Free Study Material from All Lab Experiments

Waves & Optics (B.Sc.)

Chapter - 1

Superposition of two collinear harmonic oscillations

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Superposition of Two Collinear Harmonic oscillations: Simple harmonic motion (SHM). Linearity and Superposition Principle. (1) Oscillations having equal frequencies and (2) Oscillations having different frequencies (Beats).
(6 Lectures)

Q1: what causes a system to oscillate?

The oscillation of a physical system results from *two* basic properties of the system, namely, *elasticity* and *inertia*. Consider a body in equilibrium so that forces on it balance. Let us displace it from its position of equilibrium (by doing work on it, i.e. applying a force) by a distance ψ. When it is released, a restoring force comes into play whose tendency is to 'restore' ψ to its original value, which is zero, by imparting to it an appropriate negative velocity $d\psi/dt$. *The magnitude of the restoring force is determined by the elastic properties of the system.* Inertia, on the other hand, tries to oppose any change in velocity. When the body reaches its equilibrium position ($\psi = 0$), the negative velocity is maximum which produces a negative displacement. The body then overshoots its position of equilibrium. The restoring force now becomes positive (i.e. it helps increase ψ) and it must now overcome the inertia of the negative velocity. Consequently the velocity keeps on decreasing until it is zero but by that time the displacement has become large and negative and the process is reversed. This process of the restoring force trying to bring ψ to zero by imparting a velocity and inertia preserving the velocity and making ψ to overshoot, repeats itself and the body oscillates.

Q2: Describe Simple harmonic motion (SHM).

Ans:

A periodic motion is a motion which repeats itself after regular intervals of time, and the simplest kind of periodic motion is a simple harmonic motion in which the displacement varies sinusoidally with time. To understand simple harmonic motion, we consider a point P rotating on the circumference of a circle of radius a with an angular velocity ω (see Fig. 7.1).

We choose the center of the circle as our origin, and we assume that at $t = 0$ the point P lies on the x axis (i.e., at point P_0). At an arbitrary time t the point will be at position P where $\angle POP_0 = \omega t$.

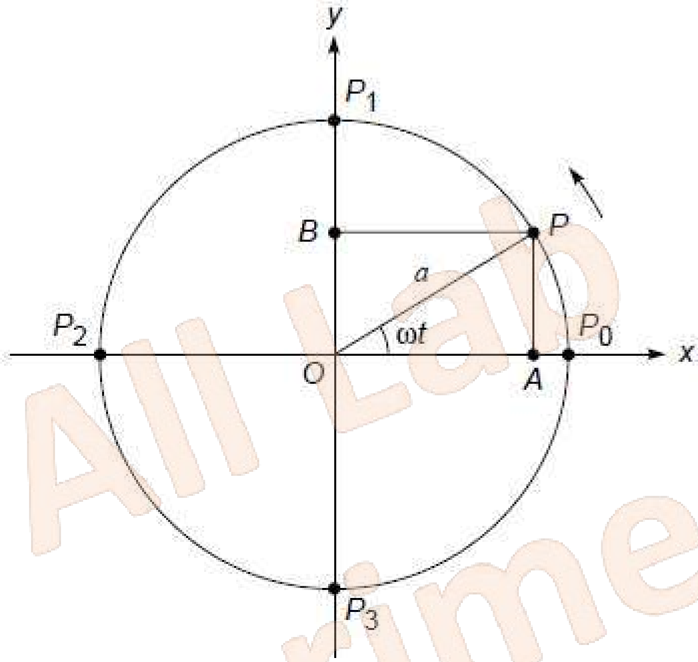


Fig. 7.1 The point P is rotating in the counterclockwise direction on the circumference of a circle of radius a, with uniform angular velocity ω. The foot of the perpendicular on any one of the diameters executes simple harmonic motion. Point P_0 is the position of the point at $t = 0$.

Let A be the foot of the perpendicular from the point P on the x axis. Clearly, the distance

$$OA = a \cos \omega t \quad \textbf{(1)}$$

and as point P rotates on the circumference of the circle, point A moves to and fro about the origin on the diameter. When point P is at P_1, then the foot of the perpendicular is at O. This can also be seen from Eq. (1) because when P coincides with P_1, $\omega t = \pi/2$ and hence $a \cos \omega t = a \cos \pi/2 = 0$. As the point still moves farther, the foot of the perpendicular will lie on the other side of the origin and thus OA will be negative, as is also evident from Eq. (1) because ωt then greater than $\pi/2$. When P coincides with P_2, then $OA = OP_2 = -a$. When point P moves from P_2 to P_3, OA starts decreasing and it finally goes to zero when P coincides with P_2. After P crosses P_3, OA starts increasing again and finally acquires the value a when P coincides with P_0. After crossing the point P_0, the motion repeats itself.

