Free Study Material from

All Lab Experiments

Mathematical Physics - I
Chapter - 2
Vector Algebra

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Vector Algebra

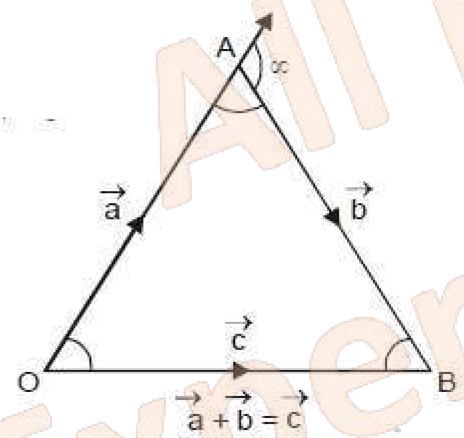
Vector Algebra: Properties of vectors. Scalar product and vector product, Scalar triple product and their interpretation in terms of area and volume respectively. Scalar and Vector fields. **(6 Lectures)**

LAWS OF VECTOR ALGEBRA. If **A**, **B** and **C** are vectors and m and n are scalars, then

1.	$\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$	Commutative Law for Addition
2.	$\mathbf{A}+(\mathbf{B}+\mathbf{C}) = (\mathbf{A}+\mathbf{B})+\mathbf{C}$	Associative Law for Addition
3.	$m\mathbf{A} = \mathbf{A}m$	Commutative Law for Multiplication
4.	$m(n\mathbf{A}) = (mn)\mathbf{A}$	Associative Law for Multiplication
5.	$(m+n)\mathbf{A} = m\mathbf{A}+n\mathbf{A}$	Distributive Law
6.	$m(\mathbf{A}+\mathbf{B}) = m\mathbf{A}+m\mathbf{B}$	Distributive Law

Q1:

If $\vec{a}$ and $\vec{b}$ be two unit vectors and α be the angle between them, then find the value of α such that $\vec{a}+\vec{b}$ is a unit vector.



Sol: Let

$\vec{OA} = \vec{a}$ be a unit vector and $\vec{AB} = \vec{b}$ is another unit vector and α be the angle between $\vec{a}$ and $\vec{b}$.

If $\vec{OB} = \vec{c} = \vec{a} + \vec{b}$ is also a unit vector then, we have

$$|\vec{OA}| = 1$$
$$|\vec{AB}| = 1$$
$$|\vec{OB}| = 1$$

OAB is an equilateral triangle.

So, each angle of $\Delta\ OAB$ is $\frac{\pi}{3}$

Hence $\propto = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Q2:

If A and B are (3, 4, 5) and (6, 8, 9), find $\vec{AB}$.

Sol:

$$\vec{AB} = \text{Position vector of } B - \text{Position vector of } A$$
$$= (6\hat{i} + 8\hat{j} + 9\hat{k}) - (3\hat{i} + 4\hat{j} + 5\hat{k}) = 3\hat{i} + 4\hat{j} + 4\hat{k}$$

DOT PRODUCT

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta, \qquad 0 \le \theta \le \pi$$

PROPERTIES OF DOT PRODUCT:

Suppose **A**, **B**, and **C** are vectors and m is a scalar. Then the following laws hold:

(i) $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ Commutative Law for Dot Products
(ii) $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$ Distributive Law
(iii) $m(\mathbf{A} \cdot \mathbf{B}) = (m\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (m\mathbf{B}) = (\mathbf{A} \cdot \mathbf{B})m$
(iv) $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1, \quad \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$
(v) If $\mathbf{A} \cdot \mathbf{B} = 0$ and **A** and **B** are not null vectors, then **A** and **B** are perpendicular.

SOME USEFUL RESULTS

$\hat{i} \cdot \hat{i} = (1)(1)\cos 0° = 1$ Similarly, $\hat{j} \cdot \hat{j} = 1$, $\hat{k} \cdot \hat{k} = 1$

$\hat{i} \cdot \hat{j} = (1)(1)\cos 90° = 0$ Similarly, $\hat{j} \cdot \hat{k} = 0$, $\hat{k} \cdot \hat{i} = 0$

