Free Study Material from

All Lab Experiments

Mathematical Physics - I
Chapter - 1
Calculus

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Calculus:
Plotting of functions. Approximation: Taylor and binomial series (statements only). First Order Differential. Equations exact and inexact differential equations and Integrating Factor. **(6 Lectures)**

Second Order Differential equations: Homogeneous Equations with constant coefficients. Wronskian and general solution. Particular Integral with operator method, method of undetermined coefficients and variation method of parameters. **(15 Lectures)**

Q1: State Taylor's theorem

Sol:

If a function $f(z)$ is analytic at all points inside a circle C, with its centre at the point a and radius R, then at each point z inside C.

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots + \frac{f^n(a)}{n!}(z-a)^n + \ldots$$

Variable separation method

If a differential equation can be written in the form

$$f(y)\,dy = \phi(x)\,dx$$

We say that variables are separable, y on left hand side and x on right hand side. We get the solution by integrating both sides.

Working Rule:

Step 1. Separate the variables as $f(y)\,dy = \phi(x)\,dx$

Step 2. Integrate both sides as $\int f(y)\,dy = \int \phi(x)\,dx$

Step 3. Add an arbitrary constant C on R.H.S.

Q2: Solve:

$$\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$$

Sol: We have,

$$\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$$

Separating the variables, we get

$$(\sin y + y\cos y)\,dy = \{x\,(2\log x + 1)\}\,dx$$

Integrating both the sides, we get $\int (\sin y + y\cos y)dy = \int \{x(2\log x + 1)\}\,dx + C$

$$-\cos y + y\sin y - \int (1)\cdot \sin y\, dy = 2\int \log x \cdot x\, dx + \int x\, dx + C$$

$$\Rightarrow \quad -\cos y + y\sin y + \cos y = 2\left[\log x \cdot \frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}\, dx\right] + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = 2\log x \cdot \frac{x^2}{2} - \int x\, dx + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = 2\log x \cdot \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = x^2 \log x + C$$

Q3: Solve the differential equation.

$$x^4 \frac{dy}{dx} + x^3 y = -\sec(x\,y).$$

Sol: we have,

$$x^4 \frac{dy}{dx} + x^3 y = -\sec(x\,y) \quad \Rightarrow \quad x^3\left(x\frac{dy}{dx} + y\right) = -\sec xy$$

$$\text{Put} \quad v = xy,\ \frac{dv}{dx} = x\frac{dy}{dx} + y \quad \Rightarrow \quad x^3 \frac{dv}{dx} = -\sec v$$

$$\Rightarrow \quad \frac{dv}{\sec v} = -\frac{dx}{x^3} \quad \Rightarrow \quad \int \cos v\, dv = -\int \frac{dx}{x^3} + c$$

$$\Rightarrow \quad \sin v = \frac{1}{2x^2} + c \quad \Rightarrow \quad \sin xy = \frac{1}{2x^2} + c$$

Q4: Solve

$$\cos(x+y)dy = dx$$

Sol:

$$\cos(x+y)\, dy = dx \quad \Rightarrow \quad \frac{dy}{dx} = \sec(x+y)$$

On putting $x + y = z$

So that $1 + \frac{dy}{dx} = \frac{dz}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dz}{dx} - 1$

$$\frac{dz}{dx} - 1 = \sec z \quad \Rightarrow \quad \frac{dz}{dx} = 1 + \sec z$$

Separating the variables, we get

$$\frac{dz}{1+\sec z} = dx$$

On integrating,

$$\int \frac{\cos z}{\cos z+1}dz = \int dx \quad \Rightarrow \quad \int \left[1-\frac{1}{\cos z+1}\right]dz = x+C$$

$$\int \left(1-\frac{1}{2\cos^2\frac{z}{2}-1+1}\right)dz = x+C$$

$$\int \left(1-\frac{1}{2}\sec^2\frac{z}{2}\right)dz = x+C \quad \Rightarrow \quad z-\tan\frac{z}{2} = x+C$$

$$x+y-\tan\frac{x+y}{2} = x+C$$

$$y-\tan\frac{x+y}{2} = C$$

HOMOGENEOUS DIFFERENTIAL EQUATIONS

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$

is called a homogeneous equation if each term of $f(x, y)$ and $\phi(x, y)$ is of the same degree *i.e.*,

$$\frac{dy}{dx} = \frac{3xy+y^2}{3x^2+xy}$$

In such case we put $\boxed{y = vx. \text{ and } \frac{dy}{dx} = v + x\frac{dv}{dx}}$

The reduced equation involves v and x only. This new differential equation can be solved by *variables separable* method.

