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Waves & Optics (B.Sc.)

Chapter - 2

Superposition of two Perpendicular Harmonic Oscillations

Chapter 2
Superposition of two perpendicular harmonic oscillations

Superposition of Two Perpendicular Harmonic Oscillations: Graphical and Analytical Methods. Lissajous Figures (1:1 and 1:2) and their uses. **(2 Lectures)**

Q: derive an expression for superposition of two perpendicular harmonic oscillations having equal frequencies using analytical and graphical method.

Ans:

Suppose a particle moves under the simultaneous influence of two perpendicular harmonic oscillations of equal frequency, one along the x-axis, the other along the y-axis. Let A_1 and A_2 respectively be the amplitudes of the x and y oscillations. For simplicity, let us assume that the phase constant of the x oscillation is zero and that of the y oscillation is δ, so that δ is the phase difference between them.[4] There is no loss of generality in doing so. Thus, the two rectangular SHMs can be written as

$$x = A_1 \cos \omega t \tag{2.24}$$

$$y = A_2 \cos (\omega t + \delta) \tag{2.25}$$

where x and y are the displacements along two mutually perpendicular directions. The resulting motion of the particle can be obtained as follows:

(a) Analytical Method

The path followed by the particle can be traced by eliminating time t from Eqs (2.24) and (2.25) so that we are left with an expression involving only x and y and the constant δ. Expanding the argument of the cosine in Eq. (2.24), we have

$$\frac{y}{A_2} = \cos \omega t \cos \delta - \sin \omega t \sin \delta$$

But from Eq. (2.24), $\cos \omega t = x/A_1$ and, therefore, $\sin \omega t = \left(1 - \frac{x^2}{A_1^2}\right)^{1/2}$

Therefore,

$$\frac{y}{A_2} = \frac{x}{A_1}\cos\delta - \left(1 - \frac{x^2}{A_1^2}\right)^{1/2}\sin\delta$$

or

$$\left(\frac{x}{A_1}\cos\delta - \frac{y}{A_2}\right) = \left(1 - \frac{x^2}{A_1^2}\right)^{1/2}\sin\delta$$

Squaring both sides we have

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1A_2}\cos\delta = \sin^2\delta \tag{2.26}$$

This is the general equation of an ellipse whose axes are inclined to the coordinate axes. Hence, *the path followed by the particle, which is subjected to two rectangular SHMs of equal frequencies, is, in general, an ellipse.*

Let us consider a few special cases :

(i) $\delta = 0$. In this case, Eq. (2.26) reduces to

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1A_2} = 0$$

or

$$\left(y - \frac{A_2}{A_1}x\right)^2 = 0$$

This represents a pair of coincident straight lines, $y = \frac{A_2}{A_1}x$, having a positive slope A_2/A_1 and passing through the origin. The resultant motion is rectilinear and takes place along a diagonal of a rectangle of sides $2A_1$ and $2A_2$ such that x and y always have the same sign, both positive or both negative (Fig 2.5). The direction of motion can be easily determined from the defining Eqs. (2.24) and (2.25) by setting $\delta = 0$,

$$x = A_1\cos\omega t$$

$$y = A_2\cos\omega t$$

which immediately give $y = \frac{A_2}{A_1}x$, the equation of the straight line of slope $\frac{A_2}{A_1}$. At time $t = 0$, we have, $x = A_1$, $y = A_2$ so that the particle is at P at $t = 0$ (see Fig 2.5). As time passes the cosines begin to

