Free Study Material from

All Lab Experiments

Mathematical Physics - III
Chapter - 1
Complex Analysis

Support us by Donating
at the link "DONATIONS" given on the Main Menu

Even the smallest contribution of you
will Help us keep Running

Chapter 1
Complex Analysis

Complex Analysis: Brief Revision of Complex Numbers and their Graphical Representation. Euler's formula, De Moivre's theorem, Roots of Complex Numbers. Functions of Complex Variables. Analyticity and Cauchy-Riemann Conditions. Examples of analytic functions. Singular functions: poles and branch points, order of singularity, branch cuts. Integration of a function of a complex variable. Cauchy's Inequality. Cauchy's Integral formula. Simply and multiply connected region. Laurent and Taylor's expansion. Residues and Residue Theorem. Application in solving Definite Integrals. (30 Lectures)

Q1:

Find real numbers x and y such that $3x + 2iy - ix + 5y = 7 + 5i$.

Sol: The given equation can be written as

$$3x + 5y + i(2y - x) = 7 + 5i.$$

Then equating real and imaginary parts,

$$3x + 5y = 7,\ 2y - x = 5.$$

Solving simultaneously,

$$x = -1,\ y = 2.$$

Q2: Prove:

$$\text{(a) } \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \text{ (b) } |z_1 z_2| = |z_1||z_2|.$$

Sol:

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then

(a) $\overline{z_1 + z_2} = \overline{x_1 + iy_1 + x_2 + iy_2} = \overline{x_1 + x_2 + i(y_1 + y_2)}$

$= x_1 + x_2 - i(y_1 + y_2) = x_1 - iy_1 + x_2 - iy_2 = \overline{x_1 + iy_1} + \overline{x_2 + iy_2} = \bar{z}_1 + \bar{z}_2$

(b) $|z_1 z_2| = |(x_1 + iy_1)(x_2 + iy_2)| = |x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2)|$

$= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} = |z_1||z_2|$

Q3: Prove:

(a) $|z_1 + z_2| \le |z_1| + |z_2|$,

(b) $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$,

(c) $|z_1 - z_2| \ge |z_1| - |z_2|$

and give a graphical interpretation.

Sol (a):

Analytically. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then we must show that

$$\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \le \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$$

Squaring both sides, this will be true if

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 \le x_1^2 + y_1^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} + x_2^2 + y_2^2$$

i.e., if
$$x_1x_2 + y_1y_2 \le \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

or if (squaring both sides again)

$$x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 \le x_1^2x_2^2 + x_1^2y_2^2 + y_1^2x_2^2 + y_1^2y_2^2$$

or
$$2x_1x_2y_1y_2 \le x_1^2y_2^2 + y_1^2x_2^2$$

But this is equivalent to $(x_1y_2 - x_2y_1)^2 \ge 0$, which is true. Reversing the steps, which are reversible, proves the result.

Graphically. The result follows graphically from the fact that $|z_1|$, $|z_2|$, $|z_1 + z_2|$ represent the lengths of the sides of a triangle (see Fig. 1-14) and that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side.

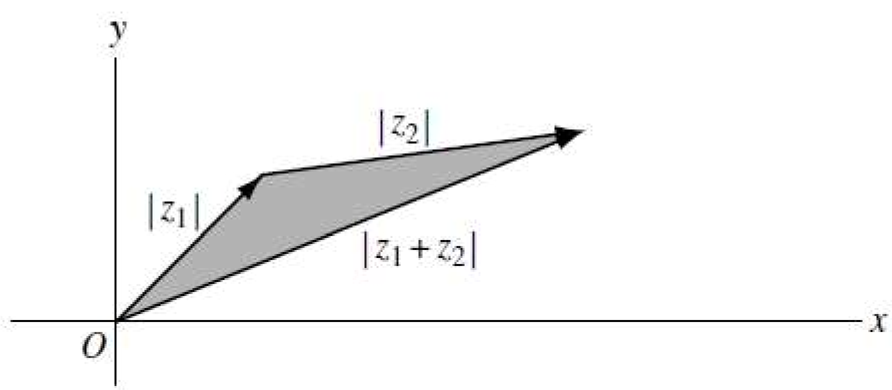


Fig. 1-14

(b) *Analytically*. By part (a),

$$|z_1 + z_2 + z_3| = |z_1 + (z_2 + z_3)| \le |z_1| + |z_2 + z_3| \le |z_1| + |z_2| + |z_3|$$

Graphically. The result is a consequence of the geometric fact that, in a plane, a straight line is the shortest distance between two points O and P (see Fig. 1-15).

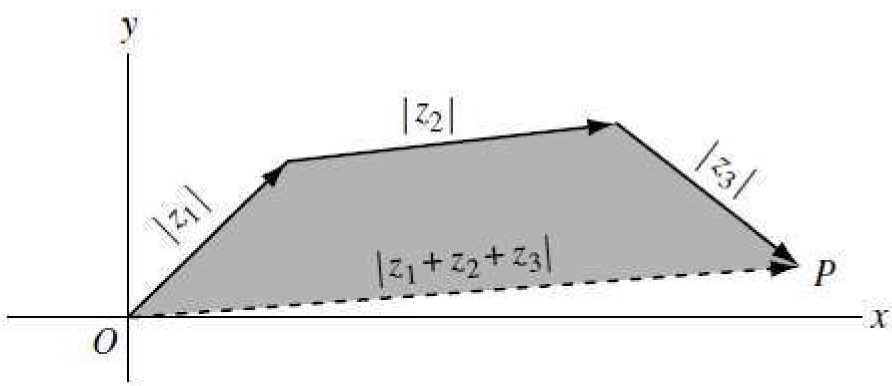


Fig. 1-15

(c)

Analytically. By part (a), $|z_1| = |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2|$. Then $|z_1 - z_2| \ge |z_1| - |z_2|$. An equivalent result obtained on replacing z_2 by $-z_2$ is $|z_1 + z_2| \ge |z_1| - |z_2|$.

Graphically. The result is equivalent to the statement that a side of a triangle has length greater than or equal to the difference in lengths of the other two sides.

Q4: Simplify the following:

(a) i^{49}, (b) i^{103}.

Sol: (a)

We divide 49 by 4 and we get

$$49 = 4 \times 12 + 1$$
$$i^{49} = i^{4 \times 12 + 1} = (i^4)^{12}\,(i^1) = (1)^{12}\,(i) = i$$

(b) we divide 103 by 4, we get

$$103 = 4 \times 25 + 3$$
$$i^{103} = i^{4 \times 25 + 3} = (i^4)^{25}\,(i^3) = (1)^{25}\,(-i) = -i$$

Q5: Multiply:

$3 + 4i$ by $7 - 3i$.

Sol:

Let $z_1 = 3 + 4i$ and $z_2 = 7 - 3i$

$$z_1.z_2 = (3 + 4i).(7 - 3i)$$
$$= 21 - 9i + 28i - 12i^2$$
$$= 21 - 9i + 28i - 12(-1) \qquad [\because i^2 = -1]$$
$$= 21 - 9i + 28i + 12$$
$$= 33 + 19i$$

Ans.

Q6: Divide:

$1 + i$ by $3 + 4i$.

Sol:

$$\frac{1+i}{3+4i} = \frac{1+i}{3+4i} \times \frac{3-4i}{3-4i}$$

$$= \frac{3 - 4i + 3i - 4i^2}{9 - 16i^2}$$

$$= \frac{3 - i + 4}{9 + 16} = \frac{7}{25} - \frac{1}{25} i$$

Q7:

If $a^2 + b^2 + c^2 = 1$ and $b + ic = (1 + a)z$,

prove that $\dfrac{a+ib}{1+c} = \dfrac{1+iz}{1-iz}$.

Sol:Here, we have

$$b + ic = (1 + a)z \quad \Rightarrow \quad z = \frac{b+ic}{1+a}$$

$$\frac{1+iz}{1-iz} = \frac{1 + i\dfrac{b+ic}{1+a}}{1 - i\dfrac{b+ic}{1+a}} = \frac{1 + a + ib - c}{1 + a - ib + c}$$

$$= \frac{[(1 + a + ib) - c]}{(1 + a + c - ib)} \times \frac{(1 + a + ib + c)}{(1 + a + c + ib)} = \frac{(1 + a + ib)^2 - c^2}{(1 + a + c)^2 + b^2}$$

$$= \frac{1 + a^2 - b^2 + 2a + 2ib + 2iab - c^2}{1 + a^2 + c^2 + 2a + 2c + 2ac + b^2} = \frac{1 + a^2 - b^2 - c^2 + 2a + 2ib + 2iab}{1 + (a^2 + b^2 + c^2) + 2a + 2c + 2ac}$$

Putting the value of $a^2 + b^2 + c^2 = 1$ in the above, we get

$$= \frac{1 + a^2 - (1 - a^2) + 2a + 2ib + 2iab}{1 + 1 + 2a + 2c + 2ac} = \frac{2(a^2 + a + ib + iab)}{2(1 + a + c + ac)} = \frac{2(1 + a)(a + ib)}{2(1 + a)(1 + c)} = \frac{a + ib}{1 + c}$$

Proved.