A motion in which the displacement varies sinusoidally with time [as in Eq. (1)] is known as a *simple harmonic motion. Thus, when a point rotates on the circumference of a circle with a uniform angular velocity, the foot of the perpendicular on any one of its diameters will execute simple harmonic motion.* The quantity a is called the amplitude of the motion, and the period of the motion T will be the time required to complete one revolution. Since the angular velocity is ω, the time taken for one complete revolution will be $2\pi/\omega$. Thus,

$$T = \frac{2\pi}{\omega} \tag{2}$$

Q3: what are the characteristics of SHM?

Ans: characteristics of SHM are as follows:

Amplitude

The amplitude of an SHM is the maximum (positive or negative) value of the displacement from the mean position. Since the maximum and minimum values of any cosine function are respectively $+1$ and -1, the maximum and minimum values of ψ in Eq. (1.7) are respectively $+A$ and $-A$. A is called the *amplitude* of SHM.

Time Period

The smallest time interval during which the oscillation repeats itself is called the *time period* (or simply, period) T of the oscillation. If time t in Eq (1.7) is advanced by $2\pi/\omega$, to $t' = t+2\pi/\omega$ the function becomes

$$\begin{aligned}\psi(t') &= A\cos(\omega t'+\phi)\\ &= A\cos\left\{\omega\left(t+\frac{2\pi}{\omega}\right)+\phi\right\}\\ &= A\cos(\omega t+\phi+2\pi)\\ &= A\cos(\omega t+\phi)\\ &= \psi(t)\end{aligned}$$

In other words, the displacement repeats itself in a time interval of $2\pi/\omega$. Therefore, the period T is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{K}} \tag{1.8}$$

Frequency of SHM, is the number of oscillations completed in a unit time interval. Therefore, by definition, frequency is the reciprocal of the time period, i e.

$$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{K}{m}} \tag{1.9}$$

Thus $$\omega = \frac{2\pi}{T} = 2\pi\nu \tag{1.10}$$

Phase

The argument $(\omega t+\phi)$ of the cosine function is called the *phase of the motion*. The constant ϕ of is called the *initial phase* (i. e. phase at $t = 0$) or the phase constant. The phase of an oscillating system at any instant is its state as regards its position and direction of motion at that instant. The knowledge of the phase constant enables us to find out how far from the mean position the system was at time $t = 0$. For example, if $\phi = 0$

$$\psi(t) = A\ \text{cost}\ \omega t$$

which means that the displacement was maximum $= A$ at time $t = 0$, i.e. when the motion was started. On the other hand, if $\phi = \pi/2$

$$\psi(t) = A\cos\left(\omega t+\frac{\pi}{2}\right) = A\sin\omega t$$

i e. the displacement was zero at time $t = 0$. In other words, the counting of time was started the moment the oscillator passed the mean position. Thus phase constant is a measure of how much time had elapsed before the oscillator last passed the mean position. Amplitude A and phase constant ϕ are determined from the *initial conditions*, i.e. the way the system is started at time $t = 0$ (... ... ...).

Q4: Derive the expression for velocity and acceleration of SHM.

It is instructive to learn how velocity and acceleration in a SHM vary with time. We know that displacement $\psi(t)$ is given by

$$\psi(t) = A \cos(\omega t + \phi)$$

Velocity V and acceleration a are given by

$$V = \frac{d\psi}{dt} = \dot{\psi} = -A\,\omega \sin(\omega t + \phi)$$

$$= \mp A\omega \left(1 - \frac{\psi^2}{A^2}\right)^{1/2} \tag{1.11}$$

and

$$a = \frac{d^2\psi}{dt^2} \equiv \ddot{\psi} = -\omega^2 A \cos(\omega t + \phi) = -\omega^2 \psi \tag{1.12}$$

We notice that when the displacement is maximum ($+A$ or $-A$) the velocity $V = 0$, because now the oscillator has to return and velocity must change its direction. But when ψ is maximum ($+A$ or $-A$), the acceleration is also maximum ($-\omega^2 A$ and $+\omega^2 A$ respectively) and is directed opposite to the displacement. When $\psi = 0$, i.e. when $\cos(\omega t + \phi) = 0$, the velocity is maximum (ωA or $-\omega A$) and the acceleration is zero.