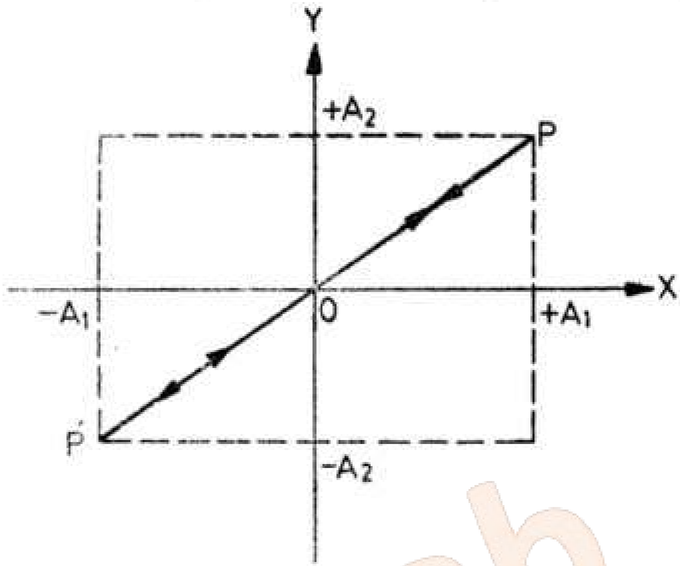

Fig. 2.5 Superposition of two perpendicular SHMs of the same frequency for phase difference $\delta = 0$

decrease until x and y become zero when $\omega t = \pi/2$. The particle moves from P to O. After this time, x and y become negative and at time when $\omega t = \pi$, x becomes $-A_1$ and y is $-A_2$. The particle moves from O to P'. After this the particle retraces its path. The particle continues to vibrate along the straight line $P\,O\,P'$. This represents what in optics is called a *linearly polarized vibration*.

(ii) $\delta = \dfrac{\pi}{2}$. In this case, Eq. (2.26) reduces to

$$\frac{x^2}{A_1^2} + \frac{y_2}{A_2^2} = 1$$

which is the equation of an ellipse whose principal axes lie along the x and y axes, as shown in Fig. 2.6. The particle moves in an elliptical path. The direction of its motion can be determined from the defining equations (2.24) and (2.25) with $\delta = \pi/2$.

$$x = A_1 \cos \omega t$$

$$y = A_2 \cos\left(\omega t + \frac{\pi}{2}\right) = -A_2 \sin \omega t$$

From these equations, the equation of the ellipse, obtained above, immediately follows. At time $t=0$, $x=A_1$ and $y=0$, so that the particle is at point P at $t=0$ (see Fig. 2.6). As time t begins to increase from zero, x begins to decrease from its maximum positive value A_1 and y immediately begins to go negative. At a time when $\omega t=\pi/2$, x becomes zero and y equals A_2. The particle moves from P to Q during this time.

The subsequent motion of the particle is indicated by arrows in the diagram. The particle traces out an ellipse in the clockwise sense. This

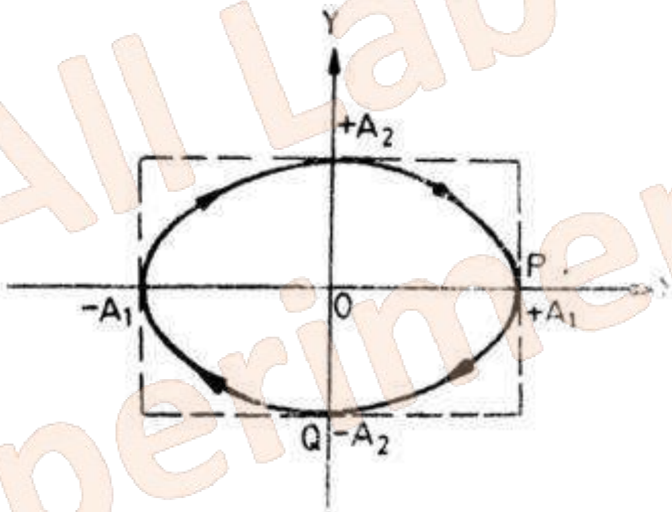

Fig. 2.6 Superposition of two perpendicular SHMs of the same frequency and phase differences $\delta=\pi/2$

represents what in optics is called the *right-handed elliptically polarized vibration*. The rotating electric field vector of the electromagnetic wave is always confined in one plane, with its tip tracing out an ellipse in the clockwise direction.

