Free Study Material from

All Lab Experiments

Mathematical Physics - III
Chapter - 3
Laplace Transform

Chapter 3
laplace transforms

Laplace Transforms: Laplace Transform (LT) of Elementary functions. Properties of LTs: Change of Scale Theorem, Shifting Theorem. LTs of 1st and 2nd order Derivatives and Integrals of Functions, Derivatives and Integrals of LTs. LT of Unit Step function, Dirac Delta function, Periodic Functions. Convolution Theorem. Inverse LT. Application of Laplace Transforms to 2nd order Differential Equations: Coupled differential equations of 1st order. Solution of heat flow along semi infinite bar using Laplace transform. (15 Lectures)

Q: Find the Laplace transform of f(t) defined as

$$f(t)=\begin{cases}\dfrac{t}{k}, & when\ \ 0<t<k\\ 1, & when\ \ t>k\end{cases}$$

Solution. $L[f(t)]=\int_0^k \frac{t}{k}e^{-st}dt+\int_k^\infty 1.e^{-st}dt=\frac{1}{k}\left[\left(t\frac{e^{-st}}{-s}\right)_0^k-\int_0^k\frac{e^{-st}}{-s}dt\right]+\left[\frac{e^{-st}}{-s}\right]_k^\infty$

$$=\frac{1}{k}\left[\frac{ke^{-ks}}{-s}-\left(\frac{e^{-st}}{s^2}\right)_0^k\right]+\frac{e^{-ks}}{s}=\frac{1}{k}\left[\frac{ke^{-ks}}{-s}-\frac{e^{-sk}}{s^2}+\frac{1}{s^2}\right]+\frac{e^{-ks}}{s}$$

$$=-\frac{e^{-sk}}{s}-\frac{1}{k}\frac{e^{-ks}}{s^2}+\frac{1}{k}\frac{1}{s^2}+\frac{e^{-ks}}{s}=\frac{1}{ks^2}[-e^{-ks}+1]$$

Q: From the first principle, find the Laplace transform of (1 + cos 2 t).

Solution. Laplace transform of (1 + cos 2 *t*)

$$=\int_0^\infty e^{-st}(1+\cos 2t)dt=\int_0^\infty e^{-st}\left(1+\frac{e^{2it}+e^{-2it}}{2}\right)dt$$

$$=\frac{1}{2}\int_0^\infty\left[2e^{-st}+e^{(-s+2i)t}+e^{(-s-2i)t}\right]dt=\frac{1}{2}\left[\frac{2e^{-st}}{-s}+\frac{e^{(-s+2i)t}}{-s+2i}+\frac{e^{(-s-2i)t}}{-s-2i}\right]_0^\infty$$

$$=\frac{1}{2}\left[\left(0+\frac{2}{s}\right)+\frac{1}{-s+2i}(0-1)+\frac{1}{-s-2i}(0-1)\right]$$

$$=\frac{1}{2}\left[\frac{2}{s}+\frac{1}{s-2i}+\frac{1}{s+2i}\right]=\frac{1}{2}\left[\frac{2}{s}+\frac{2s}{s^2+4}\right]$$

$$=\frac{1}{s}+\frac{s}{s^2+4}=\frac{2s^2+4}{s\left(s^2+4\right)}$$

Q: Find the Laplace transform of cos² t.

Solution. $\cos 2t = 2\cos^2 t - 1$

$$\therefore \quad \cos^2 t = \frac{1}{2}[\cos 2t + 1]$$

$$L(\cos^2 t) = \mathrm{L}\left[\frac{1}{2}(\cos 2t+1)\right]=\frac{1}{2}\left[\mathrm{L}(\cos 2t)+\mathrm{L}(1)\right]$$

$$=\frac{1}{2}\left[\frac{s}{s^2+(2)^2}+\frac{1}{s}\right]=\frac{1}{2}\left[\frac{s}{s^2+4}+\frac{1}{s}\right]$$

Q: Find the Laplace Transform of $t^{-(1/2)}$.

Solution. We know that $\mathrm{L}\left(t^n\right)=\frac{\left|\overline{n+1}\right.}{s^{n+1}}$

Put $n=-\frac{1}{2}, L\left(t^{-1/2}\right)=\frac{\left|\overline{-\frac{1}{2}+1}\right.}{s^{-1/2+1}}=\frac{\left|\overline{\frac{1}{2}}\right.}{\sqrt{s}}=\frac{\sqrt{\pi}}{\sqrt{s}}$, where $\left|\overline{\frac{1}{2}}\right. = \sqrt{\pi}$

Q: Find the Laplace Transform of t sin at.