Q8:

If $z = \cos\theta + i \sin\theta$, prove that

$$\frac{2}{1+z} = 1 - i \tan\frac{\theta}{2}$$

Sol: Here we have:

$$z = \cos\theta + i\sin\theta$$

$$\frac{2}{1+z} = \frac{2}{1+(\cos\theta + i\sin\theta)} = \frac{2}{(1+\cos\theta)+i\sin\theta} \times \frac{(1+\cos\theta)-i\sin\theta}{(1+\cos\theta)-i\sin\theta}$$

$$= \frac{2[(1+\cos\theta)-i\sin\theta]}{(1+\cos\theta)^2+\sin^2\theta}$$

$$= \frac{2[(1+\cos\theta)-i\sin\theta]}{2(1+\cos\theta)} = 1 - \frac{i\sin\theta}{1+\cos\theta}$$

$$= 1 - i\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)} = 1 - i\tan\left(\frac{\theta}{2}\right)$$

$$(1+\cos\theta)^2+\sin^2\theta$$
$$= 1+\cos^2\theta+2\cos\theta+\sin^2\theta$$
$$= 1+(\sin^2\theta+\cos^2\theta)+2\cos\theta$$
$$= 1+1+2\cos\theta$$
$$= 2+2\cos\theta$$
$$= 2(1+\cos\theta)$$

Proved.

Q9: Find the modulus and principal argument of the complex number

$$\frac{1+2i}{1-(1-i)^2}.$$

Sol:

$$\frac{1+2i}{1-(1-i)^2} = \frac{1+2i}{1-(1-1-2i)} = \frac{1+2i}{1+2i} = 1 = 1+0i$$

$$\left|\frac{1+2i}{1-(1-i)^2}\right| = |1+0i| = \sqrt{1^2} = 1$$

$$\text{Principal argument of } \frac{1+2i}{1-(1-i)^2} = \text{Principal argument of } 1+0i$$

$$= \tan^{-1}\frac{0}{1} = \tan^{-1}0 = 0°.$$

Hence modulus = 1 and principal argument = 0°.

Q10: Find the smallest positive integer n for which

$$\left(\frac{1+i}{1-i}\right)^n = 1.$$

Sol:

$$\left[\frac{1+i}{1-i}\right]^n = 1$$

$$\left[\frac{1+i}{1-i}\times\frac{1+i}{1+i}\right]^n = 1 \quad\Rightarrow\quad \left(\frac{1-1+2i}{1+1}\right)^n = 1$$

$$(i)^n = 1 = (i)^4 \quad\Rightarrow\quad n = 4$$

Q11: Find the square root of the complex number 5 + 12i.

Sol:

Let $\sqrt{5+12i} = x+iy$ ...(1)

Squaring both sides of (1), we get $5+12i = (x+iy)^2 = (x^2-y^2)+i\,2xy$ ...(2)

Equating real and imaginary parts of (2), we get

$$x^2 - y^2 = 5 \qquad ...(3)$$

and

$$2xy = 12 \qquad ...(4)$$

Now,

$$x^2+y^2 = \sqrt{(x^2-y^2)^2+4x^2y^2} = \sqrt{(5)^2+(12)^2}$$

$$= \sqrt{25+144} = \sqrt{169} = 13$$

$$\Rightarrow \quad x^2+y^2 = 13 \qquad ...(5)$$

Adding (3) and (5), we get $2x^2 = 5+13 = 18 \Rightarrow x = \sqrt{\frac{18}{2}} = \sqrt{9} = \pm 3$

Subtracting (3) from (5), we get $2y^2 = 13-5 = 8 \Rightarrow y = \sqrt{\frac{8}{2}} = \sqrt{4} = \pm 2$

Since, xy is positive, so x and y are of same sign. Hence, $x = \pm 3,\ y = \pm 2$

$\therefore \quad \sqrt{5+12i} = \pm 3 \pm 2i$ *i.e.* $(3+2i)$ or $-(3+2i)$ **Ans.**

Q12:Prove that:

$$\cos\theta + i\sin\theta = e^{i\theta}$$

Sol:

$$e^z = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\ldots \qquad ...(1)$$

$$\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\ldots \qquad ...(2)$$

$$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\ldots \qquad ...(3)$$

From (2) and (3), we have

$$\cos z + i\sin z = \left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\ldots\right)+i\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots\right)$$

$$= 1+\frac{(iz)^1}{1!}+\frac{(iz)^2}{2!}+\frac{(iz)^3}{3!}+\ldots = e^{iz}$$

Therefore, $\cos z + i \sin z = e^{iz}$ ...(4)

Similarly, $\cos z - i \sin z = e^{-iz}$ ...(5)

From (4) and (5), we have

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad ...(6)$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad ...(7)$$

Q13: Prove De Moivre's theorem: $(\cos\theta + i \sin\theta)^n = (\cos n\theta + i \sin n\theta)$ where n is any positive integer.

Sol: We use the principle of mathematical induction. Assume that the result is true for the particular positive integer k, i.e., assume

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta.$$

Then, multiplying both sides by $\cos\theta + i \sin\theta$, we find

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) = \cos(k+1)\theta + i\sin(k+1)\theta$$

Thus, if the result is true for n= k, then it is also true for n = k + 1. But, since the result is clearly true for n = 1, it must also be true for n = 1 + 1 = 2 and n =2 + 1 = 3, etc., and so must be true for all positive integers.

The result is equivalent to the statement $(e^{i\theta})^n = e^{ni\theta}$.

Q:14

Express $\dfrac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4}$ *in the form* $(x + iy)$.

Sol:

$$\frac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4} = \frac{(\cos\theta + i\sin\theta)^8}{(i)^4\left(\cos\theta + \frac{1}{i}\sin\theta\right)^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta - i\sin\theta)^4} = \frac{(\cos\theta + i\sin\theta)^8}{[\cos(-\theta) + i\sin(-\theta)]^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^8}{[(\cos\theta + i\sin\theta)^{-1}]^4} = \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^{-4}} = (\cos\theta + i\sin\theta)^{12}$$

$$= \cos 12\theta + i\sin 12\theta$$

Q:15 Solve $x^4 + i = 0$.

Sol: Here we have,

$$x^4 = -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$$

$$x^4 = \cos\left(2n\pi + \frac{\pi}{2}\right) - i\sin\left(2n\pi + \frac{\pi}{2}\right)$$

$$x = \left[\cos\left(2n\pi + \frac{\pi}{2}\right) - i\sin\left(2n\pi + \frac{\pi}{2}\right)\right]^{\frac{1}{4}}$$

$$= \cos(4n+1)\frac{\pi}{8} - i\sin(4n+1)\frac{\pi}{8}$$

Putting $n = 0, 1, 2, 3$ we get the roots as

$$x_1 = \cos\frac{\pi}{8} - i\sin\frac{\pi}{8}, \quad x_2 = \cos\frac{5\pi}{8} - i\sin\frac{5\pi}{8}$$

$$x_3 = \cos\frac{9\pi}{8} - i\sin\frac{9\pi}{8}, \quad x_4 = \cos\frac{13\pi}{8} - i\sin\frac{13\pi}{8}$$

Q: 16 Use De Moivre's theorem to solve the equation $x^4 - x^3 + x^2 - x + 1 = 0$.

Sol: $x^4 - x^3 + x^2 - x + 1 = 0$

$(x + 1)(x^4 - x^3 + x^2 - x + 1) = 0$

$x^5 + 1 = 0$

$x^5 = -1 = (\cos\pi + i\sin\pi) = \cos(2n\pi + \pi) + i\sin(2n\pi + \pi)$

$x = [\cos(2n+1)\pi + i\sin(2n+1)\pi]^{1/5}$

$$= \cos\frac{(2n+1)\pi}{5} + i\sin\frac{(2n+1)\pi}{5}$$

When $n = 0, 1, 2, 3, 4$, the values are

$$\cos\frac{\pi}{5} + i\sin\frac{\pi}{5},\ \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5},\ \cos\pi + i\sin\pi,\ \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5},\ \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}.$$

$\cos\pi + i\sin\pi = -1$, which is rejected as it is corresponding to $x + 1 = 0$.