If, in addition, $A_1 = A_2 = A$, the ellipse degenerates into a circle

$$x^2+y^2 = A^2$$

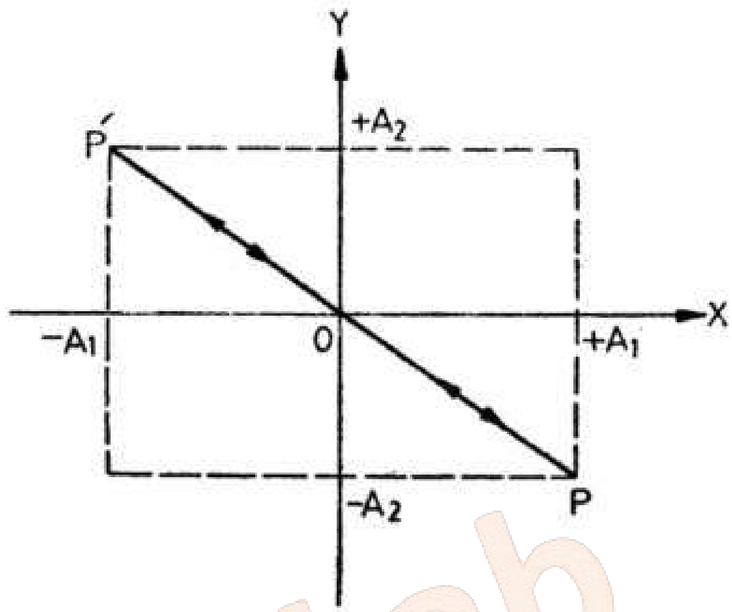

Fig. 2.7 Superposition of two perpendicular SHMs of the same frequency and phase difference $\delta = \pi$

Thus, *two harmonic oscillations, at right angles to each other, of equal amplitudes and equal frequencies but with phases differing by $\pi/2$ are equivalent to a uniform circular motion, the radius of the circle being equal to the amplitude of either oscillation.* Conversely, *a uniform circular motion can be resolved into two SHMs, at right angles to each other, their amplitudes being equal while their phases differ by $\pi/2$* (see also Sec. 1.9, Ch. 1).

(iii) $\delta = \pi$. In this case, Eq. (2.26) becomes

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} + \frac{2xy}{A_1A_2} = 0$$

or

$$\left(y + \frac{A_2}{A_1}x\right)^2 = 0$$

This represents a pair of coincident straight lines, $y = -\frac{A_2}{A_1}x$, having a negative slope $-A_2/A_1$ and passing through the origin. The ellipse degenerates into a straight line, as shown in Fig. 2.7.

(iv) $\delta = \dfrac{3\pi}{2}$. In this case, we have

$$x = A_1 \cos \omega t$$

$$y = A_2 \cos\left(\omega t + \frac{3\pi}{2}\right) = A_2 \sin \omega t$$

which give

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$$

We have an ellipse of the same form as in case (ii), but the motion is now *counter-clockwise*. In optics, such a vibration is called the *left-handed elliptically polarized vibration.*

The sequence of motions for a few values of δ in the range 0 to 2π is illustrated in Fig. 2.8. Notice that the resulting motion is the same for δ = 0 or 2π. This is expected since for δ = 0 or 2π