Q: Derive the expression for energy of SHM.

Consider a system at rest at its position of equilibrium. When it is displaced from this position (by doing work on it) it acquires potential energy. When the system is released, it begins to move with a velocity, thus acquiring kinetic energy. At any instant of time, the kinetic energy o a system of mass m executing SHM is given by [using Eq. (1.11)]

$$\text{Kinetic energy (KE)} = \tfrac{1}{2} mV^2 = \tfrac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi) \tag{1.13}$$

The kinetic energy of the oscillator varies periodically. It is maximum ($= \frac{1}{2} m\omega^2 A^2$) when the velocity is maximum ($= \pm \omega A$) and displacement is zero. When the displacement is maximum ($= \pm A$), velocity $V = 0$ and KE = 0. At these extreme positions, the energy is all potential. At intermediate positions (ψ lying between O and $\pm A$), the energy is partly kinetic and partly potential.

$$\psi(t) = A \cos(\omega t + \phi)$$

A closer look at above equation reveals that the total energy of the oscillator must remain constant because the maximum displacement is regained after every half cycle. If no energy is dissipated (we have neglected dissipative or non-conservative forces like friction), then all the potential energy becomes kinetic and vice versa.

The energy of the oscillator may decrease not only due to friction in the system but also due to radiation. The oscillating body imparts periodic motion to the particles of the medium in which it oscillates thus producing waves. For example, a tuning fork or a string produces sound waves in the medium which results in a decrease in energy.

Let us now compute potential energy at any instant of time t. Let ψ be the displacement at time t. The potential energy is given by the amount of work required to move the system from $\psi = 0$ to ψ, by applying a force.

The force must be just enough to oppose the restoring force $F = -K\psi$

In other words, the force to be applied must be $K\psi$.

Work required to give an infinitesimal displacement $d\psi = K\psi\, d\psi$

Therefore, the total work done to displace the system from

$$0 \text{ to } \psi = \int_0^\psi K\psi d\psi = \frac{1}{2} K\psi^2.$$

Thus

$$\text{Potential energy (PE)} = \frac{1}{2} K\psi^2 = \frac{1}{2} m\,\omega^2 A^2 \cos^2(\omega t + \phi) \quad (1.14)$$

where we have used Eqs. (1.3) and (1.7).

Equations (1.13) and (1.14) give the instantaneous values of kinetic and potential energy.

The total energy E in SHM is, therefore, given by

$$E = \text{KE} + \text{PE}$$

$$= \frac{1}{2} m\omega^2 A^2\{\sin^2(\omega t+\phi)+\cos^2(\omega t+\phi)\}$$

or $$E = \frac{1}{2} m\omega^2 A^2$$

which is constant as we would expect. It is obvious that the maximum values of kinetic and potential energy are equal (both equal to $\frac{1}{2} m\omega^2 A^2$) indicating that the energy exchange is complete. Figure 1.3 shows how the kinetic and potential energy of the harmonic oscillator vary with time where, for simplicity, we have set $\phi = 0$.

Q: Describe superposition principle of harmonic oscillations

Ans:

We know that for small oscillations, a simple pendulum executes simple harmonic motion. Let us reconsider this motion and release the bab at the instant $t = 0$ when it has initial displacement a_1. Let the displacement at a subsequent time t be x_1. Let us repeat the experiment with an initial displacement a_2. Let the displacement after the same interval of time t be x_2. Now if we take the initial displacement to be the sum of the earlier displacemen'ts, viz. $a_1 + a_2$, then according to the superposition principle, the displacement x_3 after the same interval of time t will be

$$x_3 = x_1 + x_2.$$

You can perform this activity by taking three identical simple pendulums. Release all three bobs simullaneously such that their initial velocities are zero and initial displacements of the first, second and the third pendulum are a_1, a_2 and $a_1 + a_2$, respectively. You will find that at any time the displacement x_3 of the third pendulum will be the algebraic sum of the displacements of the other two. In general, the initial velocities may be non-zero. Thus, the principle of superposition can be stated as follows:

When we superpose the initial conditions corresponding to velocities and amplitudes, the resultant displacement of two (or more) harmonic displacements will be simply the algebraic sum of the individual displacements at all subsequent times.