Hence, the required roots are

$$\cos\frac{\pi}{5} + i\sin\frac{\pi}{5},\ \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5},\ \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5},\ \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}.$$

Q17: Prove that

$(\cosh x - \sinh x)^n = \cosh nx - \sinh nx.$

Sol:

$$\text{L.H.S.} = (\cosh x - \sinh x)^n$$
$$= \left[\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right]^n = \left[\frac{2e^{-x}}{2}\right]^n = (e^{-x})^n = e^{-nx} \quad \ldots(1)$$
$$\text{R.H.S.} = \cosh nx - \sinh nx$$
$$= \left(\frac{e^{nx} + e^{-nx}}{2} - \frac{e^{nx} + e^{-nx}}{2}\right) = \frac{2e^{-nx}}{2} = e^{-nx} \quad \ldots(2)$$

From (1) and (2), we have

LHS = RHS

Q18: Prove that the general value of ϴ which satisfies the equation:

$(\cos\theta + i\sin\theta).(\cos 2\theta + i\sin 2\theta)\ldots(\cos n\theta + i\sin n\theta) = 1$ is $\frac{4m\pi}{n(n+1)}$, *where m is any integer.*

Sol:

$$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\ldots(\cos n\theta + i\sin n\theta) = 1$$
$$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^2\ldots(\cos\theta + i\sin\theta)^n = 1$$
$$(\cos\theta + i\sin\theta)^{1+2\ldots+n} = 1$$
$$(\cos\theta + i\sin\theta)^{\frac{n(n+1)}{2}} = (\cos 2m\pi + i\sin 2m\pi)$$
$$\cos\frac{n(n+1)}{2}\theta + i\sin\frac{n(n+1)}{2}\theta = \cos 2m\pi + i\sin 2m\pi$$
$$\frac{n(n+1)}{2}\theta = 2m\pi \Rightarrow \theta = \frac{4m\pi}{n(n+1)}$$

Q19: Separate the following into real and imaginary parts:

(i) $\sin(x+iy)$

(ii) $\cos(x+iy)$

(iii) $\tan(x+iy)$

Solution. (*i*) $\sin(x+iy) = \sin x \cos iy + \cos x \sin(iy) = \sin x \cosh y + i \cos x \sinh y.$

(*ii*) $\cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy) = \cos x \cosh y - i \sin x \sinh y.$

(*iii*) $$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$

$$= \frac{\sin 2x + \sin(2iy)}{\cos 2x + \cos 2iy} = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

$$\left\{\begin{array}{l}\because \quad 2\sin A.\cos B = \sin(A+B) + \sin(A-B) \\ \text{and } 2\cos A.\cos B = \cos(A+B) + \cos(A-B)\end{array}\right\}$$

Q20:If tan (A + iB) = x + iy, prove that

$$\tan 2A = \frac{2x}{1-x^2-y^2} \quad \text{and} \quad \tanh 2B = \frac{2x}{1+x^2+y^2}$$

Sol:

$\tan(A+iB) = x+iy$; $\tan(A-iB) = x-iy$

$\tan 2A = \tan(A+iB+A-iB)$

$$= \frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$\tan 2A = \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)} = \frac{2x}{1-\left(x^2+y^2\right)} = \frac{2x}{1-x^2-y^2}$$

$$\tan 2iB = \tan(A+iB-A+iB) = \frac{\tan(A+iB)-\tan(A-iB)}{1+\tan(A+iB)\tan(A-iB)}$$

$$\tan 2iB = \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = \frac{(2y)i}{1+x^2+y^2}$$

$$\tanh 2B = \frac{2y}{1+x^2+y^2} \qquad \tan ix = i\tanh x$$

Q21: If sin (α + iβ) = x + iy, prove that

$$(a)\ \frac{x^2}{\cosh^2\beta}+\frac{y^2}{\sinh^2\beta}=1 \qquad (b)\ \frac{x^2}{\sin^2\alpha}-\frac{y^2}{\cos^2\alpha}=1$$

Sol:

(*a*) $x+iy = \sin(\alpha+i\beta) = \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta$

Equating real and imaginary parts, we get

$$x = \sin\alpha\cosh\beta,\ y = \cos\alpha\sinh\beta$$

$$\sin\alpha = \frac{x}{\cosh\beta} \text{ and } \cos\alpha = \frac{y}{\sinh\beta}$$

Squaring and adding, $\sin^2\alpha + \cos^2\alpha = \frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta}$

$$\Rightarrow \quad 1 = \frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta}$$

(*b*) Again $\cosh\beta = \frac{x}{\sin\alpha}$ and $\sinh\beta = \frac{y}{\cos\alpha}$

$$\cosh^2\beta - \sinh^2\beta = \frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\alpha}$$

$$1 = \frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\alpha}$$

Q22: Separate log (x + iy) into its real and imaginary parts.

Sol: let

$$x = r\cos\theta \qquad \ldots(1)$$

$$y = r\sin\theta \qquad \ldots(2)$$

Squaring and adding (1) and (2) we have $x^2 + y^2 = r^2$

$$\therefore \quad r = \sqrt{x^2+y^2},$$

We have, $\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$ [Dividing (2) by (1)]

$$\therefore \quad \log(x+iy) = \log[r(\cos\theta + i\sin\theta)]$$

$$= [\log r + \log(\cos\theta + i\sin\theta)]$$

$$\log(x+iy) = \log r + \log[\cos(2n\pi+\theta) + i\sin(2n\pi+\theta)]$$

$$= \log r + \log e^{i(2n\pi+\theta)} = \log r + i(2n\pi+\theta)$$

$$\text{Log}(x+iy) = \log\sqrt{x^2+y^2} + i\left(2n\pi + \tan^{-1}\frac{y}{x}\right)$$

$$\log(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$$

Q23:

Show that $\log \frac{x+iy}{x-iy} = 2\, i \tan^{-1} \frac{y}{x}$.

Sol:

Let $\log (x+iy) = \log (r\cos\theta + ir\sin\theta) = \log r\, e^{i\theta}$

$$= \log r + i\,\theta \qquad \begin{bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{bmatrix}$$

$$\log (x - iy) = \log r - i\,\theta$$

$$\log \frac{x+iy}{x-iy} = \log (x+iy) - \log (x-iy) = (\log r + i\theta) - (\log r - i\theta) = 2i\,\theta$$

$$= 2\, i \tan^{-1} \frac{y}{x}.$$

Proved.

Q24:

Consider the function

$$f(z) = 4x + y + i(-x + 4y)$$

and discuss $\frac{df}{dz}$

Sol:

Here, $f(z) = 4x + y + i(-x+4y) = u + iv$

$u = 4x + y$ and $v = -x + 4y$

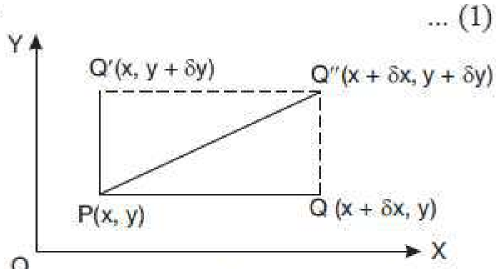
$$f(z+\delta z) = 4(x+\delta x) + (y+\delta y) - i(x+\delta x) + 4i(y+\delta y)$$

$$f(z+\delta z) - f(z) = 4(x+\delta x) + (y+\delta y) - i(x+\delta x) + 4i(y+\delta y) - 4x - y + ix - 4iy$$

$$= 4\delta x + \delta y - i\,\delta x + 4i\delta y$$

$$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{4\delta x + \delta y - i\,\delta x + 4\,i\delta y}{\delta x + i\delta y}$$

$$\Rightarrow \quad \frac{\delta f}{\delta z} = \frac{4\delta x + \delta y - i\delta x + 4i\delta y}{\delta x + i\delta y} \qquad \ldots (1)$$



(*a*) **Along real axis:** If Q is taken on the horizontal line through P (x, y) and Q then approaches P along this line, we shall have $\delta y = 0$ and $\delta z = \delta x$.

$$\frac{\delta f}{\delta z} = \frac{4\delta x - i\delta x}{\delta x} = 4 - i$$

(*b*) **Along imaginary axis:** If Q is taken on the vertical line through P and then Q approaches P along this line, we have

$$z = x + iy = 0 + iy,\ \delta z = i\delta y,\ \delta x = 0.$$

Putting these values in (1), we have

$$\frac{\delta f}{\delta z} = \frac{\delta y + 4i\delta y}{i\delta y} = \frac{1}{i}(1+4i) = 4 - i$$

(*c*) **Along a line** $y = x$: If Q is taken on a line $y = x$.

$$z = x + iy = x + ix = (1 + i)x$$

$$\delta z = (1 + i)\delta x \quad \text{and} \quad \delta y = \delta x$$

On putting these values in (1), we have

$$\frac{\delta f}{\delta z} = \frac{4\delta x + \delta x - i\delta x + 4i\delta x}{\delta x + i\delta x} = \frac{4 + 1 - i + 4i}{1 + i} = \frac{5 + 3i}{1 + i} = \frac{(5 + 3i)(1 - i)}{(1 + i)(1 - i)} = 4 - i$$

In all the three different paths approaching Q from P, we get the same values of $\frac{\delta f}{\delta z} = 4 - i$.