$$y = A_2 \cos(\omega t + 0) = A_2 \cos(\omega t + 2\pi) = A_2 \cos \omega t$$

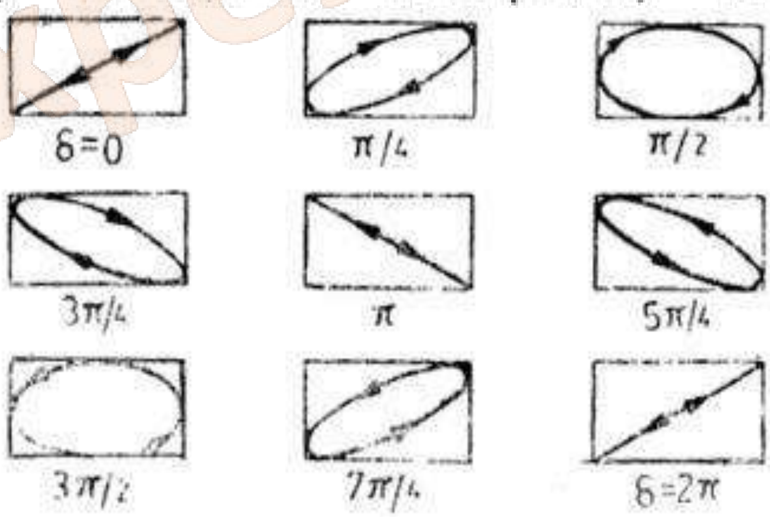

Fig. 2.8 Superposition of two perpendicular SHMs of the same frequency for various phase differences

(b) *Graphical Method*

The above results can also be obtained graphically by a double application of the rotating-vector technique. This is done as shown in Fig 2.9. Draw two circles of radii A_1 and A_2, the amplitudes of the two perpendicular SHMs. The circle of radius A_1 defines the SHM along the x-axis. Let O_1P_1 be the position of the rotating vector at a certain instant of time t. The projection of O_1P_1 on the x-axis ($O_1N_1 = OX = x$) gives the instantaneous displacement,

$$x = A_1 \cos \omega t$$

The circle of radius A_2 defines the SHM along the y-axis. Let O_2P_2 be the position of the rotating vector at time t. The projection of O_2P_2 on the y-axis ($O_2N_2 = OY = y$) gives the instantaneous perpendicular displacement

$$y = A_2 \cos (\omega t + \delta)$$

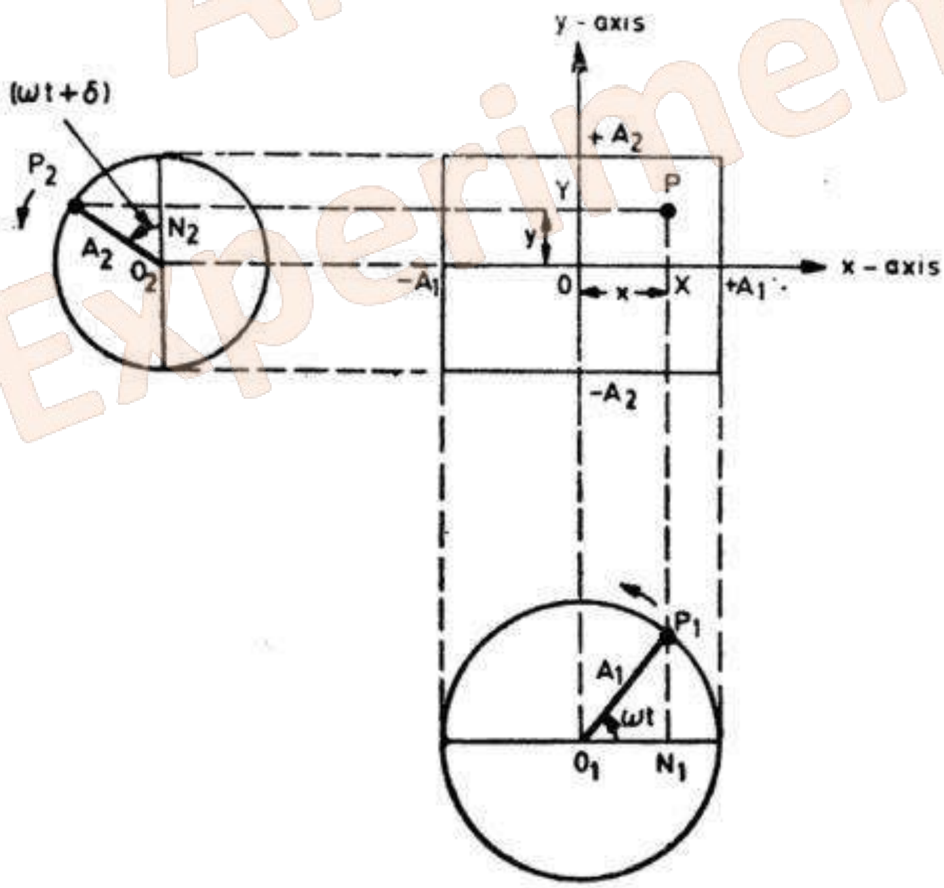

Fig. 2.9 Geometrical representation of the superposition of two SHMs at right angles to each other