Solution. $L(t\sin at)=L\left(t\frac{e^{iat}-e^{-iat}}{2i}\right)=\frac{1}{2i}\left[L\left(t.e^{iat}\right)-L\left(t.e^{-iat}\right)\right]$

$$=\frac{1}{2i}\left[\frac{1}{(s-ia)^2}-\frac{1}{(s+ia)^2}\right]=\frac{1}{2i}\left[\frac{(s+ia)^2-(s-ia)^2}{(s-ia)^2(s+ia)^2}\right]$$

$$=\frac{1}{2i}\frac{\left(s^2+2ias-a^2\right)-\left(s^2-2ias-a^2\right)}{\left(s^2+a^2\right)^2}$$

$$=\frac{1}{2i}\frac{4ias}{\left(s^2+a^2\right)^2}=\frac{2as}{\left(s^2+a^2\right)^2}$$

Q: Find the Laplace Transform of t^2 cos at.

Solution. $$L(t^2\cos at) = L\left(t^2\,\frac{e^{iat}+e^{-iat}}{2}\right) = \frac{1}{2}\left[L\left(t^2.e^{iat}\right)+L\left(t^2e^{-iat}\right)\right]$$

$$= \frac{1}{2}\left[\frac{2!}{(s-ia)^3}+\frac{2!}{(s+ia)^3}\right] = \frac{(s+ia)^3+(s-ia)^3}{(s-ia)^3(s+ia)^3}$$

$$= \frac{\left(s^3+3ias^2-3a^2s-ia^3\right)+\left(s^3-3ias^2-3a^2s+ia^3\right)}{\left(s^2+a^2\right)^3}$$

$$= \frac{2s^3-6a^2s}{\left(s^2+a^2\right)^3} = \frac{2s\left(s^2-3a^2\right)}{\left(s^2+a^2\right)^3}$$

Q: Find the Laplace transform of t sinh at.

Solution. $$L(\sinh at) = \frac{a}{s^2-a^2}$$

$$\therefore \quad L[t\sinh at] = -\frac{d}{ds}\left(\frac{a}{s^2-a^2}\right)$$

$$\Rightarrow \quad L[t\sinh at] = \frac{2as}{\left(s^2-a^2\right)}$$

Q: Find the Laplace transform of t^2 cos at

Solution. $$L(\cos at) = \frac{a}{s^2+a^2}$$

$$L\left(t^2\cos at\right) = (-1)^2\frac{d^2}{ds^2}\left[\frac{s}{s^2+a^2}\right] = \frac{d}{ds}\,\frac{\left(s^2+a^2\right).1-s(2s)}{\left(s^2+a^2\right)^2} = \frac{d}{ds}\,\frac{a^2-s^2}{\left(s^2+a^2\right)^2}$$

$$= \frac{\left(s^2+a^2\right)^2(-2s)-\left(a^2-s^2\right).2\left(s^2+a^2\right)(2s)}{\left(s^2+a^2\right)^4} = \frac{-2s^3-2a^2s-4a^2s+4s^3}{\left(s^2+a^2\right)^3}$$

$$= \frac{2s\left(s^2-3a^2\right)}{\left(s^2+a^2\right)^3}$$

Ans.

Q: Obtain the Laplace transform of

$t^2 e^t . \sin 4t$

Solution. $L(\sin 4t) = \dfrac{4}{s^2+16}$, $L(e^t \sin 4t) = \dfrac{4}{(s-1)^2+16}$

$$L(t\, e^t \sin 4t) = -\frac{d}{ds}\frac{4}{s^2-2s+17} = \frac{4(2s-2)}{\left(s^2-2s+17\right)^2}$$

$$L(t^2 e^t \sin 4t) = -4\frac{d}{ds}\frac{2s-2}{\left(s^2-2s+17\right)^2}$$

$$= -4\frac{\left(s^2-2s+17\right)^2 2-(2s-2)2\left(s^2-2s+17\right)(2s-2)}{\left(s^2-2s+17\right)^4}$$

$$= \frac{-4\left(2s^2-4s+34-8s^2+16s-8\right)}{\left(s^2-2s+17\right)^3}$$

$$= \frac{-4\left(-6s^2+12s+26\right)}{\left(s^2-2s+17\right)^3} = \frac{8\left(3s^2-6s-13\right)}{\left(s^2-2s+17\right)^3}$$

Q: Find the Laplace transform of

$\dfrac{\sin 2t}{t}$.