In such a case, the function is said to be differentiable at the point z in the given region.

Q25:

Determine whether $\frac{1}{z}$ is analytic or not?

Sol:

Let $w = f(z) = u + iv = \frac{1}{z} \quad \Rightarrow \quad u + iv = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$

Equating real and imaginary parts, we get

$$u = \frac{x}{x^2 + y^2}, \qquad v = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2 + y^2).1 - x.\,2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \qquad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2}.$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}, \qquad \frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

Thus, $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Thus C – R equations are satisfied. Also partial derivatives are continuous except at (0, 0).

Therefore $\frac{1}{z}$ is analytic everywhere except at $z = 0$.

Also $$\frac{dw}{dz} = -\frac{1}{z^2}$$

This again shows that $\frac{dw}{dz}$ exists everywhere except at $z = 0$. Hence $\frac{1}{z}$ is analytic everywhere except at $z = 0$. **Ans.**

Q26:

Show that the function $e^x (\cos y + i \sin y)$ is an analytic function, find its derivative.

Solution. Let $e^x(\cos y + i\sin y) = u + iv$

So, $e^x \cos y = u$ and $e^x \sin y = v$ then $\frac{\partial u}{\partial x} = e^x \cos y$, $\frac{\partial v}{\partial y} = e^x \cos y$

$$\frac{\partial u}{\partial y} = -e^x \sin y, \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

Here we see that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

These are $C - R$ equations and are satisfied and the partial derivatives are continuous. Hence, $e^x(\cos y + i\sin y)$ is analytic.

$$f(z) = u + iv = e^x(\cos y + y\sin y) \text{ and } \frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial v}{\partial x} = e^x \sin y$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = e^x(\cos y + i\sin y) = e^x.e^{iy} = e^{x+iy} = e^z.$$

Which is the required derivative.

Q27: Show that the real and imaginary parts of the function w = log z satisfy the Cauchy-Riemann equations when z is not zero. Find its derivative.

Solution. To separate the real and imaginary parts of $\log z$, we put $x = r\cos\theta;\ y = r\sin\theta$

$$w = \log z = \log(x + iy)$$

$$\Rightarrow \quad u + iv = \log(r\cos\theta + ir\sin\theta) = \log r(\cos\theta + i\sin\theta) = \log_e r.e^{i\theta}$$

$$= \log_e r + \log_e e^{i\theta} = \log r + i\theta = \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x} \qquad \left[\begin{array}{l} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x} \end{array}\right]$$

So $$u = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2), \quad v = \tan^{-1}\frac{y}{x}$$

On differentiating u, v, we get

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}.(2x) = \frac{x}{x^2+y^2} \qquad \ldots (1)$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} \qquad \ldots (2)$$

From (1) and (2), $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ... (A)

Again differentiating u, v, we have

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2} \qquad \ldots (3)$$

$\partial y \quad 2\; x^2+y^2 \qquad\qquad x^2+y^2$

$$\frac{\partial v}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2} \qquad \text{... (4)}$$

From (3) and (4), we have

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \qquad \text{... (B)}$$

Equations (A) and (B) are $C-R$ equations and partial derivatives are continuous.
Hence, $w = \log z$ is an analytic function except
when $\quad x^2 + y^2 = 0 \Rightarrow x = y = 0 \Rightarrow x + iy = 0 \Rightarrow z = 0$
Now $\quad w = u + iv$

$$\frac{dw}{dz}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}=\frac{x-iy}{x^2+y^2}$$

$$=\frac{x-iy}{(x+iy)(x-iy)}=\frac{1}{x+iy}=\frac{1}{z}$$

Which is the required derivative.

Q28: Discuss the analyticity of the function

$$f(z)=z\bar{z}.$$

Sol:

$$f(z)=z\bar{z}=(x+iy)(x-iy)=x^2-i^2y^2=x^2+y^2$$

$$f(z)=x^2+y^2=u+iv.$$

$$u=x^2+y^2,\; v=0$$

At origin, $\displaystyle \frac{\partial u}{\partial x}=\lim_{h\to 0}\frac{u(0+h,\;0)-u(0,0)}{h}=\lim_{h\to 0}\frac{h^2}{h}=0$

$$\frac{\partial u}{\partial y}=\lim_{k\to 0}\frac{u(0,0+k)-u(0,0)}{k}=\lim_{k\to 0}\frac{k^2}{k}=0$$

Also, $\displaystyle \frac{\partial v}{\partial x}=\lim_{h\to 0}\frac{v(0+h,0)-v(0,0)}{h}=0$

$$\frac{\partial v}{\partial y}=\lim_{k\to 0}\frac{v(0,0+k)-v(0,0)}{k}=0$$

Thus, $\displaystyle \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\displaystyle \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

Hence, C – R equations are satisfied at the origin.

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{(x^2 + y^2) - 0}{x + iy}$$

Let $z \to 0$ along the line $y = mx$, then

$$f'(0) = \lim_{x \to 0} \frac{(x^2 + m^2 x^2)}{(x + imx)} = \lim_{x \to 0} \frac{(1 + m^2) x}{1 + im} = 0$$

Therefore, $f'(0)$ is unique. Hence the function $f(z)$ is analytic at $z = 0$.

Q29: Examine the nature of the function:

$$f(z) = \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}}; z \neq 0$$

$$f(0) = 0$$

in the region including the origin.

Sol: here

$$f(z) = u + iv = \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}}; z \neq 0$$

Equating real and imaginary parts, we get

$$u = \frac{x^3 y^5}{x^4 + y^{10}}, \quad v = \frac{x^2 y^6}{x^4 + y^{10}}$$

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0 + h, 0) - u(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{0}{h^4}}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0, 0 + k) - u(0, 0)}{k} = \lim_{k \to 0} \frac{\frac{0}{k^{10}}}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0 + h, 0) - v(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{0}{h^4}}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0, 0 + k) - v(0, 0)}{k} = \lim_{k \to 0} \frac{\frac{0}{k^{10}}}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

From the above results, it is clear that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, C-R equations are satisfied at the origin.

But $$f'(0) = \lim_{z \to 0} \frac{f(0+z) - f(0)}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \left[\frac{x^2 y^5 (x+iy)}{x^4 + y^{10}} - 0 \right] \cdot \frac{1}{x+iy} \quad \text{(Increment} = z)$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y^5}{x^4 + y^{10}}$$

Let $z \to 0$ along the radius vector $y = mx$, then

$$f'(0) = \lim_{x \to 0} \frac{m^5 x^7}{x^4 + m^{10} x^{10}} = \lim_{x \to 0} \frac{m^5 x^3}{1 + m^{10} x^6} = \frac{0}{1} = 0 \quad \dots (1)$$

Again let $z \to 0$ along the curve $y^5 = x^2$

$$f'(0) = \lim_{x \to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2} \quad \dots (2)$$

(1) and (2) shows that $f'(0)$ does not exist. Hence, $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there. **Ans.**

Q30: Derive C-R equations in polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$$

Sol:

We know that $x = r\cos\theta$, and u is a function of x and y.

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$u + iv = f(z) = f(re^{i\theta}) \quad \dots (1)$$

Differentiating (1) partially w.r.t., "r", we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}).e^{i\theta} \quad \dots (2)$$

Differentiating (1) w.r.t. "θ", we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, r e^{i\theta} i \quad \dots (3)$$

Substituting the value of $f'(re^{i\theta})e^{i\theta}$ from (2) in (3), we obtain

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)i \quad \text{or} \quad \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

Equating real and imaginary parts, we get

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}} \quad \Rightarrow \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$$

Q31:prove that if f (z) = u + iv is an analytic function, then u and v are both harmonic functions.