If the particle has SHM only along the x-axis, its displacement at time t would be $x = OX$ where X is the projection of P_1 on the x-axis. On the other hand, if the particle has SHM only along the y-axis, its displacement would be $y = OY$, where Y is the projection of P_2 on the y-axis. Consequently, if the particle was subjected to both the SHMs simultaneously, its resultant displacement at time t would be OP. Point P is the intersection of perpendiculars drawn from P_1 and P_2 on the x and y axes respectively. The two displacements together describe the instantaneous position of the point P with respect to the origin O that lies at the centre of a rectangle of sides $2A_1$ and $2A_2$. The path followed by point P, as time passes, gives the resultant motion. We shall now construct the resultant motion for a few special values of the phase difference δ.

(i) $\delta = 0$: In this case, the two perpendicular motions are

$$x = A_1 \cos \omega t$$

$$y = A_2 \cos \omega t$$

The application of the above method to this particular case is shown in Fig. 2.10. Each *reference* circle is divided into the *same* number of equal parts, say, eight. Since the frequency of the two SHMs is the same,

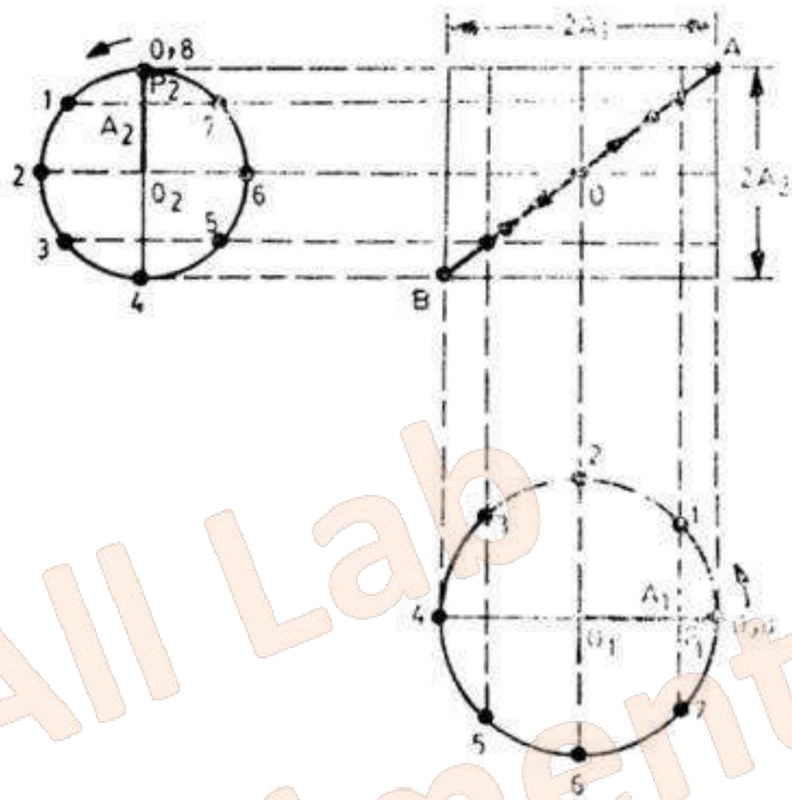

Fig. 2.10 Superposition of two perpendicular SHMs of the same frequency and zero phase difference

each of these parts of the reference circles will be described by each rotating vector in the same time which is one eighth of a period (i.e. $\pi/4\omega$). The positions of the points P_1, P_2, on the reference circles, are shown at a number of instants separated by one-eighth of a period. The points are numbered 0, 1, 2 . 8, beginning with $t = 0$, when O_1P_1 (see Fig. 2.9) is parallel to the x-axis and O_2P_2 parallel to the y-axis, so that the phase difference δ is zero. The projections from these corresponding positions of P_1 and P_2 then give us a set of intersections, as shown in Fig. 2.10, representing the instanteneous positions of the point P (see Fig. 2.9) as it moves within the rectangle. The locus defined by these points is a straight line AOB with a positive slope.