Q5: Solve the following differential equation

$$(2xy + x^2)\, y = 3y^2 + 2xy$$

Sol: we have

$$(2xy + x^2)\frac{dy}{dx} = 3y^2 + 2xy \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3y^2+2xy}{2xy+x^2}$$

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

On substituting, the given equation becomes

$$v + x\frac{dv}{dx} = \frac{3v^2x^2+2vx^2}{2vx^2+x^2} = \frac{3v^2+2v}{2v+1}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{3v^2+2v-2v^2-v}{2v+1} \qquad \Rightarrow \quad x\frac{dv}{dx}=\frac{v^2+v}{2v+1} \quad \Rightarrow \quad \left(\frac{2v+1}{v^2+v}\right)dv=\frac{dx}{x}$$

$$\Rightarrow \quad \int\left(\frac{2v+1}{v^2+v}\right)dv=\int\frac{dx}{x} \qquad \Rightarrow \quad \log(v^2+v)\log x+\log c$$

$$\Rightarrow \quad v^2+v=cx \qquad \Rightarrow \quad \frac{y^2}{x^2}+\frac{y}{x}=cx$$

$$\Rightarrow \quad y^2+xy=cx^3$$

Q6: Solve the equation:

$$\frac{dy}{dx}=\frac{y}{x}+x\sin\frac{y}{x} \qquad \text{......................... (1)}$$

Sol: Putting

$$y=vx \text{ in (1) so that } \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=v+x\sin v$$

$$x\frac{dv}{dx}=x\sin v \qquad \Rightarrow \qquad \frac{dv}{dx}=\sin v$$

Separating the variable, we get

$$\frac{dv}{\sin v}=dx \qquad \Rightarrow \qquad \int \operatorname{cosec} v\, dv=\int dx+C$$

$$\log\tan\frac{v}{2}=x+C \qquad \Rightarrow \qquad \log\tan\frac{y}{2x}=x+C$$

Q7: Solve:

$$\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$$

Putting

$x=X+h, \qquad y=Y+k.$

The given equation reduces to

$$\therefore \qquad \frac{dY}{dX}=\frac{(X+h)+2(Y+k)-3}{2(X+h)+(Y+k)-3} \qquad \left(\frac{1}{2}\neq\frac{2}{1}\right)$$

$$=\frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)} \qquad \text{.... (1)}$$

Now choose h and k so that $h+2k-3=0$, $2h+k-3=0$

Solving these equations we get $h=k=1$

$$\frac{dY}{dX} = \frac{X+2Y}{2X+Y} \qquad \dots (2)$$

Put $Y = vX$, so that $\frac{dY}{dX} = v + X\frac{dv}{dX}$

The equation (2) is transformed as

$$v + X\frac{dv}{dX} = \frac{X+2vX}{2X+vX} = \frac{1+2v}{2+v}$$

$$X\frac{dv}{dX} = \frac{1+2v}{2+v} - v = \frac{1-v^2}{2+v} \quad \Rightarrow \quad \left(\frac{2+v}{1-v^2}\right)dv = \frac{dX}{X}$$

$$\Rightarrow \quad \frac{1}{2}\frac{1}{(1+v)}dv + \frac{3}{2}\frac{1}{1-v}dv = \frac{dX}{X} \qquad \text{(Partial fractions)}$$

On integrating, we have

$$\frac{1}{2}\log(1+v) - \frac{3}{2}\log(1-v) = \log X + \log C$$

$$\Rightarrow \quad \log\frac{1+v}{(1-v)^3} = \log C^2X^2 \quad \Rightarrow \quad \frac{1+v}{(1-v)^3} = C^2X^2$$

$$\frac{1+\frac{Y}{X}}{\left(1-\frac{Y}{X}\right)^3} = C^2X^2 \quad \Rightarrow \quad \frac{X+Y}{(X-Y)^3} = C^2 \text{ or } X + Y = C^2(X-Y)^3$$

Put $X = x - 1$ and $Y = y - 1$ $\Rightarrow$ $x + y - 2 = a(x-y)^3$ **Ans.**

Q8: Solve:

$$(x + 2y)(dx - dy) = dx + dy$$

Sol:

$$(x + 2y)(dx - dy) = dx + dy \quad \Rightarrow \quad (x + 2y - 1)\,dx - (x + 2y + 1)\,dy = 0$$

$$\frac{dy}{dx} = \frac{x+2y-1}{x+2y+1} \qquad \dots(1)$$

Hence $\frac{a}{A} = \frac{b}{B}$ *i.e.,* $\left(\frac{1}{1} = \frac{2}{2}\right)$ (Case of failure)

Now put $x + 2y = z$ so that $1 + 2\frac{dy}{dx} = \frac{dz}{dx}$

Equation (1) becomes

$$\frac{1}{2}\frac{dz}{dx} - \frac{1}{2} = \frac{z-1}{z+1} \quad \Rightarrow \quad \frac{dz}{dx} = 2\frac{(z-1)}{z+1} + 1 = \frac{3z-1}{z+1}$$

$$\Rightarrow \quad \frac{z+1}{3z-1}dz = dx \quad \Rightarrow \quad \left(\frac{1}{3} + \frac{4}{3}\frac{1}{3z-1}\right)dz = dx$$

On integrating, $\frac{z}{3} + \frac{4}{9}\log(3z-1) = x + C$

$$3z + 4\log(3z-1) = 9x + 9C$$

$$\Rightarrow \quad 3(x+2y) + 4\log(3x+6y-1) = 9x + 9C$$

$$3x - 3y + a = 2\log(3x+6y-1)$$

LINEAR DIFFERENTIAL EQUATIONS

$$\boxed{\frac{dy}{dx} + Py = Q}$$

Q9: Solve:

$$(x+1)\frac{dy}{dx} - y = e^x(x+1)^2$$

Sol:

$$\frac{dy}{dx} - \frac{y}{x+1} = e^x(x+1)$$

Integrating factor $= e^{-\int\frac{dx}{x+1}} = e^{-\log(x+1)} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$