Sol:

Let $f(z) = u + iv$, be an analytic function, then we have

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \quad \ldots(1)\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \quad \ldots(2)\end{aligned}\right\} C-R \text{ equations.}$$

Differentiating (1) with respect to x, we get $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y}$ ... (3)

Differentiating (2) w.r.t. 'y' we have $\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial y \partial x}$ ... (4)

Adding (3) and (4) we have $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 v}{\partial x \partial y} - \dfrac{\partial^2 v}{\partial y \partial x}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \left(\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}\right)$$

Similarly $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

Therefore both u and v are harmonic functions.

Such functions u, v are called **Conjugate harmonic functions if $u + iv$ is also analytic function.**

Q32:

Prove that $u = x^2 - y^2$ *and* $v = \dfrac{y}{x^2 + y^2}$ *are harmonic functions of* (x, y), *but are not harmonic conjugates.*

Solution. We have, $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$u(x, y)$ satisfies Laplace equation, hence $u(x, y)$ is harmonic

$$v = \frac{y}{x^2 + y^2}, \qquad \frac{\partial v}{\partial x} = -\frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2 + y^2)^2(-2y) - (-2xy)2(x^2 + y^2)2x}{(x^2 + y^2)^4}$$

$$= \frac{(x^2 + y^2)(-2y) - (-2xy)4x}{(x^2 + y^2)^3} = \frac{6x^2y - 2y^3}{(x^2 + y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2 + y^2)\cdot 1 - y(2y)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \qquad \ldots (1)$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2 + y^2)^2(-2y) - (x^2 - y^2)2(x^2 + y^2)(2y)}{(x^2 + y^2)^4} = \frac{(x^2 + y^2)(-2y) - (x^2 - y^2)(4y)}{(x^2 + y^2)^3}$$

$$= \frac{-2x^2y - 2y^3 - 4x^2y + 4y^3}{(x^2 + y^2)^3} = \frac{-6x^2y + 2y^3}{(x^2 + y^2)^3} \qquad \ldots (2)$$

On adding (1) and (2), we get $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

$v(x, y)$ also satisfies Laplace equations, hence $v(x, y)$ is also harmonic function.

But $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} \neq -\dfrac{\partial v}{\partial x}$

Therefore u and v are not harmonic conjugates. **Proved.**

Q33:

[redacted]. *Prove that* $u = x^2 - y^2 - 2xy - 2x + 3y$ *is harmonic. Find a function* v *such that* $f(z) = u + iv$ *is analytic. Also express* $f(z)$ *in terms of* z.

Sol: we have,

$$u = x^2 - y^2 - 2xy - 2x + 3y$$

$$\frac{\partial u}{\partial x} = 2x - 2y - 2 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y - 2x + 3 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

Since Laplace equation is satisfied, therefore u is harmonic.

We know that,

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad \ldots(1) \quad \left[\because \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ and } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}\right]$$

Putting the values of $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x}$ in (1), we get

$$dv = -(-2y - 2x + 3)\,dx + (2x - 2y - 2)dy$$

$\Rightarrow$ $$v = \int (2y + 2x - 3)dx + \int (-2y - 2)dy + C \quad \text{(Ignoring } 2x\text{)}$$

Hence, $$v = 2xy + x^2 - 3x - y^2 - 2y + C$$ **Ans.**

Now,
$$\begin{aligned}
f(z) &= u + iv \\
&= (x^2 - y^2 - 2xy - 2x + 3y) + i\,(2xy + x^2 - 3x - y^2 - 2y) + iC \\
&= (x^2 - y^2 + 2ixy) + (ix^2 - iy^2 - 2xy) - (2 + 3i)\,x - i\,(2 + 3i)\,y + iC \\
&= (x^2 - y^2 + 2ixy) + i\,(x^2 - y^2 + 2ixy) - (2 + 3i)\,x - i\,(2 + 3i)y + iC \\
&= (x + iy)^2 + i\,(x + iy)^2 - (2 + 3i)\,(x + iy) + iC \\
&= z^2 + iz^2 - (2 + 3i)\,z + iC \\
&= (1 + i)\,z^2 - (2 + 3i)\,z + iC
\end{aligned}$$

Which is the required expression of $f(z)$ in terms of z. A

Q34:

If $w = \phi + i\psi$ represents the complex potential for an electric field and

$$\psi = x^2 - y^2 + \frac{x}{x^2 + y^2},$$

determine the function ϕ.

Sol:

Solution. $w = \phi + i\psi$ and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$

$$\frac{\partial \psi}{\partial x} = 2x + \frac{(x^2 + y^2)\cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial \psi}{\partial y} = -2y - \frac{x(2y)}{(x^2 + y^2)^2} = -2y - \frac{2xy}{(x^2 + y^2)^2}$$

We know that, $d\phi = \dfrac{\partial \phi}{\partial x}dx + \dfrac{\partial \phi}{\partial y}dy = \dfrac{\partial \psi}{\partial y}dx - \dfrac{\partial \psi}{\partial x}dy$

$$= \left(-2y - \frac{2xy}{(x^2 + y^2)^2}\right)dx - \left(2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}\right)dy$$

$$\phi = \int\left[-2y - \frac{2xy}{(x^2 + y^2)^2}\right]dx + c$$

This is an exact differential equation.

$$\phi = -2xy + \frac{y}{x^2 + y^2} + C$$

Which is the required function.

Q35:

If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and
$u - v = e^{-x}\,[(x - y)\sin y - (x + y)\cos y]$

Find f(z).

Sol: we know that

$$f(z) = u + iv$$
$$i\,f(z) = i\,u - v$$
$$F(z) = U + iV$$
$$U = u - v = e^{-x}\,[(x - y)\sin y - (x + y)\cos y]$$
$$\frac{\partial U}{\partial x} = -\,e^{-x}\,[(x - y)\sin y - (x + y)\cos y] + e^{-x}\,[\sin y - \cos y]$$
$$\frac{\partial U}{\partial y} = e^{-x}\,[(x - y)\cos y - \sin y - (x + y)(-\sin y) - \cos y]$$

We know that,

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy = -\frac{\partial U}{\partial y}dx + \frac{\partial U}{\partial x}dy \qquad \text{[C – R equations]}$$

$$= -e^{-x}[(x-y)\cos y - \sin y + (x+y)\sin y - \cos y]\,dx$$
$$- e^{-x}[(x-y)\sin y - (x+y)\cos y - \sin y + \cos y]\,dy$$

$$= -e^{-x}x\{(\cos y + \sin y)\,dx - e^{-x}(-y\cos y - \sin y + y\sin y - \cos y)\,dx$$
$$- e^{-x}[(x-y)\sin y - (x+y)\cos y - \sin y + \cos y]\,dy$$

$$V = (\cos y + \sin y)(x e^{-x} + e^{-x}) + e^{-x}(-y\cos y - \sin y + y\sin y - \cos y) + C$$

$$F(z) = U + iV$$

$$F(z) = e^{-x}[(x-y)\sin y - (x+y)\cos y] + i\,e^{-x}[x\cos y + \cos y + x\sin y + \sin y$$
$$- y\cos y - \sin y + y\sin y - \cos y] + iC$$

$$= e^{-x}[\{x\sin y - y\sin y - x\cos y - y\cos y\} + i\{x\cos y + x\sin y - y\cos y + y\sin y\}] + iC$$

$$= e^{-x}[(x+iy)\sin y - (x+iy)\cos y + (-y+ix)\sin y + (-y+ix)\cos y] + iC$$

$$= e^{-x}[(x+iy)\sin y - (x+iy)\cos y + i(x+iy)\sin y + i(x+iy)\cos y] + iC$$

$$= e^{-x}(x+iy)[\sin y - \cos y + i\sin y + i\cos y] + iC$$

$$= e^{-x}(x+iy)[(1+i)\sin y + i(1+i)\cos y] + iC$$

$$(1+i)f(z) = e^{-x}(x+iy)(1+i)(\sin y + i\cos y) + iC$$

$$f(z) = e^{-x}(x+iy)(\sin y + i\cos y) + \frac{iC}{1+i}$$

$$= i\,z\,e^{-x}(\cos y - i\sin y) + \frac{iC}{1+i}$$

$$= i\,z\,e^{-x}e^{-iy} = i\,z\,e^{-(x+iy)} = i\,z\,e^{-z} + \frac{iC}{1+i} \qquad \textbf{Ans.}$$