(ii) $\delta = \frac{\pi}{4}$ The application of the rotating vector method to this particular case is shown in Fig. 2.11. The positions of the points P_1 and P_2, on the two reference circles, are shown at a number of instants separated by one-eighth of the period of each component motion. The points are numbered 0, 1, 2, ..., 8 in sequence, starting with $t = 0$ when O_1P_1 (see Fig. 2.9) is parallel to the x-axis, and O_2P_2 is at angle $\delta = \frac{\pi}{4}$ or 45° measured in counterclockwise sense, from the y-axis, so that the phase difference δ is $\pi/4$. The projections of these corresponding positions of P_1 and P_2 give us a set of points of intersection, as shown in Fig. 2.11. These intersections represent the instantaneous positions of the point P as it moves within the rectangle of sides $2A_1$ and $2A_2$. The locus of these points is an inclined ellipse, described in the clockwise sense as shown. The exact shape of the curve can be ascertained by dividing the reference circles into 16, 32 ... etc. parts instead of 8.

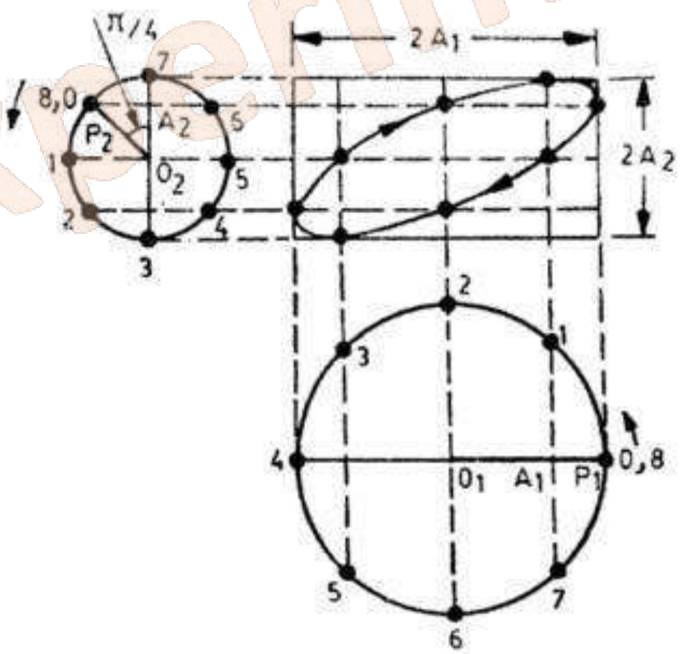

Fig. 2.11 Superposition of two perpendicular SMSs of the same frequency and a phase difference of $\pi/4$.

The resulting motion for other values of the phase difference can be similarly constructed. The sequence of motions is shown in Fig. 2.8.

Q: derive an expression for superposition of two perpendicular harmonic oscillations having different frequencies of ratios 1:2.

Ans:

When the frequencies of the two perpendicular SHMs are not equal, the resulting motion becomes more complicated. The patterns, that are traced by a particle which is subjected simultaneously to two perpendicular SHMs of different frequencies, are known as *Lissajous figures*, after J.A. Lissajous (1822-1880) who made an extensive study of these motions.

Let us first consider the case when the frequency ω_2 of the y oscillation is twice the frequency ω_1 the x oscillation, i.e. $\omega_1 = \omega$ and $\omega_2 = 2\omega$. The two SHMs are then given by

$$x = A_1 \cos \omega t \tag{2.27}$$

$$y = A_2 \cos (2\omega t + \delta) \tag{2.28}$$

where A_1 and A_2 are their respective amplitudes and δ is the phase difference between them.