Solution. $L(\sin 2t) = \dfrac{2}{s^2+4}$

$$L\left(\frac{\sin 2t}{t}\right) = \int_s^\infty \frac{2}{s^2+4}ds = 2\cdot\frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty$$

$$= \left[\tan^{-1}\infty - \tan^{-1}\frac{s}{2}\right] = \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$$

$$= \cot^{-1}\frac{s}{2}$$

Q: Find the Laplace transform of

$$f(t) = \int_0^t \frac{\sin t}{t}dt.$$

Solution. $L\ \sin t = \dfrac{1}{s^2+1}$

$$L\frac{\sin t}{t} = \int_s^\infty \frac{1}{s^2+1}ds = \left[\tan^{-1}s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}s \ = \cot^{-1}s$$

$$L\int_0^t \frac{\sin t}{t}dt = \frac{1}{s}\cot^{-1}s$$

Q: Find the Laplace transform of

$$\frac{1-\cos t}{t^2}.$$

Solution. $L\ (1-\cos t) = L(1) - L\ (\cos t) = \dfrac{1}{s} - \dfrac{s}{s^2+1}$

$$L\frac{(1-\cos t)}{t} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right)ds = \left[\log s - \frac{1}{2}\log\left(s^2+1\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s^2 - \log\left(s^2+1\right)\right]_s^\infty = \frac{1}{2}\left[\log\frac{s^2}{s^2+1}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2}{s^2\left(1+\frac{1}{s^2}\right)}\right]_s^\infty = \frac{1}{2}\left[0 - \log\frac{s^2}{s^2+1}\right] = -\frac{1}{2}\log\frac{s^2}{s^2+1}$$

Again, $L\left[\dfrac{1-\cos t}{t^2}\right] = -\dfrac{1}{2}\displaystyle\int_s^\infty \log\frac{s^2}{s^2+1}ds = -\frac{1}{2}\int_s^\infty\left(\log\frac{s^2}{s^2+1}\cdot 1\right)ds$

Integrating by parts, we have

$$= -\frac{1}{2}\left[\log\frac{s^2}{s^2+1}\cdot s - \int\frac{s^2+1}{s^2}\frac{\left(s^2+1\right)2s - s^2\left(2s\right)}{\left(s^2+1\right)^2}\cdot s ds\right]_s^\infty$$

$$= -\frac{1}{2}\left[s\log\frac{s^2}{s^2+1} - 2\int\frac{1}{s^2+1}ds\right]_s^\infty = -\frac{1}{2}\left[s\log\frac{s^2}{s^2+1} - 2\tan^{-1}s\right]_s^\infty$$

$$= -\frac{1}{2}\left[0 - 2\left(\frac{\pi}{2}\right) - s\log\frac{s^2}{s^2+1} + 2\tan^{-1}s\right] = -\frac{1}{2}\left[-\pi - s\log\frac{s^2}{s^2+1} + 2\tan^{-1}s\right]$$

$$= \frac{\pi}{2} + \frac{s}{2}\log\frac{s^2}{s^2+1} - \tan^{-1} s$$

$$= \left(\frac{\pi}{2} - \tan^{-1} s\right) + \frac{s}{2}\log\frac{s^2}{s^2+1} = \cot^{-1} s + \frac{s}{2}\log\frac{s^2}{s^2+1}.$$

Q: Evaluate

$$L\left[e^{-4t}\frac{\sin 3t}{t}\right].$$

Solution. $L\sin 3t = \frac{3}{s^2+3^2} \Rightarrow L\frac{\sin 3t}{t} = \int_s^\infty \frac{3}{s^2+9}ds = \left[\frac{3}{3}\tan^{-1}\frac{s}{3}\right]_s^\infty$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{3} = \cot^{-1}\frac{s}{3}$$

$$L\left[e^{-4t}\frac{\sin 3t}{t}\right] = \cot^{-1}\frac{s+4}{3} = \tan^{-1}\frac{3}{s+4}$$

