Free Study Material from

All Lab Experiments

Waves & Optics (B.Sc.)
Chapter - 1
Wave Motion

Chapter-3
Wave motion

Waves Motion- General: Transverse waves on a string. Travelling and standing waves on a string. Normal Modes of a string. Group velocity, Phase velocity. Plane waves. Spherical waves, Wave intensity. **(8 Lectures)**

Q:derive an expression for displacement in transverse waves on a stretched string.

Ans:

Let us consider a stretched string having a tension T. In its equilibrium position the string is assumed to lie on the x axis. If the string is pulled in the y direction, then forces will act on the string which will tend to bring it back to its equilibrium position. Let us consider a small length AB of the string and calculate the net force acting on it in the y direction. Due to the tension T, the endpoints A and B experience force in the direction of the arrows shown in Fig. 11.6. The force at A in the upward direction is

$$-T\sin\theta_1 \approx -T\tan\theta_1 = -T\left.\frac{\partial y}{\partial x}\right|_x$$

Similarly, the force at B in the upward direction is

$$T\sin\theta_2 \approx T\tan\theta_2 = T\left.\frac{\partial y}{\partial x}\right|_{x+dx}$$

where we have assumed θ_1 and θ_2 to be small. Thus the net force acting on AB in the y direction is

$$T\left[\left(\frac{\partial y}{\partial x}\right)_{x+dx}-\left(\frac{\partial y}{\partial x}\right)_{x}\right]=T\frac{\partial^2 y}{\partial x^2}dx \qquad \textbf{(33)}$$

where we have used the Taylor series expansion of $(\partial y/\partial x)_{x+dx}$ about the point x

$$\left(\frac{\partial y}{\partial x}\right)_{x+dx}=\left(\frac{\partial y}{\partial x}\right)_{x}+\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\bigg|_{x} dx$$

and have neglected higher-order terms because dx is infinitesimal. The equation of motion is therefore

$$\Delta m\,\frac{\partial^2 y}{\partial t^2}=T\frac{\partial^2 y}{\partial x^2}dx$$

where Δm is the mass of element AB. If ρ is the mass per unit length, then

$$\Delta m=\rho\, dx$$

and we get

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{T/\rho}\frac{\partial^2 y}{\partial t^2}$$

which is the one-dimensional wave equation. Thus we may conclude that transverse waves can propagate through a stretched string, and if we compare the above equation with Eq. (25), we obtain the following expression for the speed of the transverse waves:

$$v = \sqrt{\frac{T}{\rho}} \quad (36)$$

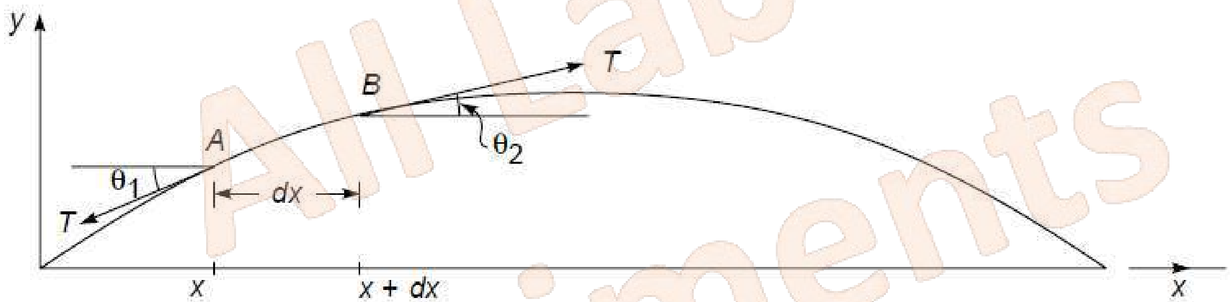
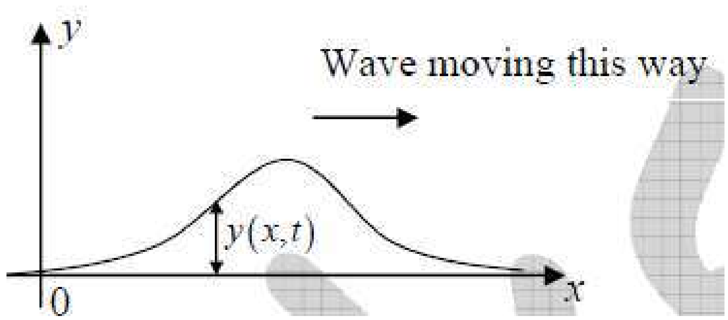


Fig. 11.6 Transverse vibrations of a stretched string.