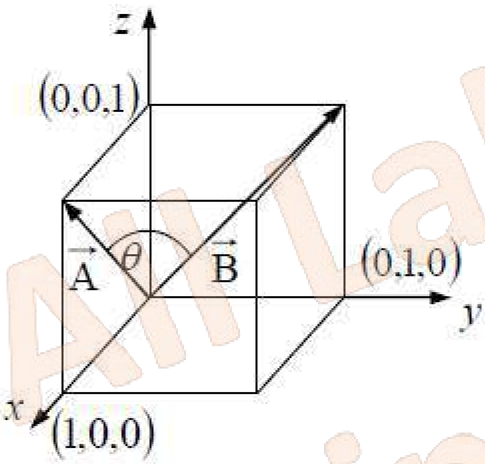
Q3: Prove

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

Sol:

$$\vec{b}\cdot\vec{a} = |\vec{b}||\vec{a}|\cos(-\theta)$$
$$= |\vec{a}||\vec{b}|\cos\theta$$
$$= \vec{a}\cdot\vec{b} \qquad \textbf{Proved.}$$

Q4: Find the angle between the face diagonals of a cube.



Sol:

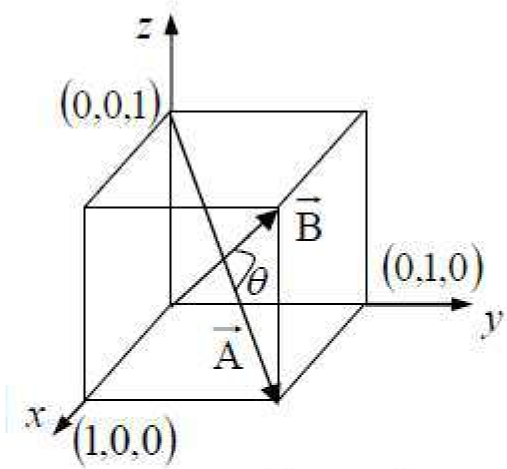
The face diagonals $\vec{A}$ and $\vec{B}$ are

$$\vec{A} = 1\hat{x} + 0\hat{y} + 1\hat{z}; \quad \vec{B} = 0\hat{x} + 1\hat{y} + 1\hat{z}$$

So, $\Rightarrow \vec{A}.\vec{B} = 1$

Also, $\Rightarrow \vec{A}.\vec{B} = AB\cos\theta = \sqrt{2}\sqrt{2}\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^0$

Q5: Find the angle between the body diagonals of a cube.



Solution: The body diagonals $\vec{A}$ and $\vec{B}$ are

$$\vec{A} = \hat{x} + \hat{y} - \hat{z}; \quad \vec{B} = \hat{x} + \hat{y} + \hat{z}$$

So, $\Rightarrow \vec{A}.\vec{B} = 1 + 1 - 1 = 1$

Also, $\Rightarrow \vec{A}.\vec{B} = AB\cos\theta = \sqrt{3}\sqrt{3}\cos\theta \Rightarrow \cos\theta = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right)$

CROSS PRODUCT

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, . \, \hat{\eta}$$

PROPERTIES OF CROSS PRODUCT

Suppose **A**, **B**, and **C** are vectors and m is a scalar. Then the following laws hold:

(i) $\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$ Commutative Law for Cross Products Fails

(ii) $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$ Distributive Law

(iii) $m(\mathbf{A} \times \mathbf{B}) = (m\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (m\mathbf{B}) = (\mathbf{A} \times \mathbf{B})m$

(iv) $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$, $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

(v) If $\mathbf{A} \times \mathbf{B} = 0$ and **A** and **B** are not null vectors, then **A** and **B** are parallel.

(vi) The magnitude of $\mathbf{A} \times \mathbf{B}$ is the same as the area of a parallelogram with sides **A** and **B**.

USEFUL RESULTS:

Since $\hat{i}, \hat{j}, \hat{k}$ are three mutually perpendicular unit vectors, then

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$$

$$\hat{i}\times\hat{j} = -\hat{j}\times\hat{i} = \hat{k}$$

$$\hat{j}\times\hat{k} = -\hat{k}\times\hat{j} = \hat{i}$$

$$\hat{k}\times\hat{i} = -\hat{i}\times\hat{k} = \hat{j}$$

and

$$\hat{j}\times\hat{i} = -\hat{i}\times\hat{j}$$

$$\hat{k}\times\hat{j} = -\hat{j}\times\hat{k}$$

$$\hat{i}\times\hat{k} = -\hat{k}\times\hat{i}$$

Q6: Find the area of a parallelogram whose adjacent sides are i – 2j + 3 k and 2i + j – 4k.