Q: Express the following function in terms of units step functions and find its Laplace transform:

$$f(t) = \begin{cases} 8, & t<2 \\ 6, & t>2 \end{cases}$$

Sol:

$$f(t) = \begin{cases} 8+0, & t<2 \\ 8-2, & t>2 \end{cases}$$

$$= 8 + \begin{cases} 0, & t<2 \\ -2, & t>2 \end{cases} = 8 + (-2)\begin{cases} 0, & t<2 \\ 1, & t>2 \end{cases}$$

$$= 8 - 2u(t-2)$$

$$\mathrm{L}\,f(t) = 8\,\mathrm{L}\,(1) - 2\,\mathrm{L}\,u(t-2) = \frac{8}{s} - 2\frac{e^{-2s}}{s}$$

Q: Express the following function in terms of unit step function and find its Laplace transform:

$$f(t) = \begin{cases} E, & a<t<b \\ 0, & t>b \end{cases}$$

Sol:

$$f(t) = E\begin{cases} 1, & a < t < b \\ 0, & t > b \end{cases} = E\,[u(t-a) - u(t-b)]$$

$$\mathrm{L}\, f(t) = E\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right]$$

Q: Express the following function in terms of unit step function:

$$f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$$

and find its Laplace transform.

Sol:

$$f(t) = \begin{bmatrix} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{bmatrix}$$

$$= (t-1)[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$$

$$= (t-1)u(t-1) - (t-1)u(t-2) + (3-t)u(t-2) + (t-3)u(t-3)$$

$$= (t-1)u(t-1) - 2(t-2)\,u(t-2) + (t-3)\,u(t-3)$$

$$Lf(t) = \frac{e^{-s}}{s^2} - 2\frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$

Q:

$$\mathrm{L}\, f(t)\, u(t-a) = e^{-as}\,\mathrm{L}\,[f(t+a)]$$

Sol:

$$\mathrm{L}f(t)\,.u(t-a) = \int_0^\infty e^{-st}\,[f(t)\,.u(t-a)]\,dt$$

$$= \int_0^a e^{-st}\left[f(t).u(t-a)\right]dt + \int_a^\infty e^{-st}.\left[f(t).u(t-a)\right]dt$$

$$= 0 + \int_a^\infty e^{-st}.f(t)(1)\,dt$$

$$= \int_0^\infty e^{-s(y+a)}\cdot f(y+a)\,dy = e^{-as}\int_0^\infty e^{-sy}\cdot f(y+a)\,dy \qquad (t-a=y)$$

$$= e^{-as}\int_0^\infty e^{-st}\cdot f(t+a)\,dt = e^{-as}\,\mathrm{L}f(t+a)$$ **Proved.**

Q: Find the Laplace transform of t^2 u (t – 3).

Solution. $t^2 .u(t-3) = [(t-3)^2 + 6(t-3) + 9]u(t-3)$

$= (t-3)^2.\ u(t-3) + 6(t-3).\ u(t-3) + 9u(t-3)$

$\mathrm{L}\ t^2.u(t-3) = \mathrm{L}\ (t-3)^2.\ u(t-3) + 6\mathrm{L}(t-3).\ u(t-3) + 9\mathrm{L}\ u(t-3)$

$$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

Q: Find the Laplace transform of e^{-2} t u_π(t).

$$u_\pi(t) = \begin{cases} 0: & t < \pi \\ 1: & t > \pi \end{cases}$$

Solution. $u_\pi(t) = \begin{cases} 0: & t < \pi \\ 1: & t > \pi \end{cases}$

$= u(t-\pi)$

$\mathrm{L}e^{-2t}u_\pi(t) = \mathrm{L}e^{-2t}u(t-\pi)$ $f(t) = e^{-2t}$

$= e^{-\pi s}\mathrm{L}f(t+\pi)$ $f(t+\pi) = e^{-2(t+\pi)}$

$= e^{-\pi s}\mathrm{L}e^{-2(t+\pi)} = e^{-\pi s}e^{-2\pi}\mathrm{L}e^{-2t}$

$$= e^{-(\pi s + 2\pi)}\frac{1}{s+2}$$

$$= \frac{e^{-\pi(s+2)}}{s+2}$$

Q: Find the Laplace transform of

$t^3\delta\ (t-4)$.