You will note that the principle of superposition holds for any number of simple harmonic ascillations. These may be in the same or mutually perpendicular directions, i.e. in two dimensions.

Q: derive an expression for superposition of two collinear harmonic oscillations having equal frequencies.

Ans:

Suppose we have two SHMs of equal frequencies but of different amplitudes and phase constants acting on a particle (or a system) in the x direction. The displacements x_1 and x_2 of the two harmonic motions, of the same angular frequency ω, are given by

$$x_1 = A_1 \cos(\omega t + \phi_1) \tag{2.6}$$

and

$$x_2 = A_2 \cos(\omega t + \phi_2) \tag{2.7}$$

where A_1 and A_2 are the amplitudes and ϕ_1 and ϕ_2 are the phase constants of the two motions. The resultant motion of the system, which moves in the x direction under the simultaneous effect of the two harmonic oscillations, can be found by the following methods.

We use the superposition principle which states that the resultant displacement x is equal to the sum of the individual displacements x_1 and x_2, i.e.

$$x = x_1 + x_2$$

$$= A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2)$$

Using the trignometric identity $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$, this equation can be rewritten as

$$x = (A_1 \cos\phi_1 + A_2 \cos\phi_2) \cos\omega t - (A_1 \sin\phi_1 + A_2 \sin\phi_2) \sin\omega t \tag{2.8}$$

Now let (see also Fig. 2.1)

$$A_1 \sin\phi_1 + A_2 \sin\phi_2 = A \sin\delta \tag{2.9}$$

and

$$A_1 \cos\phi_1 + A_2 \cos\phi_2 = A \cos\delta \tag{2.10}$$

where A and δ are constants to be determined. Using the transformations (2.9) and (2.10) in Eq. (2.8) we have

$$x = A \cos(\omega t + \delta) \tag{2.11}$$

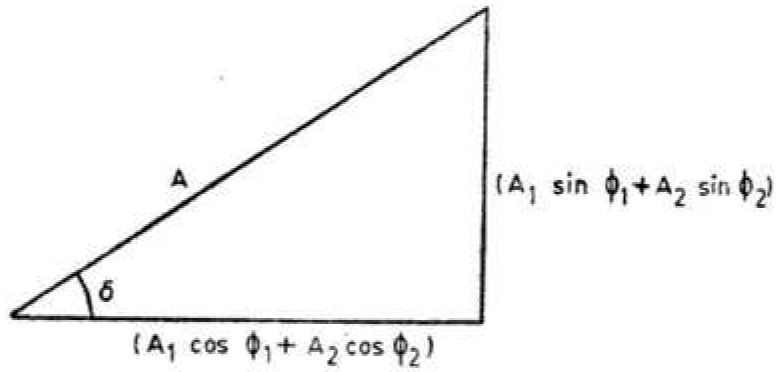


Fig. 2.1 Transformation into A and δ.

Equation (2.11) shows that the resulting motion is simple harmonic with an angular frequency ω, the same as that of the individual SHMs. The resulting motion has an amplitude A and a phase constant δ. A and δ can be evaluated from Eqs. (2.9) and (2.10). Squaring these equations and adding, we find that the resultant amplitude A is given by

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos(\phi_2 - \phi_1) \tag{2.12}$$

Dividing Eqs. (2.9) and (2.10) we find that the phase constant of the resulting motion is given by

$$\tan \delta = \frac{A_1 \sin \phi_1 + A_2 \sin \phi_2}{A_1 \cos \phi_1 + A_2 \cos \phi_2} \tag{2.13}$$

Thus we conclude that the resultant effect of two collinear SHMs of equal frequencies is a simple harmonic motion of the same frequency but having amplitude and phase constant given respectively by Eqs. (2.12) and (2.13).

It is evident from Eq. (2.12) that the amplitude of the resulting oscillations is maximum given by

$$A_{max} = A_1 + A_2$$

if $\cos(\phi_2 - \phi_1) = +1$ or $\phi_2 - \phi_1 = 2m\pi$, where m is an integer with values $m = 0, 1, 2, 3, \ldots$. On the other hand, the resultant amplitude is minimum given by

$$A_{min} = A_1 - A_2$$

if $\cos(\phi_2 - \phi_1) = -1$ or $\phi_2 - \phi_1 = (2m+1)\pi$. For other values of the phase difference $(\phi_2 - \phi_1)$ the resultant amplitude A lies between A_{max} and A_{min}.

Q: derive an expression for superposition of two collinear harmonic oscillations having different frequencies.