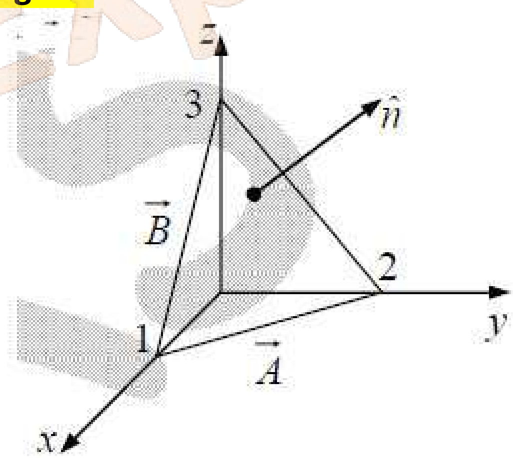
Sol : Vector area of parallelogram =

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 1 & -4 \end{vmatrix}$$

$$= (8-3)\hat{i} - (-4-6)\hat{j} + (1+4)\hat{k} = 5\hat{i} + 10\hat{j} + 5\hat{k}$$

Area of parallelogram $= \sqrt{(5)^2 + (10)^2 + (5)^2} = 5\sqrt{6}$ **Ans.**

Q7: Find the components of the unit vector n̂ perpendicular to the plane shown in the figure.



Sol: The vectors $\vec{A}$ and $\vec{B}$ can be defined as

$$\vec{A} = -\hat{x} + 2\hat{y}; \quad \vec{B} = -\hat{x} + 3\hat{z} \Rightarrow \hat{n} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \frac{6\hat{x} + 3\hat{y} + 2\hat{z}}{7}$$

Q8:

Show that $|\mathbf{A}\times\mathbf{B}|^2 + |\mathbf{A}\cdot\mathbf{B}|^2 = |\mathbf{A}|^2|\mathbf{B}|^2$.

Sol:

$$|\mathbf{A}\times\mathbf{B}|^2 + |\mathbf{A}\cdot\mathbf{B}|^2 = |AB\sin\theta\,\mathbf{u}|^2 + |AB\cos\theta|^2 = A^2B^2\sin^2\theta + A^2B^2\cos^2\theta$$
$$= A^2B^2 = |\mathbf{A}|^2|\mathbf{B}|^2$$

Q9: Determine a unit vector perpendicular to the plane of A = 2i - 6j - 3k and B = 4i + 3j - k .

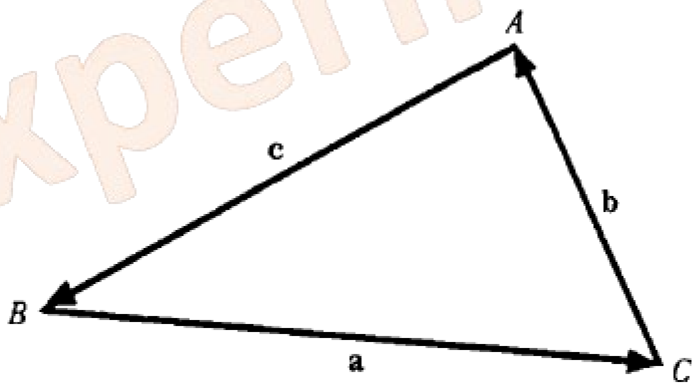
Sol: A x B is a vector perpendicular to the plane of A and B .

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = 15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}$$

A unit vector parallel to $\mathbf{A}\times\mathbf{B}$ is $\dfrac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|} = \dfrac{15\mathbf{i}-10\mathbf{j}+30\mathbf{k}}{\sqrt{(15)^2+(-10)^2+(30)^2}} = \dfrac{3}{7}\mathbf{i} - \dfrac{2}{7}\mathbf{j} + \dfrac{6}{7}\mathbf{k}$

Q10: Prove the law of sines for plane triangles.