Sol:

$$\mathrm{L}\ t^3\delta\ (t-4) = \int_0^\infty e^{-st}t^3\delta(t-4)\,dt$$

$$= 4^3 e^{-4s}$$

Q:

$$Evaluate \int_{-\infty}^{\infty} e^{-5t}\delta(t-2)$$

Solution. $\int_{-\infty}^{\infty} e^{-5t}\delta(t-2) e^{-5\times 2} = e^{-10}$

Q: If f (t) is a periodic function with Period T, then prove that

$$\mathrm{L}\left[f(t)\right] = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

Proof. $\mathrm{L}\left[f(t)\right] = \int_0^{\infty} e^{-st} f(t)\,dt$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + ..$$

Substituting $t = u+T$ in second integral and $t = u + 2T$ in third integral, and so on.

$$\mathrm{L}\left[f(t)\right] = \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du + ...$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du + ...$$

$$\left[f(u) = f(u+T) = f(u+2T) = f(u+3T) = ...\right]$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-st} f(t)\,dt + e^{-2sT}\int_0^T e^{-st} f(t)\,dt + ...$$

$$= \left[1+e^{-sT}+e^{-2sT}+e^{-3sT}+...\right]\int_0^T e^{-st} f(t)\,dt \qquad \left[1+a+a^2+a^3+... = \frac{1}{1-a}\right]$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt.$$

Q: Find the Laplace transform of the waveform

$$f(t) = \left(\frac{2t}{3}\right), 0 \le t \le 3.$$

Solution. $$L\left[f(t)\right] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$L\left[\frac{2t}{3}\right]=\frac{1}{1-e^{-3s}}\int_0^3 e^{-st}\left(\frac{2}{3}t\right)dt=\frac{1}{1-e^{-3s}}\frac{2}{3}\left[\frac{te^{-st}}{-s}-(1)\frac{e^{-st}}{s^2}\right]_0^3$$

$$=\frac{2}{3}\frac{1}{1-e^{-3s}}\left[\frac{3e^{-3s}}{-s}-\frac{e^{-3s}}{s^2}+\frac{1}{s^2}\right]=\frac{2}{3}.\frac{1}{1-e^{-3s}}\left[\frac{3e^{-3s}}{-s}+\frac{1-e^{-3s}}{s^2}\right]$$

$$=\frac{2e^{-3s}}{-s\left(1-e^{-3s}\right)}+\frac{2}{3s^2}$$

Q: Find the Laplace Transform of the Periodic function (saw tooth wave)

$$f(t)=\frac{kt}{T}\text{ for }0<t<T,\qquad f(t+T)=f(t)$$

Solution. $L[f(t)]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}\frac{kt}{T}dt$

$$=\frac{1}{1-e^{-sT}}\frac{k}{T}\int_0^T e^{-st}.t\,dt=\frac{k}{T\left(1-e^{-sT}\right)}\left[t\frac{e^{-st}}{-s}-\int 1.\frac{e^{-st}}{-s}dt\right]_0^T$$

Integrating by parts

$$=\frac{k}{T\left(1-e^{-sT}\right)}\left[\frac{te^{-st}}{-s}-\frac{e^{-st}}{s^2}\right]_0^T=\frac{k}{T\left(1-e^{-sT}\right)}\left[\frac{Te^{-sT}}{-s}-\frac{e^{-sT}}{s^2}+\frac{1}{s^2}\right]$$

$$=\frac{k}{T\left(1-e^{-sT}\right)}\left[\frac{Te^{-sT}}{-s}+\frac{1}{s^2}\left(1-e^{-sT}\right)\right]=-\frac{ke^{-sT}}{s\left(1-e^{-sT}\right)}+\frac{k}{Ts^2}$$

Q: A semi-infinite solid x > 0 is initially at temperature zero. At time t = 0, a constant temperature u_0 is applied and maintained at the face x = 0. Find the temperature at any point of the solid and at any time t > 0.