The solution is $y.\frac{1}{x+1} = \int e^x.(x+1).\frac{1}{x+1}dx = \int e^x dx$

$$\frac{y}{x+1} = e^x + C$$

Q10: Solve a differential equation

$$(x^3 - x)\frac{dy}{dx} - (3x^2 - 1)y = x^5 - 2x^3 + x.$$

Sol: We have,

$$(x^3 - x)\frac{dy}{dx} - (3x^2 - 1)y = x^5 - 2x^3 + x.$$

$$\frac{dy}{dx} - \frac{3x^2-1}{x^3-x}y = \frac{x^5-2x^3+x}{x^3-x} \quad \Rightarrow \quad \frac{dy}{dx} - \frac{3x^2-1}{x^3-x}y = x^2 - 1$$

$$\text{I.F.} = e^{\int -\frac{3x^2-1}{x^3-x}dx} = e^{-\log(x^3-x)} = e^{\log(x^3-x)^{-1}} = \frac{1}{x^3-x}$$

Its solution is

$$y(\text{I.F.}) = \int Q(I.F.)\,dx + C \qquad \Rightarrow \quad y\left(\frac{1}{x^3-x}\right) = \int \frac{x^2-1}{x^3-x}dx + C$$

$$\Rightarrow \quad \frac{y}{x^3-x} = \int \frac{x^2-1}{x(x^2-1)}dx + C \qquad \Rightarrow \quad \frac{y}{x^3-x} = \int \frac{1}{x}dx + C$$

$$\Rightarrow \quad \frac{y}{x^3-x} = \log x + C \qquad \Rightarrow \quad y = (x^3 - x)\log x + (x^3 - x)\,C$$

Q11: Solve:

$$x^2dy + y(x + y)\,dx = 0$$

Sol: We have,

$$x^2dy + y(x + y)\,dx = 0$$

$$\frac{dy}{dx} + \frac{y}{x} = -\frac{y^2}{x^2} \qquad \Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = -\frac{1}{x^2}$$

Put $-\frac{1}{y} = z$ so that $\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

The given equation reduces to a linear differential equation in z.

$$\frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$$

$$\textbf{I.F.} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log 1/x} = \frac{1}{x}.$$

Hence the solution is

$$z.\frac{1}{x} = \int -\frac{1}{x^2}.\frac{1}{x}\,dx + C \qquad \Rightarrow \quad \frac{z}{x} = \int -x^{-3}\,dx + C$$

$$\Rightarrow \quad -\frac{1}{xy} = -\frac{x^{-2}}{-2} + C \qquad \Rightarrow \quad \frac{1}{xy} = -\frac{1}{2x^2} - C$$

Q12: Solve:

$$\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x \sec y.$$

Sol:

$$\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x \sec y$$

$$\cos y\frac{dy}{dx}-\frac{\sin y}{1+x}=(1+x)e^x$$

………………… **(1)**

Put $\sin y = z,$ so that $\cos y\frac{dy}{dx}=\frac{dz}{dx}$

(1) becomes $\frac{dz}{dx}-\frac{z}{1+x}=(1+x)e^x$

$$I.F.=e^{-\int\frac{1}{1+x}dx}=e^{-\log(1+x)}=e^{\log\frac{1}{1+x}}=\frac{1}{1+x}$$

Solution is $z.\frac{1}{1+x}=\int(1+x)e^x.\frac{1}{1+x}dx+C=\int e^x dx+C$

$$\frac{\sin y}{1+x}=e^x+C$$

Q13: Solve the differential equation.
y log y dx + (x – log y) dy = 0

Sol: We have

$$y\log y\,dx+(x-\log y)\,dy=0$$

$$\frac{dx}{dy}=\frac{-x+\log y}{y\log y} \quad\Rightarrow\quad \frac{dx}{dy}=\frac{-x}{y\log y}+\frac{\log y}{y\log y}$$

$$\frac{dx}{dy}+\frac{x}{y\log y}=\frac{1}{y}$$

$$\text{I.F.}=e^{\int\frac{1}{y\log y}dy}=e^{\log(\log y)}=\log y$$

Its solution is $x.\log y=\int\frac{1}{y}(\log y)\,dy$

$$x.\log y=\frac{(\log y)^2}{2}+C$$

Q14: Solve

$$r\sin\theta-\frac{dr}{d\theta}\cos\theta=r^2$$

Sol: The given equation can be written as

$$-\frac{dr}{d\theta}\cos\theta + r\sin\theta = r^2 \qquad \ldots (1)$$

Dividing (1) by $r^2\cos\theta$, we get $-r^{-2}\frac{dr}{d\theta} + r^{-1}\tan\theta = \sec\theta$ .... **(2)**