Ans:

In the subsequent chapters we shall come across many physical phenomena in which the moving part of a system is subjected simultaneously to two harmonic oscillations of different frequencies. To analyse the resulting motion of the system, let us consider two harmonic oscillations of different amplitudes A_1 and A_2 and different angular frequencies ω_1 and ω_2. For simplicity, we assume that the two oscillations have the same phase constant which we take to be zero[3]. The two harmonic oscillations can be written as

$$x_1 = A_1 \cos \omega_1 t \quad (2.15)$$

$$x_2 = A_2 \cos \omega_2 t \quad (2.16)$$

From superposition principle, the resulting oscillation is given by

$$x = x_1 + x_2 = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t \quad (2.17)$$

We shall now recast Eq. (2.17) into a particularly simple form. Let us define on *average frequency* ω_a and a *modulation frequency* ω_m as :

$$\omega_a = \tfrac{1}{2}(\omega_1 + \omega_2) \quad \text{and} \quad \omega_m = \tfrac{1}{2}(\omega_2 - \omega_1)$$

where $\omega_2 > \omega_1$, so that

$$\omega_1 = \omega_a - \omega_m$$

$$\omega_2 = \omega_a + \omega_m$$

Substituting for ω_1 and ω_2 in Eq. (2.17) we get

$$x = A_1 \cos(\omega_a - \omega_m)\, t + A_2 \cos(\omega_a + \omega_m)\, t$$

or $$x = (A_1 + A_2) \cos \omega_m t \cos \omega_a t - (A_1 - A_2) \sin \omega_m t \sin \omega_a t \quad (2.18)$$

Now, as before, let

$$(A_1 + A_2) \cos \omega_m t = A_m \cos \delta_m \quad (2.19)$$

and $$(A_1 - A_2) \sin \omega_m t = A_m \sin \delta_m \quad (2.20)$$

Using these transformations in Eq. (2.18) gives

$$x = A_m \cos(\omega_a t + \delta_m) \quad (2.21)$$

where A_m and δ_m are given by [use Eqs. (2.19) and (2.20)]

and $$A_m^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos(2\omega_m t) \quad (2.22)$$

$$\tan \delta_m = \frac{(A_1 - A_2) \sin \omega_m t}{(A_1 + A_2) \cos \omega_m t} \tag{2.23}$$

The formal resemblance of Eq. (2.21) with the equation of SHM is misleading. In fact, the oscillation described by Eq. (2.21) is not harmonic since its amplitude A_m and phase constant δ_m both vary with time according to Eqs. (2.22) and (2.23) respectively. This oscillation can, at best, be described as periodic with an angular frequency ω_a, the average of the two component frequencies.

Q: Explain the formation of beats.

Ans:

Beats. Recasting of the superposition (2.17) in the form of Eq. (2.21) becomes useful if ω_1 and ω_2 are nearly equal, i.e.

$$\omega_2 \simeq \omega_1$$

so that $$\omega_m \ll \omega_a$$

In that case, the 'modulated' amplitude A_m and 'modulated' phase δ_m vary only slightly with time and may be treated as sensibly constant during the time scale of interest, which in our case, is the period $(2\pi/\omega_a)$ of the fast oscillation. Therefore, Eq. (2.21) will represent an 'almost' harmonic oscillation at an angular frequency ω_a. The resulting oscillation, in the case when the two frequencies of the SHMs are nearly equal, exhibits what are called *beats*.

The amplitude A_m of the resulting motion is maximum $(= A_1 + A_2)$ when [see Eq. (2.22)]

$$\cos (2\omega_m t) = +1$$

or $$2\omega_m t = 0, 2\pi, 4\pi, \ldots$$

or $$(\omega_2 - \omega_1) t = 0, 2\pi, 4\pi, \ldots$$

or $$2\pi(\nu_2 - \nu_1) t = 0, 2\pi, 4\pi, \ldots$$

or when $$t = 0, \frac{1}{\nu_2 - \nu_1}, \frac{2}{\nu_2 - \nu_1}, \ldots$$

Here $\nu_1 (= \omega_1/2\pi)$ and $\nu_2 (= \omega_2/2\pi)$ are the frequencies of the two SHMs expressed in hertz.

Hence the time interval t_b between two consecutive maxima $= \frac{1}{\nu_2 - \nu_1}$. The frequency ν_b of the maxima $= \nu_2 - \nu_1$.

