and30^0 ≤Φ≤ 120^0.



Sol: From figure, surface area is required for a cylinder when r = 3m, z = 0 to 6m,

$$30^0 \le \phi \le 120^0 \text{ or } \frac{\pi}{6} \le \phi \le \frac{2\pi}{3}$$

In cylindrical coordinate system, the elemental surface area as scalar is

$$d\vec{a} = r\, d\phi\, dz\hat{r}$$

Taking the magnitude only

$$A = \int_S da = \int_{\phi=\pi/6}^{2\pi/3} \int_{z=0}^{6} r\, d\phi\, dz = 3\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)6 = 9\pi\, m^2$$