Putting $r^{-1} = v$ so that $-r^{-2}\frac{dr}{d\theta} = \frac{dv}{d\theta}$ in (2), we get

$$\frac{dv}{d\theta} + v\tan\theta = \sec\theta$$

$$\text{I.F.} = e^{\int \tan\theta\, d\theta} = e^{\log\sec\theta} = \sec\theta$$

Solution is $v\sec\theta = \int \sec\theta, \sec\theta + C \quad \Rightarrow \quad v\sec\theta = \int \sec^2\theta\, d\theta + C$

$$\frac{\sec\theta}{r} = \tan\theta + C \qquad \Rightarrow \qquad r^{-1} = (\sin\theta + C\cos\theta)$$

$\therefore$
$$r = \frac{1}{\sin\theta + C\cos\theta}$$

EXACT DIFFERENTIAL EQUATION

An exact differential equation is formed by directly differentiating its primitive (solution) without any other process

$$Mdx + Ndy = 0$$

is said to be an exact differential equation if it satisfies the following condition

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

where $\frac{\partial M}{\partial y}$ denotes the differential co-efficient of M with respect to y keeping x constant and $\frac{\partial N}{\partial x}$, the differential co-efficient of N with respect to x, keeping y constant.

Q15: Solve

$$(5x^4 + 3x^2y^2 - 2xy^3)\, dx + (2x^3y - 3x^2y^2 - 5y^4)\, dy = 0$$

Sol: Here

$M = 5x^4 + 3x^2y^2 - 2xy^3, \qquad N = 2x^3y - 3x^2y^2 - 5y^4$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2, \quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

Since, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

Now $\int M\,dx + \int (\text{terms of } N \text{ is not containing } x)\,dy = C$

$$\int \left(5x^4 + 3x^2y^2 - 2xy^3\right)dx + \int -5y^4 dy = C$$

$$\Rightarrow \quad x^5 + x^3y^2 - x^2y^3 - y^5 = C$$

Q16: Solve:

$$[1 + \log(x\,y)]\,dx + \left[1 + \frac{x}{y}\right]dy = 0$$

Sol:

$$[1 + \log(x\,y)]\,dx + \left[1 + \frac{x}{y}\right]dy = 0$$

which is in the form $M\,dx + N\,dy = 0$

$$M = [1 + \log x + \log y] \quad \text{and} \quad N = 1 + \frac{x}{y}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{1}{y} \quad \text{and} \quad \Rightarrow \frac{\partial N}{\partial x} = \frac{1}{y} \quad \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the given differential equation is exact.

$\therefore$ Solution is $\displaystyle\int_{y\text{ constant}} M\,dx + \int N(\text{terms not containing } x)\,dy = C$

y constant

$$\therefore \quad \int (1 + \log x + \log y)\,dx + \int dy = C$$

$$\Rightarrow \quad x + \int \log x\,dx + \int \log y\,dx + y = C \qquad \ldots (1)$$

Now, $\displaystyle\int \log x\,dx = \int \log x.(1)\,dx = (\log x)x - \int\left[\frac{d}{dx}(\log x)x\right]dx = x\log x - \int \frac{1}{x}.x\,dx$

$$= x\log x - \int dx = x\log x - x = x[\log x - 1]$$

$\therefore$ Equation (1) becomes $\Rightarrow \quad x + x\log x - x + x\log y + y = C$

$$x\,[\log x + \log y] + y = C \;\Rightarrow\; x\log xy + y = C$$

Q17: Solve:

$(2x\log x - xy)\,dy + 2y\,dx = 0$ **............... (1)**

Sol:

$$M = 2y, \qquad N = 2x\log x - xy$$
$$\frac{\partial M}{\partial y} = 2, \qquad \frac{\partial N}{\partial x} = 2(1+\log x) - y$$

Here, $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2-2-2\log x + y}{2x\log x - xy} = \frac{-(2\log x - y)}{x(2\log x - y)} = -\frac{1}{x} = f(x)$$

$$\text{I.F.} = e^{\int f(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

On multiplying the given differential equation (1) by $\frac{1}{x}$, we get

$$\frac{2y}{x}dx + (2\log x - y)dy = 0 \qquad \Rightarrow \qquad \int \frac{2y}{x}dx + \int -y\,dy = c$$

$$\Rightarrow \qquad 2y\log x - \frac{1}{2}y^2 = c$$

Q18: Solve:

$$(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0 \qquad \text{……………… (1)}$$

Sol:

Here $M = y^4 + 2y$; $\qquad N = xy^3 + 2y^4 - 4x$

$$\frac{\partial M}{\partial y} = 4y^3 + 2; \qquad \frac{\partial N}{\partial x} = y^3 - 4$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{(y^3-4)-(4y^3+2)}{y^4+2y} = \frac{-3(y^3+2)}{y(y^3+2)} = -\frac{3}{y} = f(y)$$

$$\text{I.F.} = e^{\int f(y)dy} = e^{\int -\frac{3}{y}dy} = e^{-3\log y} = e^{\log y^{-3}} = y^{-3} = \frac{1}{y^3}$$

On multiplying the given equation (1) by $\frac{1}{y^3}$ we get the exact differential equation.