The shape of the Lissajous figure can be obtained either by analytical or graphical method. In the analytical method, we find the locus of the instantaneous particle positions by eliminating time t from the above equations. Expanding the argument of the cosine in Eq. (2.28), we have

$$\frac{y}{A_2} = \cos 2\omega t \cos \delta - \sin 2\omega t \sin \delta$$

$$= (2\cos^2 \omega t - 1)\cos \delta - 2 \sin \omega t \cos \omega t \sin \delta$$

But from Eq. (2.27), $\cos\omega t = \dfrac{x}{A_1}$ and $\sin \omega t = \left(1 - \dfrac{x^2}{A_1^2}\right)^{1/2}$. Therefore,

$$\frac{y}{A_2} = \left(\frac{2x^2}{A_1^2} - 1\right)\cos \delta - \frac{2x}{A_1}\left(1 - \frac{x^2}{A_1^2}\right)^{1/2} \sin \delta$$

Rearranging we have

$$\left(\frac{y}{A_2} + \cos \delta\right) - \frac{2x^2}{A_1^2}\cos \delta = -\frac{2x}{A_1}\left(1 - \frac{x^2}{A_1^2}\right)^{1/2} \sin \delta$$

which, on squaring and upon simplification, reduces to

$$\left(\frac{y}{A_2} + \cos \delta\right)^2 + \frac{4x^2}{A_1^2}\left(\frac{x^2}{A_1^2} - 1 - \frac{y}{A_2}\cos \delta\right) = 0 \tag{2.29}$$

This is an equation of the fourth degree which, in general, represents a closed curve having two loops. For a given value of δ, the curve corresponding to Eq. (2.29) can be traced using the knowledge of coordinate geometry. Equation (2.29) reduces to a particularly simple form for $\delta = 0$. Setting $\cos\delta = 1$ in this equation, we have

$$\left(\frac{y}{A_2}+1\right)^2 + \frac{4x^2}{A_1^2}\left(\frac{x^2}{A_1^2} - 1 - \frac{y}{A_2}\right) = 0$$

$$\left(\frac{y}{A_2} + 1 - \frac{2x^2}{A_1^2}\right)^2 = 0$$

This represents two coincident parabolas with their vertices at $(0, -A_2)$ as shown in Fig. 2.12, the equation of each parabola being

$$\frac{y}{A_2} + 1 - \frac{2x^2}{A_1^2} = 0$$

or $$x^2 = \frac{A_1^2}{2A_2}(y + A_2)$$

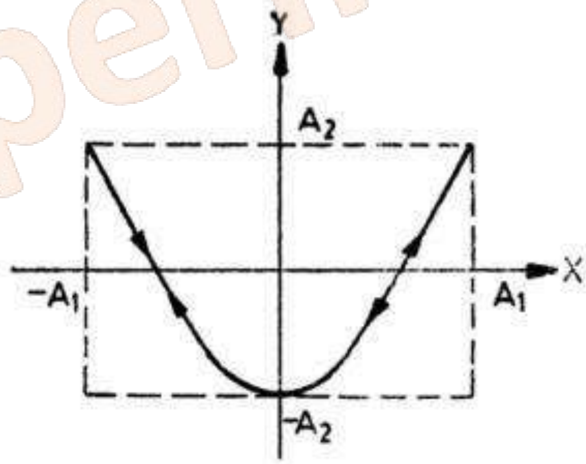

Fig. 2.12 Superposition of two perpendicular SHMs with frequencies in the ratio 1 : 2 and phase difference equal to zero