The amplitude A_m of the resulting motion is minimum $(= A_2 - A_1)$ when

$$\cos(2\omega_m t) = -1$$

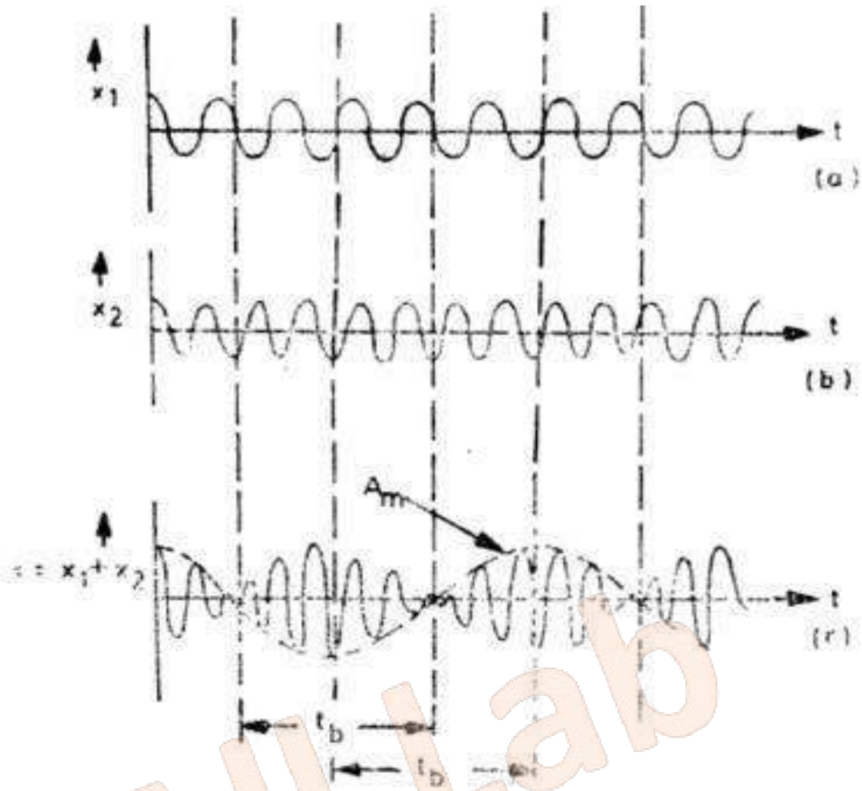
or when $t = \frac{1}{2(\nu_2 - \nu_1)}, \frac{3}{2(\nu_2 - \nu_1)}, \frac{5}{2(\nu_2 - \nu_1)}$ ... etc. Hence the frequency of the minima is also $\nu_b = \nu_2 - \nu_1$. Between any two maxima, there is a minimum. The periodic variation of the amplitude of the motion, resulting from the superposition of SHMs of slightly different frequencies, is known as the *phenomenon of beats*. One maximum of amplitude followed by a minimum is technically called a *beat*. The time period t_b between the successive beats is called the *beat period* given by

$$t_b = \frac{1}{\nu_2 - \nu_1}$$

and the *beat frequency* ν_b is given by

$$\nu_b = \frac{1}{t_b} = \nu_2 - \nu_1$$

Hence the beat frequency is equal to the difference between the frequencies of the component oscillations.



(a) Harmonic oscillation at frequency ν_1.
(b) Harmonic oscillation at frequency ν_2; ($\nu_2 > \nu_1$)
(c) Superposition of (a) and (b) and an harmonic oscillation with period $t_b = 1/(\nu_2 - \nu_1)$

Figure 2.4 displays graphically the result of superposing two harmonic oscillations of different frequencies. Notice that Figs 2.4a and 2.4b are harmonic oscillations but their superposition shown in Fig. 2.4c is periodic but not harmonic.

Q: write down the applications of beats.

Ans:

The phenomenon of beats is of great practical importance. Beats can be used to determine the small difference between frequencies of two sources of sound. Musicians often make use of beats in tuning their instruments. A piano tuner uses beats to tell whether his standard tuning fork has the same frequency as the string of his instrument. If the two differ in frequency, i.e. are out of tune, he will hear beats. He adjusts the tension in the string and thus changes the frequency of the note emitted by the string and matches it with his fork. Sometimes beats are deliberately produced in a particular section of an orchestra to give a pleasing tone to the resulting sound. A more complex beat phenomenon, resulting from the superposition of many harmonic oscillations of different frequencies, is employed to transmit a signal from one place to another.