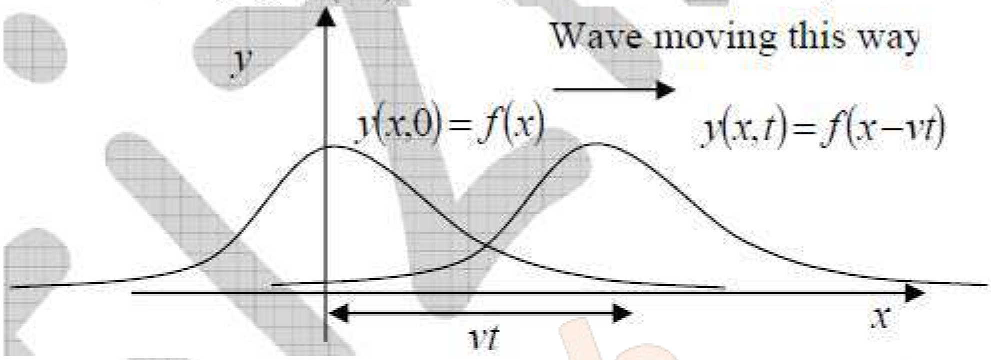
Q: derive an expression for differential equation and energy in travelling waves on a stretched string.

Ans:

Let's start with a rope, like a clothesline, stretched between two hooks. You take one end off the hook, holding the rope, and, keeping it stretched fairly tight, wave your hand up and back once. If you do it fast enough, you'll see a single bump travel along the rope: This is the simplest example of a ***traveling wave***.



Taking the rope to be stretched tightly enough that we can take it to be horizontal, we'll use it as x-axis. The y-axis is taken vertically upwards, and we only wave the rope in an up-and-down way, so actually $y(x,t)$ will be how far the rope is from its rest position at x at time t : that is, the graph $y(x,t)$ above just shows where the rope is at time t.



Taking for convenience time $t = 0$ to be the moment when the peak of the wave passes $x = 0$, we graph here the rope's position at $t = 0$, and some later time t. Denoting the first function by $y(x,0) = f(x)$, than the displacement y of the rope at any horizontal position at x at time t has the form

$$y(x,t) = f(x - vt)$$

it's *the same function*—the "same shape"—but moved over by vt, where v is the velocity of the wave in the positive x axis.

Expression for a Plane Progressive Harmonic Wave

Wave equation can be expressed in following types $y = a\sin\omega\left(t - \frac{x}{v}\right)$

In term of time period $T = \frac{2\pi}{\omega}$ and wavelength $\lambda = vT$ is $y = a\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$

In term of wave vector $k = \frac{2\pi}{\lambda}$ is

$y = a\sin\frac{2\pi}{\lambda}(vt - x)$ or $y = a\sin k(vt - x)$ or $y = a\sin(\omega t - kx)$

Differential Equation of Wave Motion

We have the equation of plane progressive wave motion is

$$y = a\sin\frac{2\pi}{\lambda}(vt - x)$$

$$\text{acceleration of the particle} = \frac{\partial^2 y}{\partial t^2} = -\frac{4\pi^2 v^2}{\lambda^2} a\sin\frac{2\pi}{\lambda}(vt - x) = -\frac{4\pi^2 v^2}{\lambda^2} y$$

Also, rate of change of compression with distance can be obtain as

$$\frac{\partial^2 y}{\partial x^2} = -\frac{4\pi^2}{\lambda^2} a\sin\frac{2\pi}{\lambda}(vt - x) = -\frac{4\pi^2}{\lambda^2} y$$

Combining above two equation, we get

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

This is the differential equation for one dimensional wave motion. We can also write

partilce acceleration = $(\text{wave velocity})^2 \times$ curvature of the dispacement curve

Energy Density of the Plane progressive Wave

In a progressive wave motion, energy from the source is passed from particles to other particles. There is a regular transmission of energy across every section of the medium.

Energy density of plane progressive wave = total energy

= {kinetic energy (K.E) + potential energy (P.E.)} per unit volume of the medium.

Let ρ is the density of the medium in which wave is moving.