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$$

$$\int\left(y + \frac{2}{y^2}\right)dx + \int 2y\,dy = c \qquad \Rightarrow \qquad x\left(y + \frac{2}{y^2}\right) + y^2 = c$$

Q19: Solve

$y\,(xy + 2x^2y^2)\,dx + x\,(xy - x^2y^2)\,dy = 0$ ………… **(1)**

Sol: Dividing (1) by xy, we get

$$y\,(1 + 2xy)\,dx + x\,(1 - xy)\,dy = 0 \qquad \ldots (2)$$

$$M = y\,f_1\,(xy), \quad N = x\,f_2\,(xy)$$

$$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{xy(1+2xy) - xy(1-xy)} = \frac{1}{3x^2y^2}$$

On multiplying (2) by $\frac{1}{3x^2y^2}$, we have an exact differential equation

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0 \quad \Rightarrow \quad \int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}dy = c$$

$$\Rightarrow \quad -\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c \quad \Rightarrow \quad -\frac{1}{xy} + 2\log x - \log y = b$$

Q20: Solve

$$(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$$

Sol: We have

$$(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$$

$$y^2\,(ydx + 2xdy) + x^2\,(-2ydx - xdy) = 0$$

Here $m = 0,\ h = 2,\ a = 1,\ b = 2, \quad m' = 2,\ n' = 0,\ a' = -2,\ b' = -1$

$$\frac{0+h+1}{1} = \frac{2+k+1}{2} \quad \text{and} \quad \frac{2+h+1}{-2} = \frac{0+k+1}{-1}$$

$$\Rightarrow \quad 2h + 2 = 2 + k + 1 \text{ and } h + 3 = 2k + 2$$

$$\Rightarrow \quad 2h - k = 1 \text{ and } h - 2k = -1$$

On solving $h = k = 1$. Integrating Factor $= xy$

Multiplying the given equation by xy, we get

$$(xy^4 - 2x^3y^2)\,dx + (2x^2y^3 - x^4y)\,dy = 0$$

which is an exact differential equation.

$$\int (xy^4 - 2x^3y^2)dx = C \quad \Rightarrow \quad \frac{x^2y^4}{2} - \frac{2x^4y^2}{4} = C$$

$$x^2y^4 - x^4y^2 = C' \quad \Rightarrow \quad x^2y^2\,(y^2 - x^2) = C'$$

Q21: Solve:

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$$

Sol:

$$(x^3 + y^3)\, dx - (xy^2)\, dy = 0 \qquad \text{.......................... (1)}$$

Here $M = x^3 + y^3, \quad N = -xy^2$

$$\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x(x^3 + y^3) - xy^2(y)} = \frac{1}{x^4}$$

Multiplying (1) by $\frac{1}{x^4}$ we get $\frac{1}{x^4}(x^3 + y^3)dx + \frac{1}{x^4}(-xy^2)dy = 0$

$\Rightarrow \left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx - \frac{y^2}{x^3}dy = 0$, which is an exact differential equation.

$$\int \left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx = c \quad \Rightarrow \quad \log x - \frac{y^3}{3x^3} = c$$

WRONSKIAN

We know that

$$W(y_1, y_2, x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

(1) If $W(y_1, y_2, x) = 0$, then $y_1(x)$ and $y_2(x)$ are linearly dependent.

(2) If $W(y_1, y_2, x) \neq 0$, then $y_1(x), y_2(x)$ are linearly independent.

Q22: Check whether the following functions are linearly independent or not

$e^x \cos x, e^x \sin x.$

Sol: We have,

$$y_1 = e^x \cos x,\ y_2 = e^x \sin x$$

$$y'_1 = e^x \cos x - e^x \sin x \text{ and } y'_2 = e^x \sin x + e^x \cos x$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix}$$

$$= e^{2x} \begin{vmatrix} \cos x & \sin x \\ \cos x - \sin x & \sin x + \cos x \end{vmatrix}$$

$$= e^{2x} (\sin x . \cos x + \cos^2 x - \sin x . \cos x + \sin^2 x)$$

$$= e^{2x} \neq 0$$

Hence, $e^x \cos x$ and $e^x \sin x$ are linearly independent.

SECOND ORDER DIFFERENTIAL EQUATION:

Complete Solution = Complementary Function + Particular Integral.