Sol: Let a, b and c represent the sides of triangle ABC as shown in the adjoining figure:



then a+b+c = 0. Multiplying by a x, b x and c x in succession, we find

$$\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$$

i.e. $ab\sin C = bc\sin A = ca\sin B$

or $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

SCALAR TRIPLE PRODUCT

$$\vec{A}.(\vec{B}\times\vec{C}) = \vec{B}.(\vec{C}\times\vec{A}) = \vec{C}.(\vec{A}\times\vec{B})$$

VECTOR TRIPLE PRODUCT

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}.\vec{C}\right)-\vec{C}\left(\vec{A}.\vec{B}\right)$$

Q11: Find the volume of parallelopiped if

$$\vec{a} = -3\,\hat{i} + 7\,\hat{j} + 5\,\hat{k}, \quad \vec{b} = -3\,\hat{i} + 7\,\hat{j} - 3\,\hat{k}, \text{ and } \vec{c} = 7\,\hat{i} - 5\,\hat{j} - 3\,\hat{k}$$

are the three co-terminous edges of the parallelopiped.

Solution.

$$\text{Volume} = \vec{a}\,.\,(\vec{b}\times\vec{c})$$

$$= \begin{vmatrix} -3 & 7 & 5 \\ -3 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix} = -3\,(-21-15) - 7\,(9+21) + 5\,(15-49)$$

$$= 108 - 210 - 170 = -272$$

$$\text{Volume} = 272 \text{ cube units.}$$

Q12: Show that the volume of the tetrahedron having $\vec{A}\vec{B}$, $\vec{B}\vec{C}$, $\vec{C}\vec{A}$ as concurrent edges is twice the volume of the tetrahendron having $\vec{A}$, $\vec{B}$, $\vec{C}$, as concurrent edges.

Solution. Volume of tetrahendron $= \frac{1}{6}(\vec{A}+\vec{B})\,.\,[(\vec{B}+\vec{C})\times(\vec{C}+\vec{A})]$

$$= \frac{1}{6}(\vec{A}+\vec{B})\,.\,[\vec{B}\times\vec{C}+\vec{B}\times\vec{A}+\vec{C}\times\vec{C}+\vec{C}\times\vec{A}] \qquad [\vec{C}\times\vec{C}=0]$$

$$= \frac{1}{6}(\vec{A}+\vec{B})\,.\,(\vec{B}\times\vec{C}+\vec{B}\times\vec{A}+\vec{C}\times\vec{A})$$

$$= \frac{1}{6}[\vec{A}.(\vec{B}\times\vec{C})+\vec{A}.(\vec{B}\times\vec{A})+\vec{A}.(\vec{C}\times\vec{A})+\vec{B}.(\vec{B}\times\vec{C})+\vec{B}.(\vec{B}\times\vec{A})+\vec{B}.(\vec{C}\times\vec{A})]$$

$$= \frac{1}{6}[\vec{A}.(\vec{B}\times\vec{C})+\vec{B}.(\vec{C}\times\vec{A})] = \frac{1}{3}\vec{A}.(\vec{B}\times\vec{C})$$

$$= 2\times\frac{1}{6}[\vec{A}\;\vec{B}\;\vec{C}]$$

$= 2$ Volume of tetrahedron having $\vec{A}, \vec{B}, \vec{C}$, as concurrent edges. **Proved.**

Q13: Evaluate

$$(2\mathbf{i}-3\mathbf{j})\cdot[(\mathbf{i}+\mathbf{j}-\mathbf{k})\times(3\mathbf{i}-\mathbf{k})]$$

Sol: the result is equal to

$$(2\mathbf{i}-3\mathbf{j})\cdot[\mathbf{i}\times(3\mathbf{i}-\mathbf{k}) + \mathbf{j}\times(3\mathbf{i}-\mathbf{k}) - \mathbf{k}\times(3\mathbf{i}-\mathbf{k})]$$
$$= (2\mathbf{i}-3\mathbf{j})\cdot[3\mathbf{i}\times\mathbf{i} - \mathbf{i}\times\mathbf{k} + 3\mathbf{j}\times\mathbf{i} - \mathbf{j}\times\mathbf{k} - 3\mathbf{k}\times\mathbf{i} + \mathbf{k}\times\mathbf{k}]$$
$$= (2\mathbf{i}-3\mathbf{j})\cdot(0 + \mathbf{j} - 3\mathbf{k} - \mathbf{i} - 3\mathbf{j} + 0)$$
$$= (2\mathbf{i}-3\mathbf{j})\cdot(-\mathbf{i}-2\mathbf{j}-3\mathbf{k}) = (2)(-1) + (-3)(-2) + (0)(-3) = 4.$$