The analytical method becomes very cumbersome for values of δ other than zero. In such cases, the resultant motion can be constructed quite conveniently by the graphical method. Figure 2.13 shows how the rotating vector technique is used to obtain the shape of the Lissajous figure when $\delta = \pi/4$ and $\omega_2 = 2\omega_1$. The rotating vector O_2P_2 subtends an angle $\pi/4$ at time $t = 0$ with the y-axis so that the y oscillation has an initial phase of $\pi/4$, but the rotating vector O_1P_1 is along the x-axis at this instant of time, so that the x oscillation has no initial phase ; the phase difference between them is thus $\pi/4$. The y oscillation is twice as fast as the x oscillation. Therefore, we choose to divide the circle of radius A_2 into 8 equal parts and the circle of radius A_1 into 16 equal parts. During the time the vector O_2P_2 describes one-eighth of its circle, the vector O_1P_1 describes only one-sixteenth of its circle. During one complete cycle of ω_2 we go through only half a cycle of ω_1 and the points on the reference circles are marked accordingly. One must, of course, go through a complete cycle of ω_2 in order to obtain one complete period of the combined motion.

The combined motion corresponding to other phase differences can be similarly constructed. Figure 2.14 shows the sequence of these motions for values of δ in the range 0 to π.

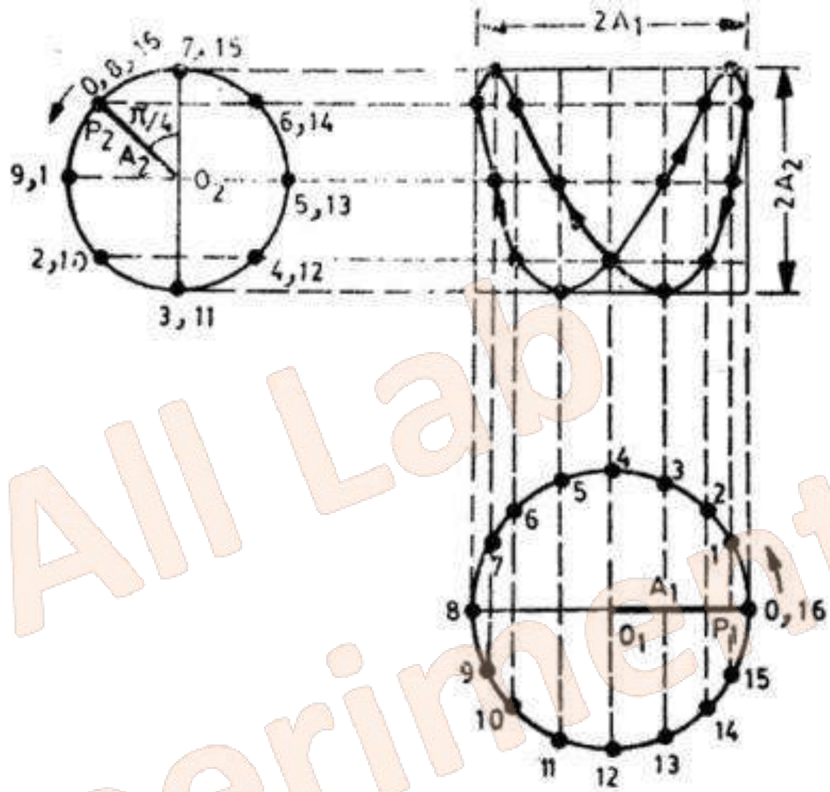

Fig. 2.13 Superposition of two perpendicular SHMs with frequencies in the ratio 1 : 2 and phase difference equal to π/4.

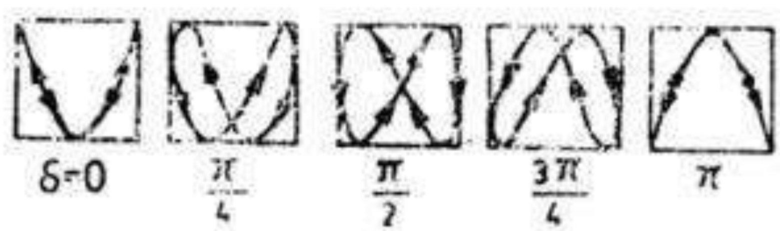

Fig. 2.14 Lissajous figures : $\omega_2 = 2\omega_1$ with various initial phase differences