Thus the kinetic energy per unit volume K_ρ is obtained as

$$K.E. = \frac{1}{2}(\text{mass})(\text{velocity})^2 = \frac{1}{2}(\text{density}\times\text{velocity})(\text{velocity})$$

$$K_\rho = \frac{K.E.}{\text{volume}} = \frac{1}{2}(\text{density})(\text{velocity})^2 = \frac{2\pi^2 v^2 \rho}{\lambda^2} a^2 \cos^2\frac{2\pi}{\lambda}(vt - x)$$

The potential energy per unit volume U_ρ is obtained as

Force acting perunit volume = density $\times$ acceleration $= \rho \frac{\partial^2 y}{\partial t^2} = \frac{4\pi^2 v^2 \rho}{\lambda^2} y$

Now, work done per unit volume in a small displacement dy

$$= \text{Force per unit volume} \times \text{displacement} = \frac{4\pi^2 v^2 \rho}{\lambda^2} y\,dy$$

Work done per unit volume for whole displacement from 0 to y

$$= \int_0^y \frac{4\pi^2 v^2 \rho}{\lambda^2} y\,dy = \frac{2\pi^2 v^2 \rho}{\lambda^2} y^2 = \frac{2\pi^2 v^2 \rho}{\lambda^2} a^2 \sin^2 \frac{2\pi}{\lambda}(vt - x)$$

This work done is stored in the medium in the form of potential energy.

Thus $U_\rho = P.E.$ per unit volume $= \frac{2\pi^2 v^2 \rho}{\lambda^2} a^2 \sin^2 \frac{2\pi}{\lambda}(vt - x)$

The total energy per unit volume of the medium or the energy density of the plane progressive wave is $E_\rho = K_\rho + U_\rho$

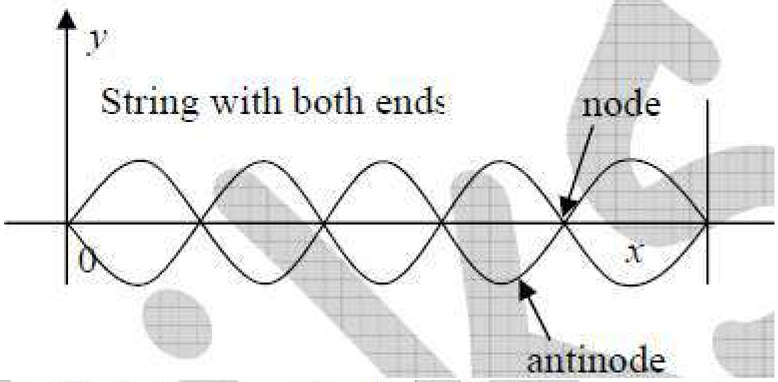
$$E_\rho = \frac{2\pi^2 v^2 \rho}{\lambda^2} a^2 \cos^2 \frac{2\pi}{\lambda}(vt - x) + \frac{2\pi^2 v^2 \rho}{\lambda^2} a^2 \sin^2 \frac{2\pi}{\lambda}(vt - x)$$

$E_\rho = \frac{2\pi^2 v^2 \rho}{\lambda^2} a^2 = 2\pi^2 \left(\frac{v}{\lambda}\right) \rho a^2 = 2\pi^2 f^2 a^2 \rho$ where, f is the frequency of the wave.

Q: derive an expression for differential equation and energy in standing waves on a stretched string.

Ans:

Another familiar kind of wave is that generated on a string fixed at both ends when it is made to vibrate. We found in class that for certain frequencies the string vibrated in a sine-wave pattern, as illustrated below, with no vibration at the ends, of course, but also no vibration at a series of equally-spaced points between the ends: these quiet places we term ***nodes***. The places of maximum oscillation are ***antinodes***.



When two identical wave either longitudinal or transverse, travel through a medium along the same line in opposite direction, they superimpose to produce a wave which appears to be stationary in space. Such waves are called stationary wave or standing wave.

Condition for Stationary waves

The presence of bounded medium is prerequisite for producing stationary wave. Medium should not be infinite in length, it should have boundary.

If the medium is infinite, the wave just continues to travel along it for an infinite time, i.e. such medium support travelling wave.

General Consideration of Stationary Wave

Consider a plane progressive wave of amplitude a travelling with velocity v along the $+x$ axis. Its equation is written as $y_1 = a\sin\frac{2\pi}{\lambda}(vt - x)$

where, y_1 is the displacement at a point x at a time t and a is the amplitude.