Solution. Let $u(x, t)$ be the temperature at any point x and at any time t. The equation governing the flow of heat in the solid is given by

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}\qquad x>0,\ t>0 \qquad \ldots(1)$$

The initial and boundary conditions are

$$u=0\text{ when }t=0\text{ for all }x\ (x\geq 0) \qquad \ldots(2)$$

$$u=u_0\text{ when }x=0\text{ for all }t, \qquad \ldots(3)$$

$$u\text{ is finite for all }x\text{ and for all }t, \qquad \ldots(4)$$

Multiplying equation (1) by e^{-st} and integrate w.r.t. 't' from 0 to ∞,

$$\int_0^\infty \frac{\partial u}{\partial t} e^{-st}\, dt = c^2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} e^{-st}\, dt \quad \text{or} \quad s\bar{u} = c^2 \frac{d^2\bar{u}}{dx^2} \qquad \dots (5)$$

($\bar{u}$ = Laplace transform of u)

Similarly Laplace transformation of equation (3) gives

$$s\bar{u} = u_0 \quad \text{when} \quad x = 0 \quad \text{or} \quad \bar{u} = \frac{u_0}{s} \qquad \dots (6)$$

Equation (5) is an ordinary differential equation and its solution is given by

$$\bar{u} = Ae^{\frac{\sqrt{s}}{c}x} + Be^{-\frac{\sqrt{s}}{c}x} \qquad \dots (7)$$

According to condition (4), u is finite at $x \to \infty$,

$\therefore$ $A = 0$

So (7) becomes $\bar{u} = Be^{-\frac{\sqrt{s}}{c}x}$

Using (6) equation $\bar{u} = \frac{u_0}{s}$ when $x = 0$, $u_0 / s = B$

Thus (8) becomes $\bar{u} = \frac{u_0}{s} e^{-\frac{\sqrt{s}}{c}x}$

To get u from $\bar{u}$, we invert the transformation.

$$u = u_0\left(1 - erf\frac{x}{2c\sqrt{t}}\right) \quad \textbf{Ans.}$$

Q: An infinitely long string having one end at x = 0 is initially at rest along x-axis. The end x = 0 is given a transverse displacement f(t), when t > 0. Find the displacement of any point of the string at any time.

Solution. Let $y(x, t)$ be the displacement, then wave equation is

$$\frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial t^2} \qquad \dots (1)$$

subject to the conditions

$y(x, 0) = 0 \quad \dots (2) \qquad \frac{\partial y}{\partial t}(x, 0) = 0 \quad \dots (3)$

$y(0, t) = f(t) \quad \dots (4) \qquad y(x, t)$ is bounded $\quad \dots (5)$

On taking Laplace transform of (1), we have

$$L\left(\frac{\partial^2 y}{\partial x^2}\right) = c^2 L \frac{\partial^2 y}{\partial t^2}$$

$$s^2\bar{y} - sy(x,0) - \frac{\partial y}{\partial t}(x,0) = c^2 \frac{d^2\bar{y}}{dx^2} \qquad \dots (6)$$

On putting $y\,(x,\,0) = 0,\ \frac{\partial y}{\partial t}(x,\,0) = 0$ in (6), we get

$$s^2\,\bar{y} = c^2 \frac{d^2\bar{y}}{dx^2} \text{ or } \frac{d^2\bar{y}}{dx^2} = \left(\frac{s}{c}\right)^2 \bar{y} \qquad \dots (7)$$

Laplace transform of (4), $\bar{y}(0,\,s) = \bar{f}(s)$ at $x = 0$ ... (8)

On solving (7), we get $\bar{y} = Ae^{\frac{sx}{c}} + Be^{-\frac{sx}{c}}$ ... (9)

According to condition (5), y is finite at $x \to \infty$, this gives $A = 0$ so (9) becomes

$$\bar{y} = Be^{-\frac{sx}{c}} \qquad \dots (10)$$

Putting the value of $\bar{y}\,(0,\,s) = \bar{f}\,(s)$ at $x = 0$ in (10), we get $\bar{f}\,(s) = B$

Thus (10) becomes $\qquad \bar{y} = \bar{f}(s)\cdot e^{-\frac{sx}{c}}$

To get y from $\bar{y}$, we use complex inversion formula

$$y = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\left(t-\frac{x}{c}\right)s}\,\bar{f}(s)\,ds$$

Hence $\qquad y = f\left(t - \frac{x}{c}\right)$ **Ans.**

Q: A uniform rod of length l is at rest in its equilibrium position with the end x = 0 fixed. At t = 0, a constant force F0 per unit area is applied at the free end. Determine the motion of the rod for t > 0.