Let $\phi_1(x, y) = -e^{-x}[(x-y)\sin y - (x+y)\cos y] + e^{-x}[\sin y - \cos y]$

$$\phi_1(z, 0) = -e^{-z}[z\sin 0 - z\cos 0] + e^{-z}[\sin 0 + \cos 0]$$
$$= -e^{-z}[z-1]$$

Let $\phi_2(x, y) = e^{-x}[(x-y)\cos y - \sin y + (x+y)\sin y - \cos y]$

$$\phi_2(z, 0) = e^{-z}[(z)\cos 0 - \sin 0 + z\sin 0 - \cos 0]$$
$$= e^{-z}[z-1]$$

$$F(z) = U + iV$$

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = f_1(z, 0) - i f_2(z, 0)$$

$$= e^{-z}(z-1) - i\,e^{-z}(z-1) = (1-i)\,e^{-z}(z-1) = (1-i)\,e^{-z}(z-1)$$

$$F(z) = (1-i)\left[z\frac{e^{-z}}{-1} - \int\frac{e^{-z}}{-1}dz\right] + C = (1-i)[-z\,e^{-z} - e^{-z}] + C$$

$$(1+i)f(z) = (-1+i)(z+1)\,e^{-z} + C$$

$$f(z) = \frac{(-1+i)}{1+i}(z+1)\,e^{-z} + C = \frac{(-1+i)(1-i)}{(1+i)(1-i)}(z+1)\,e^{-z} + C$$

$$= i(z+1)\,e^{-z} + C$$

Q36:

Find analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ such that
$v(r,\theta) = r^2 \cos 2\theta - r\cos\theta + 2.$

Sol: we have

$$v = r^2 \cos 2\theta - r\cos\theta + 2 \quad \text{... (1)}$$

Differentiating (1), we get

$$\frac{\partial v}{\partial \theta} = -2r^2 \sin 2\theta + r\sin\theta \quad \text{... (2)}$$

$$\frac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta \quad \text{... (3)}$$

Using $C-R$ equations in polar coordinates, we get

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = -2r^2 \sin 2\theta + r\sin\theta \quad \text{[From (2)]}$$

$$\Rightarrow \quad \frac{\partial u}{\partial r} = -2r\sin 2\theta + \sin\theta \quad \text{... (4)}$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r\cos 2\theta - \cos\theta \quad \text{[From (3)]}$$

$$\Rightarrow \quad \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r\cos\theta \quad \text{... (5)}$$

By total differentiation formula

$$du = \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta = (-2r\sin 2\theta + \sin\theta)dr + (-2r^2\cos 2\theta + r\cos\theta)d\theta$$

$$= -[(2r\,dr)\sin 2\theta + r^2(2\cos 2\theta\, d\theta)] + [\sin\theta \cdot dr + r(\cos\theta\, d\theta)]$$

$$= -[(2r\,dr)\sin 2\theta - \sin\theta dr] + [-r^2 2\cos 2\theta\, d\theta + r\cos\theta\, d\theta]$$

$$= -d(r^2 \sin 2\theta) + d(r\sin\theta) \quad \text{(Exact differential equation)}$$

Integrating, we get

$$u = -r^2 \sin 2\theta + r\sin\theta + c$$

Hence, $\quad f(z) = u + iv$

$$= (-r^2 \sin 2\theta + r\sin\theta + c) + i(r^2\cos 2\theta - r\cos\theta + 2)$$

$$= ir^2(\cos 2\theta + i\sin 2\theta) - ir(\cos\theta + i\sin\theta) + 2i + c$$

$$= ir^2 e^{2i\theta} - ir\,e^{i\theta} + 2i + c = i(r^2 e^{2i\theta} - r\,e^{i\theta}) + 2i + c.$$

This is the required analytic function.

Q37:

If $u = x^2 - y^2$, find a corresponding analytic function.

Solution. $\frac{\partial u}{\partial x} = 2x = \phi_1(x, y), \quad \frac{\partial u}{\partial y} = -2y = \phi_2(x, y)$

On replacing x by z and y by 0, we have

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + C$$

$$= \int [2z - i(0)]dz + c = \int 2z\, dz + c = z^2 + C$$

This is the required analytic function.

Q38:

Show that e^x (x cos y − y sin y) is a harmonic function. Find the analytic function for which e^x (x cos y − y sin y) is imaginary part.

Solution. Here $v = e^x (x \cos y - y \sin y)$

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial v}{\partial x} = e^x (x\cos y - y\sin y) + e^x \cos y = \psi_2(x, y), \quad \text{(say)} \quad \text{... (1)}$$

$$\frac{\partial v}{\partial y} = e^x (-x\sin y - y\cos y - \sin y) = \psi_1(x, y) \quad \text{(say)} \quad \text{... (2)}$$

$$\frac{\partial^2 v}{\partial x^2} = e^x (x\cos y - y\sin y) + e^x \cos y + e^x \cos y$$

$$= e^x (x\cos y - y\sin y + 2\cos y) \quad \text{... (3)}$$

and $$\frac{\partial^2 v}{\partial y^2} = e^x (-x\cos y + y\sin y - 2\cos y) \quad \text{... (4)}$$

Adding equations (3) and (4), we have

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \Rightarrow v \text{ is a harmonic function.}$$

Now putting $x = z, y = 0$ in (1) and (2), we get

$$\psi_2(z,0) = ze^z + e^z \qquad \psi_1(z,0) = 0$$

Hence by Milne-Thomson method, we have

$$f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)]\, dz + C$$

$$= \int [0 + i(ze^z + e^z)]\, dz + C = i(ze^z - e^z + e^z) + C = ize^z + C.$$

This is the required analytic function. **Ans.**

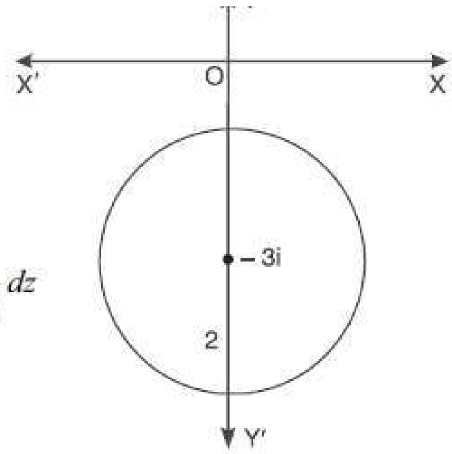
Q39:

Evaluate $\oint_C \frac{dz}{z^2+9}$, *where C is*

(i) $|z + 3i| = 2$ *(ii)* $|z| = 5$

Solution. Here $f(z) = \dfrac{1}{z^2+9}$

The poles of $f(z)$ can be determined by equating the denominator equal to zero.



(*i*) $\therefore \quad z^2 + 9 = 0 \qquad \Rightarrow \qquad z = \pm 3i$

Pole at $z = -3i$ lies in the given circle C.

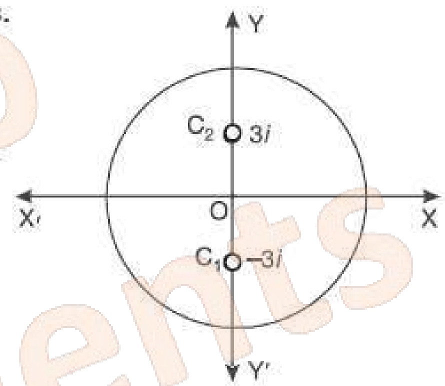
$$\int_C f(z)\,dz = \int_C \frac{1}{z^2+9}\,dz = \int_C \frac{1}{(z+3i)(z-3i)}\,dz$$

$$= \int_C \frac{\dfrac{1}{z-3i}}{z+3i}\,dz$$

$$= 2\pi i\left[\frac{1}{z-3i}\right]_{z=-3i}$$

$$= 2\pi i\left[\frac{1}{-3i-3i}\right] = \frac{-2\pi i}{6i} = -\frac{\pi}{3} \qquad \textbf{Ans.}$$

(*ii*) Both the poles $z = 3i$ and $z = -3i$ lie inside the given contour



$$\int_C f(z)\,dz = \int_C \frac{1}{z^2+9}\,dz = \int_C \frac{1}{(z+3i)(z-3i)}\,dz$$

$$= \int_{C_1} \frac{\dfrac{1}{z-3i}}{z+3i}\,dz + \int_{C_2} \frac{\dfrac{1}{z+3i}}{z-3i}\,dz$$

$$= 2\pi i\left[\frac{1}{z-3i}\right]_{z=-3i} + 2\pi i\left[\frac{1}{z+3i}\right]_{z=3i}$$

$$= 2\pi i\left[\frac{1}{-3i-3i}\right] + 2\pi i\left[\frac{1}{3i+3i}\right] = -\frac{\pi}{3} + \frac{\pi}{3} = 0 \qquad \textbf{Ans.}$$

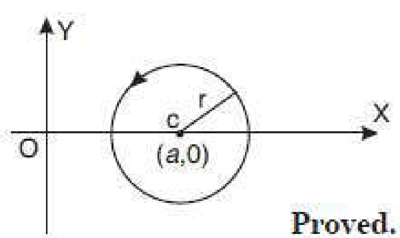
Q40:

Prove that $\displaystyle\int_C \frac{dz}{z-a} = 2\pi i$, *where C is the circle* $|z-a| = r$

Solution. We have,

$\displaystyle\int_C \frac{dz}{z-a}$, where C is the circle with centre $(a, 0)$ and radius r.