If this wave is reflected from a rigid boundary wall situated at $x = 0$. The displacement y_2 of the reflected wave will be $y_2 = a'\sin\frac{2\pi}{\lambda}(vt + x)$

Where, a' is the amplitude of the reflected wave.

Both the wave travelling in the same linear path superimpose and the equation of the resultant wave is $y = y_1 + y_2 = a\sin\frac{2\pi}{\lambda}(vt - x) + a'\sin\frac{2\pi}{\lambda}(vt + x)$

The displacement y at rigid wall (at $x = 0$) will always be zero. This give the boundary condition, $y = 0$ at $x = 0$.

Thus $0 = a\sin\frac{2\pi}{\lambda}(vt) + a'\sin\frac{2\pi}{\lambda}(vt) \quad \Rightarrow \quad a = -a'$

Thus the resultant displacement is

$$y = a\left[\sin\frac{2\pi}{\lambda}(vt - x) - \sin\frac{2\pi}{\lambda}(vt + x)\right] = -2a\sin\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda} \Rightarrow y = -2a\sin kx\cos\omega t$$

This is the equation of the resultant stationary wave. Equation shows that the resulting wave is also a simple harmonic wave of the same time period and wavelength but with the amplitude of $-2a\sin kx$.

The particle velocity is

$$u = \frac{\partial y}{\partial t} = \frac{4\pi a v}{\lambda}\sin\frac{2\pi x}{\lambda}\sin\frac{2\pi vt}{\lambda}$$

The resultant pressure variation is

$$p = -K\frac{\partial y}{\partial x} = K\frac{4\pi a}{\lambda}\cos\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda}$$

Q:what is phase velocity. Derive the expression for phase velocity.

Ans:

The phase velocity of a wave is the rate at which the phase of the wave propagates in space. This is the velocity at which the phase of any one frequency component of the wave travels. For such a component, any given phase of the wave will appear to travel at the phase velocity.

Consider a monochromatic wave with angular frequency $\omega = 2\pi\nu$ (where ν is the frequency) travelling in $+ve$ x-direction is given by

$$y = a\sin(\omega t - kx)$$

Consider a point on the sine wave at a particular instant of time the wave equation can be written as $y = a\sin\omega t$.

After certain time say t_0 if the point is displaced by a distance x the equation is given by $y = a\sin\left[\omega(t - t_0)\right]$

Velocity of the point $v = \frac{x}{t_0}$ or $t_0 = \frac{x}{v}$

Therefore $y = a\sin\left[\omega\left(t - \frac{x}{v}\right)\right] = a\sin\left[\omega t - \left(\frac{\omega}{v}\right)x\right] = a\sin(\omega t - kx)$ where $k = \frac{\omega}{v}$

Where k is defined as the propagation constant or the wave number which is also defined as $k = \frac{2\pi}{\lambda}$. 'v' mentioned above as stated earlier is associated with the individual wave and hence it is nothing but phase velocity.

Thus phase velocity $v_p = \frac{\omega}{k}$

Q:what is group velocity. Derive the expression for group velocity.

A number of waves of different frequencies, wavelengths and velocities may be superposed to form a group. Waves rarely occur as single monochromatic components; a white light pulse consists of an infinitely fine spectrum of frequencies and the motion of such a pulse would be described by its group velocity. Such a group would, of course, 'disperse' with time because the wave velocity of each component would be different in all media except free space. Only in free space would it remain as white light. Its importance is that it is the velocity with which the energy in the wave group is transmitted. For a monochromatic wave the group velocity and the wave velocity are identical.

We begin by considering a group which consists of two components of equal amplitude a but frequencies ω_1 and ω_2 which differ by a small amount.

Their separate displacements are given by $y_1 = a\cos(\omega_1 t - k_1 x)$ and $y_2 = a\cos(\omega_2 t - k_2 x)$