Solution. Let $y\,(x,\,t)$ be the displacement in the rod. Equation of motion is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad l > x > 0, \quad t > 0 \qquad \dots (1)$$

subject to the conditions

$$y\,(x,\,0) = 0 \quad \ldots (2) \qquad \frac{\partial y}{\partial t}(x, 0) = 0 \quad \ldots (3)$$

$$y\,(0,\,t) = 0 \quad \ldots (4) \qquad \frac{\partial y}{\partial x}(l,\,t) = \frac{F_0}{E} \quad \ldots (5)$$

where E = Young's modulus.

Applying Laplace transform on (1) $\qquad c^2 \dfrac{d^2 y}{dx^2} = s^2 \bar{y} \qquad \ldots (6)$

Eq. (6) is an ordinary differential equation and its solution is

$$\bar{y} = Ae^{\frac{sx}{c}} + Be^{-\frac{sx}{c}} \qquad \ldots (7)$$

Putting $y = 0$, $x = 0$ from (2) in (7), we get

$0 = A + B$ or $B = -A$, then (7) becomes

$$\bar{y} = A\left(e^{\frac{sx}{c}} - e^{-\frac{sx}{c}}\right) \qquad \ldots (8)$$

Laplace transform of (3) $\quad \dfrac{d\,\bar{y}}{dx} = \dfrac{F_0}{Es}$ at $x = l \qquad \ldots (9)$

Differentiating (8) w.r.t. 'x' we get

$$\frac{d\,\bar{y}}{dx} = A\left(\frac{s}{c}e^{\frac{sx}{c}} + \frac{s}{c}e^{-\frac{sx}{c}}\right) \qquad \ldots (10)$$

Putting the value of $\dfrac{d\,\bar{y}}{dx}$ from (9) in (10), we have

$$\frac{F_0}{ES} = A\frac{s}{c}\left(e^{\frac{s}{c}l} + e^{-\frac{s}{c}l}\right) \quad \text{or} \quad A = \frac{F_0}{Es}\frac{c}{s}\frac{1}{e^{\frac{s}{c}l} + e^{-\frac{s}{c}l}}$$

Putting the value of A in (8) we obtain

$$\bar{y} = \frac{cF_0}{Es^2}\,\frac{e^{\frac{s}{c}x} - e^{-\frac{s}{c}x}}{e^{\frac{sl}{c}} + e^{-\frac{sl}{c}}} = \frac{cF_0}{E.s^2}\,\frac{1 - e^{-\frac{2s}{c}x}}{1 + e^{-\frac{2sl}{c}}}\,\frac{e^{\frac{s}{c}x}}{e^{\frac{s}{c}l}}$$

$$= \frac{cF_0}{Es^2}\left[\left(1 - e^{\frac{-2s}{c}x}\right)\left(1 + e^{-\frac{2sl}{c}}\right)^{-1}\right]e^{\frac{s}{c}(x-l)}$$

$$= \frac{cF_0}{Es^2}\left[\left(1-e^{-\frac{2s}{c}x}\right)\left(1-e^{-\frac{2sl}{c}}+\dots\right)\right]\times e^{\frac{s}{c}(x-l)}$$

$$\bar{y} = \frac{cF_0}{Es^2}\left[1-e^{-\frac{2s}{c}x}-e^{-\frac{2sl}{c}}+e^{-\frac{2s}{c}(x+l)}+\dots\right]e^{\frac{s}{c}(x-l)} \qquad \dots (11)$$

Putting $x = l$ in (11) we get

$$x = l\ \bar{y} = \frac{cF_0}{Es^2}\left[1-e^{-\frac{2s}{c}l}-e^{-\frac{2sl}{c}}+e^{-\frac{2s(l+l)}{c}}\dots\right] \qquad \dots (12)$$

Applying inversion transformation on (12) we get

$$y = \frac{F_0 c}{E}t, \qquad 0 < t < \frac{2l}{c} \qquad \dots (13)$$

$$y = \frac{F_0 c}{E}t - \frac{2F_0 c}{E}\left(t-\frac{2l}{c}\right); \qquad \frac{2l}{c} < t < \frac{4l}{c}$$

Putting $\frac{2l}{c} = \lambda$ in (13), we have

$$y = \begin{cases} At & 0 < t < \lambda \\ At - 2A\,(t-\lambda),\ \text{where } A = \frac{F_0 C}{E}, & \lambda < t < 2\lambda \end{cases}$$ **Ans.**