By Cauchy Integral Formula



$$\left[\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\; f(a)\right]$$

$$\int_C \frac{dz}{z-a} = 2\pi i \quad (1)$$

$$\Rightarrow \qquad \int_C \frac{dz}{z-a} = 2\pi i$$

Proved.

Q41: Use Cauchy's integral formula to calculate

$$\int_C \frac{2z+1}{z^2+z}\,dz \quad \text{where } C \text{ is } |z| = \frac{1}{2}.$$

Solution. Poles are given by

$$z^2 + z = 0$$

$$\Rightarrow \qquad z(z+1) = 0 \qquad \Rightarrow \qquad z = 0, -1$$

$|z| = \frac{1}{2}$ is a circle with centre at origin and radius $\frac{1}{2}$.

Therefore it encloses only one pole $z = 0$.

$$\therefore \int_C \frac{2z+1}{z(z+1)}\,dz = \int_C \frac{\frac{2z+1}{z+1}}{z}\,dz = 2\pi i \left[\frac{2z+1}{z+1}\right]_{z=0} = 2\pi i$$

Q42:

Evaluate: $\int_C \frac{e^z}{(z-1)(z-4)}\,dz$ *where C is the circle* $|z| = 2$ *by using Cauchy's Integral Formula.*

Sol: we have,

$$\int_C \frac{e^z}{(z-1)(z-4)}\,dz \quad \text{where C is the circle with centre at origin and radius 2.}$$

Poles are given by putting the denominator equal to zero.

$$(z-1)(z-4) = 0$$

$$\Rightarrow \qquad z = 1, 4$$

Here there are two simple poles at z = 1 and z = 4.

There is only one pole at z = 1 inside the contour. Therefore

$$\int_C \frac{e^z}{(z-1)(z-4)}\,dz = \int \frac{\frac{e^z}{(z-4)}}{(z-1)}\,dz = 2\pi i \left[\frac{e^z}{z-4}\right]_{z=1}$$

$$= 2\pi i \left(\frac{e}{1-4}\right) = -\frac{2\pi i\, e}{3}$$

Which is the required value of the given integral.

Q43:

Evaluate $\int_C \frac{e^{3iz}}{(z+\pi)^3}\, dz$

where C is the circle $|z - \pi| = 3.2$

Solution. Here, $I = \int_C \frac{e^{3iz}}{(z+\pi)^3}\, dz$

Where C is a circle $\{|z-\pi| = 3.2\}$ with centre $(\pi, 0)$ and radius 3.2.

Poles are determined by putting the denominator equal to zero.

$$(z+\pi)^3 = 0 \quad \Rightarrow \quad z = -\pi, -\pi, -\pi$$

There is a pole at $z - \pi$ of order 3. But there is no pole within C.

By Cauchy Integral Formula $\int_C \frac{e^{3iz}}{(z+\pi)^3}\, dz = 0$

Q44: Evaluate using Cauchy's integral formula

$$\int_C \frac{\log z}{(z-1)^3}\, dz \quad \text{where } C \text{ is } |z-1| = \frac{1}{2}.$$

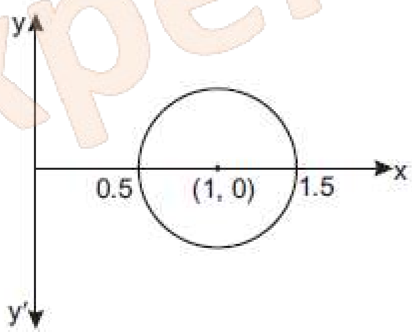
Sol: Using Cauchy's Integral formula,

$$\int_C \frac{\log z}{(z-1)^3} dz \qquad C: |z-1| = \frac{1}{2}$$

Poles are determined by putting denominater equal to zero.

$$(z - 1)^3 = 0 \quad \Rightarrow \quad z = 1, 1, 1$$

There is one pole of order three at $z = 1$ which is inside the circle C.



$$\int \frac{f(z)}{(z-a)^3}\,dz = 2\pi i f^2(a)$$

$$= 2\pi i\left[\frac{d^2}{dz^2}\log z\right]_{z=1} = 2\pi i\left[\frac{d}{dz}\left(\frac{1}{z}\right)\right]_{z=1}$$

$$= 2\pi i\left(-\frac{1}{z^2}\right)_{z=1} = -2\pi i$$

Q45:

. Find the residue at $z = 0$ of $z\cos\frac{1}{z}$.

Solution. Expanding the function in powers of $\frac{1}{z}$, we have

$$z\cos\frac{1}{z} = z\left[1 - \frac{1}{2!\,z^2} + \frac{1}{4!\,z^4} - \ldots\right] = z - \frac{1}{2z} + \frac{1}{24z^3} - \ldots$$

This is the Laurent's expansion about $z = 0$.

The coefficient of $\frac{1}{z}$ in it is $-\frac{1}{2}$. So the residue of $z\cos\frac{1}{z}$ at $z = 0$ is $-\frac{1}{2}$.

Q46:

Find the residue of $f(z) = \frac{z^3}{z^2-1}$ at $z = \infty$.

Solution. We have, $f(z) = \frac{z^3}{z^2-1}$

$$f(z) = \frac{z^3}{z^2\left(1-\frac{1}{z^2}\right)} = z\left(1-\frac{1}{z^2}\right)^{-1} = z\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\ldots\right) = z + \frac{1}{z} + \frac{1}{z^3} + \ldots$$

$$\text{Residue at infinity} = -\left(\text{coeff. of } \frac{1}{z}\right) = -1.$$

Q47: Determine the pole and residue at the pole of the function

$$f(z) = \frac{z}{z-1}$$

Solution. The poles of $f(z)$ are given by putting the denominator equal to zero.

$\therefore \quad z - 1 = 0 \Rightarrow z = 1$

The function $f(z)$ has a simple pole at $z = 1$.

Residue is calculated by the formula

$$\text{Residue} = \lim_{z \to a} (z - a) f(z)$$

$$\text{Residue of } f(z) \text{ (at } z = 1) = \lim_{z \to 1} (z - 1)\left(\frac{z}{z-1}\right) = \lim_{z \to 1} (z) = 1$$

Hence, $f(z)$ has a simple pole at $z = 1$ and residue at the pole is 1.

Q48: Find the residue of a function

$$f(z) = \frac{z^2}{(z+1)^2 (z-2)} \text{ at its double pole.}$$

Solution. We have, $f(z) = \dfrac{z^2}{(z+1)^2 (z-2)}$

Poles are determined by putting denominator equal to zero.

i.e.; $(z+1)^2 (z-2) = 0$

$\Rightarrow \quad z = -1, -1 \quad \text{and} \quad z = 2$

The function has a double pole at $z = -1$

$$\text{Residue at } (z = -1) = \lim_{z \to -1} \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z+1)^2 \frac{z^2}{(z+1)^2 (z-2)}\right\}\right]$$

$$= = \left[\frac{d}{dz}\left(\frac{z^2}{z-2}\right)\right]_{z=-1} = \left(\frac{(z-2)\,2z - z^2 \cdot 1}{(z-2)^2}\right)_{z=-1} = \left[\frac{z^2 - 4z}{(z-2)^2}\right]_{z=-1} = \frac{(-1)^2 - 4(-1)}{(-1-2)^2}$$

$$\text{Residue at } (z = -1) = \frac{1+4}{9} = \frac{5}{9}$$

Q49:

Find the residue of $\dfrac{z^3}{(z-1)^4 (z-2)(z-3)}$ *at a pole of order 4.*

Solution. The poles of $f(z)$ are determined by putting the denominator equal to zero.