$\Rightarrow$ $$y = C.F. + P.I.$$

Q23: Solve

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0.$$

Sol: Given equation can be written as

$$(D^2 - 8D + 15)\, y = 0$$

Here auxiliary equation is $m^2 - 8m + 15 = 0$

$\Rightarrow$ $(m-3)(m-5) = 0$ $\therefore\ m = 3, 5$

Hence, the required solution is

$$y = C_1 e^{3x} + C_2 e^{5x}$$

Q24: Solve

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

Sol: Given equation can be written as

$$(D^2 - 6D + 9)\, y = 0$$

A.E. is $m^2 - 6m + 9 = 0$ $\Rightarrow$ $(m-3)^2 = 0$ $\Rightarrow$ $m = 3, 3$

Hence, the required solution is

$$y = (C_1 + C_2 x)\, e^{3x}$$

Q25: Solve

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0,$$

$$y = 2 \text{ and } \frac{dy}{dx} = \frac{d^2y}{dx^2} \text{ when } x = 0.$$

Sol: Here the auxiliary equation is

$$m^2 + 4m + 5 = 0$$

Its root are $-2 \pm i$

The complementary function is

$$y = e^{-2x}(A\cos x + B\sin x) \qquad \ldots(1)$$

On putting $y = 2$ and $x = 0$ in (1), we get

$$2 = A$$

On putting $A = 2$ in (1), we have

$$y = e^{-2x}[2\cos x + B\sin x] \quad ...(2)$$

On differentiating (2), we get

$$\frac{dy}{dx} = e^{-2x}[-2\sin x + B\cos x] - 2e^{-2x}[2\cos x + B\sin x]$$
$$= e^{-2x}[(-2B-2)\sin x + (B-4)\cos x]$$

$$\frac{d^2y}{dx^2} = e^{-2x}[(-2B-2)\cos x - (B-4)\sin x]$$
$$- 2e^{-2x}[(-2B-2)\sin x + (B-4)\cos x]$$
$$= e^{-2x}[(-4B+6)\cos x + (3B+8)\sin x]$$

But $$\frac{dy}{dx} = \frac{d^2y}{dx^2}$$

$$e^{-2x}[(-2B-2)\sin x + (B-4)\cos x] = e^{-2x}[(-4B+6)\cos x + (3B+8)\sin x]$$

On putting $x = 0$, we get

$$B - 4 = -4B + 6 \quad \Rightarrow \quad B = 2$$

(2) becomes, $$y = e^{-2x}[2\cos x + 2\sin x]$$
$$y = 2e^{-2x}[\sin x + \cos x]$$

Q26: Solve

$$: \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$$

Sol:

$$(D^2 + 6D + 9)y = 5e^{3x}$$

Auxiliary equation is

$$m^2 + 6m + 9 = 0 \quad \Rightarrow \quad (m+3)^2 = 0 \quad \Rightarrow \quad m = -3, -3,$$

$$\text{C.F.} = (C_1 + C_2x)e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^2 + 6D + 9}.5.e^{3x} = 5\frac{e^{3x}}{(3)^2 + 6(3) + 9} = \frac{5e^{3x}}{36}$$

The complete solution is $$y = (C_1 + C_2x)e^{-3x} + \frac{5e^{3x}}{36}$$

Q27: Solve:

$(D^2 + 4)y = \cos 2x$

Solution. $(D^2 + 4)y = \cos 2x$

Auxiliary equation is $m^2 + 4 = 0$

$m = \pm 2i$, C.F. $= A\cos 2x + B\sin 2x$

$$\text{P.I.} = \frac{1}{D^2+4}\cos 2x = x.\frac{1}{2D}\cos 2x = \frac{x}{2}\left(\frac{1}{2}\sin 2x\right) = \frac{x}{4}\sin 2x$$

Complete solution is $y = A\cos 2x + B\sin 2x + \frac{x}{4}\sin 2x$

Q28: Solve

$$(D^2 - 4D + 4)\,y = x^3\,e^{2x}$$

Sol:

A.E. is $m^2 - 4m + 4 = 0 \quad \Rightarrow \quad (m-2)^2 = 0 \quad \Rightarrow \quad m = 2, 2$

C.F. $= (C_1 + C_2\,x)\,e^{2x}$

$$\text{P.I.} = \frac{1}{D^2-4D+4}x^3 \cdot e^{2x} = e^{2x}\frac{1}{(D+2)^2-4(D+2)+4}x^3$$

$$= e^{2x}\frac{1}{D^2}x^3 = e^{2x}.\frac{1}{D}\left(\frac{x^4}{4}\right) = e^{2x}.\frac{x^5}{20}$$

The complete solution is $y = (C_1 + C_2 x)e^{2x} + e^{2x}.\frac{x^5}{20}$

Q29: Solve the differential equation

$$\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 10\frac{dy}{dx} = e^{2x}\sin x$$

Solution. $\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 10\frac{dy}{dx} = e^{2x}\sin x \quad \Rightarrow D^3y - 7D^2y + 10\,Dy = e^{2x}\sin x$

A.E. is

$$m^3 - 7m^2 + 10\,m = 0 \quad \Rightarrow \quad (m-2)\,(m^2 - 5m) = 0$$

$$\Rightarrow \quad m\,(m-2)\,(m-5) = 0 \quad \Rightarrow \quad m = 0, 2, 5$$

C.F $= C_1e^{0x} + C_2e^{2x} + C_3e^{5x}$

$$\text{P.I.} = \frac{1}{D^3-7D^2+10D}e^{2x}\sin x = e^{2x}\frac{1}{(D+2)^3-7\,(D+2)^2+10(D+2)}.\sin x$$

$$= e^{2x}\frac{1}{D^3+6D^2+12D+8-7D^2-28D-28+10D+20}.\sin x$$

$$= e^{2x}\frac{1}{D^3-D^2-6D}\sin x = e^{2x}\frac{1}{(-1^2)\,D-(-1^2)-6D}\sin x$$

$$= e^{2x}\frac{1}{-D+1-6D}\sin x = e^{2x}\frac{1}{1-7D}\sin x = e^{2x}\frac{1+7D}{1-49D^2}\sin x = e^{2x}\frac{1+7D}{1-49\,(-1^2)}\sin x$$

$$= e^{2x}\frac{1+7D}{50}\sin x = \frac{e^{2x}}{50}(\sin x + 7\cos x)$$

Complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \quad y = C_1 + C_2 e^{2x} + C_3 e^{5x} + \frac{e^{2x}}{50}(\sin x + 7\cos x)$$