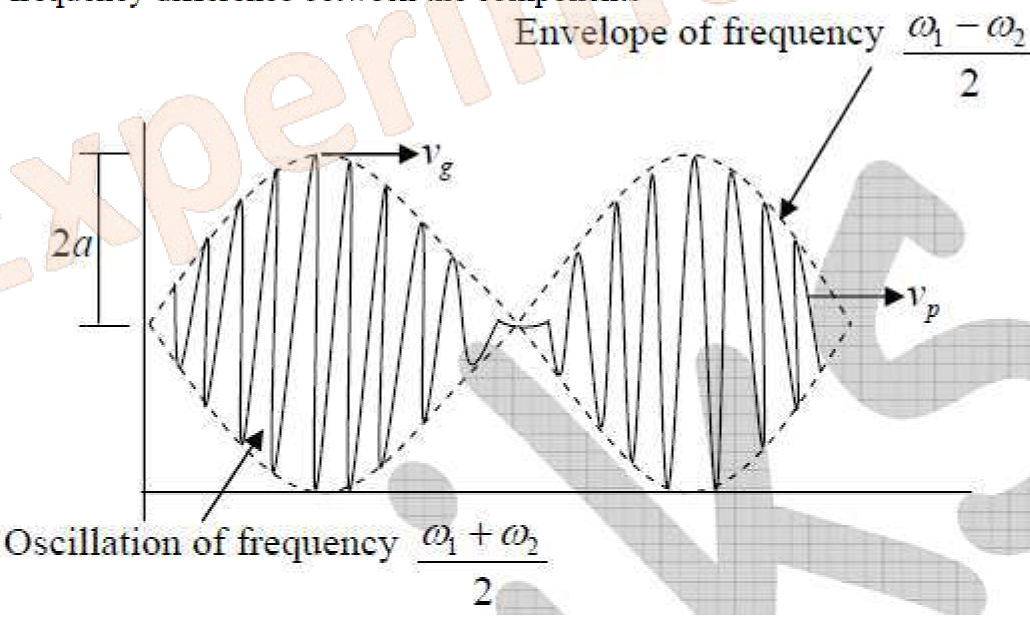
Superposition of amplitude and phase gives

$$y = y_1 + y_2 = 2a\cos\left[\frac{(\omega_1-\omega_2)}{2}t - \frac{(k_1-k_2)}{2}x\right]\cos\left[\frac{(\omega_1+\omega_2)}{2}t - \frac{(k_1+k_2)}{2}x\right]$$

a wave system with a frequency $\frac{(\omega_1+\omega_2)}{2}$ which is very close to the frequency of either component but with a maximum amplitude of $2a$, modulated in space and time by a very slowly varying envelope of frequency $\frac{(\omega_1-\omega_2)}{2}$ and wave number $\frac{(k_1-k_2)}{2}$.

Then $\frac{(\omega_1-\omega_2)}{2}t - \frac{(k_1-k_2)}{2}x = \text{constant} \Rightarrow v_g = \frac{dx}{dt} = \frac{\omega_1-\omega_2}{k_1-k_2}$

Figures shows that the faster oscillation occurs at the average frequency of the two components $\frac{(\omega_1+\omega_2)}{2}$ and the slowly varying group envelope has a frequency $\frac{(\omega_1-\omega_2)}{2}$, half the frequency difference between the components



Q: What are the values of phase velocity and group velocity respectively of the de-Broglie wave describing a free electron with velocity v?

Solution: According to de-Broglie, every moving object is associated with a wave, known as de-Broglie wave. If m is mass of the object moving with velocity v then associated wavelength λ is given as $\lambda = \frac{h}{mv}$ where h is Planck's constant

The velocity of wavefront of constant phase is called phase velocity and is equal to the velocity of the wave

Hence, phase velocity $= \lambda \nu$, by energy equation $mc^2 = h\nu \Rightarrow \nu = \frac{mc^2}{h}$

Thus Phase velocity $= \lambda \times \nu = \frac{h}{mv} \times \frac{mc^2}{h} = \frac{c^2}{v}$

Now, the group velocity is given as $v_g = \frac{d\omega}{dk}$

$\because \omega = 2\pi\nu = \frac{2\pi mc^2}{h}$ *and* $k = \frac{2\pi}{\lambda} = \frac{2\pi mv}{h}$ where $m = \frac{m_0}{\sqrt{1-v^2/c^2}}$

Thus, $\omega = \frac{2\pi m_0 c^2}{h\sqrt{1-v^2/c^2}}$ *and* $k = \frac{2\pi m_0 v}{h\sqrt{1-v^2/c^2}}$

By differentiation we get,

$$\frac{d\omega}{dv} = \frac{2\pi m_0 v}{h\left(1-v^2/c^2\right)^{3/2}} \quad and \quad \frac{dk}{dv} = \frac{2\pi m_0}{h\left(1-v^2/c^2\right)^{3/2}}$$

$$\Rightarrow v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv} = \frac{2\pi m_0 v}{h\left(1-v^2/c^2\right)^{3/2}} / \frac{2\pi m_0}{h\left(1-v^2/c^2\right)^{3/2}} = v$$