Q14:

If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$ show that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Sol:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{A}\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$
$$= (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot[(B_2C_3 - B_3C_2)\mathbf{i} + (B_3C_1 - B_1C_3)\mathbf{j} + (B_1C_2 - B_2C_1)\mathbf{k}]$$
$$= A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Q15: Prove that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}).$$

Sol: from previous question, we have

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

By a theorem of determinants which states that interchange of two rows of a determinant changes its sign, we have

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\begin{vmatrix} B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$$

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\begin{vmatrix} C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

Q16: Find the volume of tetrahedron having vertices

$$(-\hat{j}-\hat{k}),\quad (4\hat{i}+5\hat{j}+q\hat{k}),\quad (3\hat{i}+9\hat{j}+4\hat{k}) \text{ and } 4(-\hat{i}+\hat{j}+\hat{k}).$$

Also find the value of q for which these four points are coplanar.

Sol:

Let $\vec{A} = -\hat{j}-\hat{k},\ \bar{B}=4\hat{i}+5\hat{j}+q\hat{k},\ \bar{C}=3\hat{i}+9\hat{j}+4\hat{k},\ \bar{D}=4(-\hat{i}+\hat{j}+\hat{k})$

$$\overline{AB} = \vec{B}-\vec{A} = 4\hat{i}+5\hat{j}+q\hat{k}-(-\hat{j}-\hat{k}) = 4\hat{i}+6\hat{j}+(q+1)\hat{k}$$

$$\overline{AC} = \vec{C}-\vec{A} = (3\hat{i}+9\hat{j}+4\hat{k})-(-\hat{j}-\hat{k}) = 3\hat{i}+10\hat{j}+5\hat{k}$$

$$\overline{AD} = \vec{D}-\vec{A} = 4(-\hat{i}+\hat{j}+\hat{k})-(-\hat{j}-\hat{k}) = -4\hat{i}+5\hat{j}+5\hat{k}$$

$$\text{Volume of the tetrahedron} = \frac{1}{6}[\overline{AB}\ \overline{AC}\ \overline{AD}]$$

$$= \frac{1}{6}\begin{vmatrix} 4 & 6 & q+1 \\ 3 & 10 & 5 \\ -4 & 5 & 5 \end{vmatrix} = \frac{1}{6}\{4(50-25)-6(15+20)+(q+1)(15+40)\}$$

$$= \frac{1}{6}\{100-210+55(q+1)\} = \frac{1}{6}(-110+55+55q)$$

$$= \frac{1}{6}(-55+55q) = \frac{55}{6}(q-1)$$

If four points A, B, C and D are coplanar, then $(\overline{AB}\ \overline{AC}\ \overline{AD}) = 0$

i.e., Volume of the tetrahedron $= 0$

$$\Rightarrow \quad \frac{55}{6}(q-1) = 0 \quad \Rightarrow \quad q = 1$$

Q17: Prove that

$$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b}) = 0$$

Sol: Here ,we have

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$$

$$= [(\vec{a} \,.\, \vec{c})\, \vec{b} - (\vec{a} \,.\, \vec{b})\, \vec{c}] + [(\vec{b} \,.\, \vec{a})\, \vec{c} - (\vec{b} \,.\, \vec{c})\, \vec{a}] + [(\vec{c} \,.\, \vec{b})\, \vec{b} - (\vec{c} \,.\, \vec{a})\, \vec{b}]$$

$$= [(\vec{b} \,.\, \vec{a})\, \vec{c} - (\vec{a} \,.\, \vec{b})\, \vec{c}] + [(\vec{c} \,.\, \vec{b})\, \vec{a} - (\vec{b} \,.\, \vec{c})\, \vec{a}] + [(\vec{a} \,.\, \vec{c})\, \vec{b} - (\vec{c} \,.\, \vec{a})\, \vec{b}]$$

$$= [(\vec{a} \,.\, \vec{b})\, \vec{c} - (\vec{a} \,.\, \vec{b})\, \vec{c}] + [(\vec{b} \,.\, \vec{c})\, \vec{a} - (\vec{b} \,.\, \vec{c})\, \vec{a}] + [(\vec{c} \,.\, \vec{a})\, \vec{b} - (\vec{c} \,.\, \vec{a})\, \vec{b}]$$

$$= 0 + 0 + 0 = 0$$

Proved.