Q30: Solve

$$(D^2 + 6D + 9)y = \frac{e^{-3x}}{x^3}.$$

Sol:

A.E. is $m^2 + 6m + 9 = 0$

$(m+3)^2 = 0 \quad \therefore \quad m = -3, -3$

C.F. $= (C_1 + C_2 x)\, e^{-3x}$

$$\text{P.I.} = \frac{1}{D^2 + 6D + 9}\frac{e^{-3x}}{x^3} = e^{-3x}\frac{1}{(D-3)^2 + 6(D-3) + 9}\frac{1}{x^3}$$

$$= e^{-3x}\frac{1}{D^2 - 6D + 9 + 6D - 18 + 9}\frac{1}{x^3} = e^{-3x}\frac{1}{D^2}(x^{-3})$$

$$= e^{-3x}\frac{1}{D}\left(\frac{x^{-2}}{-2}\right) = e^{-3x}\frac{x^{-1}}{(-2)(-1)} = \frac{e^{-3x}x^{-1}}{2} = \frac{e^{-3x}}{2x}$$

Hence, the solution is $y = (C_1 + C_2 x)e^{-3x} + \dfrac{e^{-3x}}{2x}$

Q31: Solve

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\sin x$$

Sol: Auxiliary equation is

$$m^2 - 2m + 1 = 0 \text{ or } m = 1, 1$$

C.F. $= (C_1 + C_2 x)\, e^x$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1}x \cdot \sin x \quad (e^{ix} = \cos x + i\sin x)$$

$= \text{Imaginary part of } \dfrac{1}{D^2 - 2D + 1}x(\cos x + i\sin x) = \text{Imaginary part of } \dfrac{1}{D^2 - 2D + 1}x \cdot e^{ix}$

$= \text{Imaginary part of } e^{ix}\dfrac{1}{(D+i)^2 - 2(D+i) + 1} \cdot x = \text{Imaginary part of } e^{ix}\dfrac{1}{D^2 - 2(1-i)D - 2i} \cdot x$

$= \text{Imaginary part of } e^{ix}\dfrac{1}{-2i}\left[1 - (1+i)D - \dfrac{1}{2i}D^2\right]^{-1} \cdot x$

$= \text{Imaginary part of } (\cos x + i\sin x)\left(\dfrac{i}{2}\right)[1 + (1+i)D]x = \text{Imaginary part of } \dfrac{1}{2}(i\cos x - \sin x)[x + 1 + i]$

$$\text{P.I.} = \frac{1}{2}x\cos x + \frac{1}{2}\cos x - \frac{1}{2}\sin x$$

Complete solution is $y = (C_1 + C_2x)e^x + \frac{1}{2}(x\cos x + \cos x - \sin x)$

METHOD OF VARIATION OF PARAMETERS

Q32: Apply the method of variation of parameters to solve

$$\frac{d^2y}{dx^2} + y = \tan x$$

Sol: We have

$$\frac{d^2y}{dx^2} + y = \tan x$$

$$(D^2 + 1)y = \tan x$$

A.E. is $m^2 = -1$ or $m = \pm i$

C. F. $y = A\cos x + B\sin x$

Here, $y_1 = \cos x,\quad y_2 = \sin x$

$$y_1 \cdot y'_2 - y'_1 \cdot y_2 = \cos x(\cos x) - (-\sin x)\sin x = \cos^2 x + \sin^2 x = 1$$

P. I. $= u.\,y_1 + v.\,y_2$ where

$$u = \int \frac{-y_2\tan x}{y_1 \cdot y'_2 - y'_1 \cdot y_2}dx = -\int \frac{\sin x\tan x}{1}dx = -\int \frac{\sin^2 x}{\cos x}dx = -\int \frac{1-\cos^2 x}{\cos x}dx$$

$$= \int(\cos x - \sec x)\,dx = \sin x - \log(\sec x + \tan x)$$

$$v = \int \frac{y_1\tan x}{y_1 \cdot y'_2 - y'_1 \cdot y_2}dx = \int \frac{\cos x \cdot \tan x}{1}dx = \int \sin x\,dx = -\cos x$$

P. I. $= u.\,y_1 + v.\,y_2$

$$= [\sin x - \log(\sec x + \tan x)]\cos x - \cos x\sin x = -\cos x\log(\sec x + \tan x)$$

Complete solution is

$$y = A\cos x + B\sin x - \cos x\log(\sec x + \tan x)$$

Q33: Solve by method of variation of parameters:

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

Solution. $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

A. E. is $(m^2 - 1) = 0$

$m^2 = 1, \quad m = \pm 1$

$C.F. = C_1 e^x + C_2 e^{-x}$

$\therefore \quad P.I. = uy_1 + vy_2$

Here, $y_1 = e^x, \quad y_2 = e^{-x}$

and $y_1 \cdot y'_2 - y'_1 \cdot y_2 = -e^x \cdot e^{-x} - e^x . e^{-x} = -2$

$$u = \int \frac{-y_2 X}{y_1 . y'_2 - y'_1 . y_2} dx = -\int \frac{e^{-x}}{-2} \times \frac{2}{1+e^x} dx$$

$$= \int \frac{e^{-x}}{1+e^x} dx = \int \frac{dx}{e^x(1+e^x)} = \int \left(\frac{1}{e^x} - \frac{1}{1+e^x} \right) dx$$

$$= \int e^{-x} dx - \int \frac{e^{-x}}{e^{-x}+1} dx = -e^{-x} + \log(e^{-x}+1)$$

$$v = \int \frac{y_1 X}{y_1 . y'_2 - y'_1 . y_2} dx = \int \frac{e^x}{-2} \frac{2}{1+e^x} dx = -\int \frac{e^x}{1+e^x} dx = -\log(1+e^x)$$

P.I. $= u. y_1 + v. y_2 = [-e^{-x} + \log(e^{-x}+1)] e^x - e^{-x} \log(1+e^x)$

$= -1 + e^x \log(e^{-x}+1) - e^{-x} \log(e^x+1)$

Complete solution $= y = C_1 e^x + C_2 e^{-x} - 1 + e^x \log(e^{-x}+1) - e^{-x} \log(e^x+1)$

Q34: Solve by method of variation of parameters:

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x.$$

Sol: Given equation can be written as

$$(D^2 + 1) y = \operatorname{cosec} x$$

A.E. is $m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$

$$C.F. = A \cos x + B \sin x$$

Here $y_1 = \cos x, \qquad y_2 = \sin x$

$$\text{P.I.} = y_1\, u + y_2\, v$$

where $$u = \int \frac{-y_2 \,.\, \text{cosec}\, x\, dx}{y_1 \,.\, y'_2 - y'_1 \,.\, y_2} = \int \frac{-\sin x \,.\, \text{cosec}\, x\, dx}{\cos x\,(\cos x) - (-\sin x)\,(\sin x)}$$

$$= \int \frac{-\sin x \,.\, \frac{1}{\sin x}\, dx}{\cos^2 x + \sin^2 x} = -\int dx = -x$$

$$v = \int \frac{y_1 \,.\, X\, dx}{y_1 \,.\, y'_2 - y'_1\, y_2} = \int \frac{\cos x \,.\, \text{cosec}\, x\, dx}{\cos x\,(\cos x) - (-\sin x)\,(\sin x)}$$

$$= \int \frac{\cos x \,.\, \frac{1}{\sin x}}{\cos^2 x + \sin^2 x}\, dx = \int \frac{\cot x\, dx}{1} = \log \sin x$$

$$P.I. = uy_1 + vy_2 = -x \cos x + \sin x\, (\log \sin x)$$

General solution = C.F. + P.I.

$$y = A \cos x + B \sin x - x \cos x + \sin x\, .(\log \sin x)$$

Q35: The equations of motions of a particle are given by

$$\frac{dx}{dt} + \omega y = 0 \qquad \Rightarrow \qquad \frac{dy}{dt} - \omega x = 0$$

Find the path of the particle and show that it is a circle.

Solution. On putting $\frac{d}{dt} \equiv D$ in the equations, we have

$$Dx + \omega y = 0 \qquad ...(1)$$

$$-\omega x + Dy = 0 \qquad ...(2)$$

On multiplying (1) by w and (2) by D, we get

$$\omega Dx + \omega^2 y = 0 \qquad ...(3)$$

$$-\omega Dx + D^2 y = 0 \qquad ...(4)$$

On adding (3) and (4), we obtain

$$\omega^2 y + D^2 y = 0 \qquad \Rightarrow \qquad (D^2 + \omega^2)\, y = 0 \qquad ...(5)$$

Now, we have to solve (5) to get the value of y.

A.E. is $m^2 + \omega^2 = 0 \quad \Rightarrow \quad m^2 = -\omega^2 \quad \Rightarrow \quad m = \pm i\omega$

$$y = A \cos wt + B \sin wt \qquad ...(6)$$

$\Rightarrow Dy = -A\,\omega \sin \omega t + B\omega \cos wt$

On putting the value of Dy in (2), we get $-\omega x - A\omega \sin \omega t + B\omega \cos \omega t = 0$

$$\Rightarrow \qquad \omega x = -A\omega \sin \omega t + B\omega \cos \omega t$$

$$\Rightarrow \qquad x = -A \sin \omega t + B \cos \omega t \qquad ...(7)$$

On squaring (6) and (7) and adding , we get

$$x^2 + y^2 = A^2(\cos^2\omega t + \sin^2\omega t) + B^2(\cos^2\omega t + \sin^2\omega t)$$

$$\Rightarrow \qquad x^2 + y^2 = A^2 + B^2$$

This is the equation of circle. **Proved.**