$\therefore \quad (z-1)^4 (z-2)(z-3) = 0 \quad \Rightarrow \quad z = 1, 2, 3$

Here $z = 1$ is a pole of order 4.

$$f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)} \qquad ...(1)$$

Putting $z-1=t$ or $z=1+t$ in (1), we get

$$f(1+t) = \frac{(1+t)^3}{t^4(t-1)(t-2)} = \frac{1}{t^4}(t^3+3t^2+3t+1)(1-t)^{-1}\frac{1}{2}\left(1-\frac{t}{2}\right)^{-1}$$

$$= \frac{1}{2}\left(\frac{1}{t}+\frac{3}{t^2}+\frac{3}{t^3}+\frac{1}{t^4}\right)(1+t+t^2+t^3+...)\times\left(1+\frac{t}{2}+\frac{t^2}{4}+\frac{t^3}{8}...\right)$$

$$= \frac{1}{2}\left(\frac{1}{t}+\frac{3}{t^2}+\frac{3}{t^3}+\frac{1}{t^4}\right)\left(1+\frac{3}{2}t+\frac{7}{4}t^2+\frac{15}{8}t^3+...\right) = \frac{1}{2}\left(\frac{1}{t}+\frac{9}{2t}+\frac{21}{4}\frac{1}{t}+\frac{15}{8}\frac{1}{t}\right)+...$$

$$= \frac{1}{2}\left(1+\frac{9}{2}+\frac{21}{4}+\frac{15}{8}\right)\frac{1}{t} \qquad \left[\text{Res } f(a) = \text{coeffi. of } \frac{1}{t}\right]$$

Coefficient of $\frac{1}{t} = \frac{1}{2}\left(1+\frac{9}{2}+\frac{21}{4}+\frac{15}{8}\right) = \frac{101}{16}$,

Hence, the residue of the given function at a pole of order 4 is $\frac{101}{16}$.

Q50: Determine the poles of the function and residue at the poles.

$$f(z) = \frac{z}{\sin z}$$

Solution. $f(z) = \frac{z}{\sin z}$

Poles are determined by putting $\sin z = 0 = \sin n\pi \quad \Rightarrow z = n\pi$

$$\text{Residue} = \left(\frac{z}{\cos z}\right)_{z=n\pi}$$

$$= \frac{n\pi}{\cos n\pi} = \frac{n\pi}{(-1)^n}$$

Hence, the residue of the given function at pole $z = n\pi$ is $\frac{n\pi}{(-1)^n}$.

Q51:

Find the residue of $f(z) = \frac{z\,e^z}{(z-a)^3}$ *at its pole.*

Solution. The pole of $f(z)$ is given by $(z-a)^3 = 0$ *i.e.*, $z = a$

Here $z = a$ is a pole of order 3.

Putting $z - a = t$ where t is small.

$$f(z) = \frac{ze^z}{(z-a)^3} \Rightarrow f(z) = \frac{(a+t)e^{a+t}}{t^3} = \left(\frac{a}{t^3} + \frac{1}{t^2}\right)e^{a+t} = e^a\left(\frac{a}{t^3} + \frac{1}{t^2}\right)e^t \qquad (z = a + t)$$

$$= e^a\left(\frac{a}{t^3} + \frac{1}{t^2}\right)\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \ldots\right) = e^a\left[\frac{a}{t^3} + \frac{a}{t^2} + \frac{a}{2t} + \frac{1}{t^2} + \frac{1}{t} + \frac{1}{2} + \ldots\right]$$

$$= e^a\left[\frac{1}{2} + \left(\frac{a}{2} + 1\right)\frac{1}{t} + (a+1)\frac{1}{t^2} + (a)\frac{1}{t^3} + \ldots\right]$$

Coefficient of $\frac{1}{t} = e^a\left(\frac{a}{2} + 1\right)$

Hence the residue at $z = a$ is $e^a\left(\frac{a}{2} + 1\right)$.

Q52: Using Residue theorem, evaluate

$$\frac{1}{2\pi i}\int_C \frac{e^{zt}\,dz}{(z^2 + 2z + 2)}$$

where C is the cirlce | z | = 3.

Solution. Here, we have

$$\frac{1}{2\pi i}\int_C \frac{e^{zt}\,dz}{z^2\,(z^2 + 2z + 2)}$$

Poles are given by

$z = 0$ (double pole)

$z = -1 \pm i$ (simple poles)

All the four poles are inside the given circle.

Residue at z = 0 is $\lim_{z \to 0} \frac{d}{dz} z^2 \frac{e^{zt}}{z^2\,(z^2 + 2z + 2)}$

$$= \lim_{z \to 0} \frac{d}{dz} \frac{e^{zt}}{z^2 + 2z + 2}$$

$$= \lim_{z \to 0} \frac{(z^2 + 2z + 2)\,t e^{zt} - (2z + 2)e^{zt}}{(z^2 + 2z + 2)^2}$$

$$= \frac{2t e^0 - 2e^0}{4} = \frac{(t-1)}{2}$$

Residue at $z = -1 + i$

$$= \lim_{z \to -1+i} \frac{(z+1-i)\, e^{zt}}{z^2\,(z+1-i)\,(z+1+i)} = \lim_{z \to -1+i} \frac{e^{zt}}{z^2\,(z+1+i)}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2\,(-1+i+1+i)} = \frac{e^{(-1+i)t}}{(1-2i-1)\,(2i)} = \frac{e^{(-1+i)t}}{4}$$

$$\int \frac{e^{2zt}}{z^2\,(z^2+2z+2)}\,dz = 2\pi i \quad \text{(Sum of the Residues)}$$

$$\Rightarrow \quad \frac{1}{2\pi i}\int \frac{e^{2zt}}{z^2\,(z^2+2z+2)}\,dz = \frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{4}\,(e^{it} + e^{-it}) = \frac{t-1}{2} + \frac{e^{-t}}{4}\,(2\cos t)$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{2}\cos t$$

Q53: Evaluate the integral:

$$\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$$

Solution. $$\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \int_0^{2\pi} \frac{d\theta}{5 - 3\left(\dfrac{e^{i\theta}+e^{-i\theta}}{2}\right)}$$

$$= \int_0^{2\pi} \frac{2\,d\theta}{10 - 3e^{i\theta} - 3e^{-i\theta}}$$

$$= \int_C \frac{1}{10 - 3z - \dfrac{3}{z}}\,\frac{dz}{iz} = \frac{1}{i}\int_C \frac{dz}{10z - 3z^2 - 3}$$

[C is the unit circle $|z| = 1$]

$$= -\frac{1}{i}\int_C \frac{dz}{3z^2 - 10z + 3}$$

$$= -\frac{1}{i}\int_C \frac{dz}{(3z-1)\,(z-3)} = i\int_C \frac{dz}{(3z-1)\,(z-3)}$$

Let $I = \int_C \frac{dz}{(3z-1)(z-3)}$

Poles of the integrand are given by

$$(3z-1)(z-3) = 0 \quad \Rightarrow \quad z = \frac{1}{3}, 3$$

There is only one pole at $z = \frac{1}{3}$ inside the unit circle C.

$$\text{Residue at } \left(z = \frac{1}{3}\right) = \lim_{z \to \frac{1}{3}} \left(z - \frac{1}{3}\right) f(z) = \lim_{z \to \frac{1}{3}} \frac{\left(z - \frac{1}{3}\right)}{(3z-1)(z-3)} = \lim_{z \to \frac{1}{3}} \frac{1}{3(z-3)}$$

$$= \frac{1}{3\left(\frac{1}{3} - 3\right)} = -\frac{1}{8}$$

Hence, by Cauchy's Residue Theorem

$$I = 2\pi i \text{ (Sum of the residues within Contour)} = 2\pi i \left(-\frac{1}{8}\right) = -\frac{\pi i}{4}$$

$$\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta} = i\left(\frac{-\pi i}{4}\right) = \frac{\pi}{4}$$

Q54: Use the complex variable technique to find the value of the integral :

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}.$$

Solution. Let $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_0^{2\pi} \frac{d\theta}{2 + \frac{e^{i\theta} + e^{-i\theta}}{2}} = \int_0^{2\pi} \frac{2\,d\theta}{4 + e^{i\theta} + e^{-i\theta}}$

Put $e^{i\theta} = z$ so that $e^{i\theta}(i\,d\theta) = dz \Rightarrow iz\,d\theta = dz \Rightarrow d\theta = \frac{dz}{iz}$

$$I = \int_C \frac{2\frac{dz}{iz}}{4 + z + \frac{1}{z}} \qquad \text{where } c \text{ denotes the unit circle } |z| = 1.$$

$$= \frac{1}{i} \int_c \frac{2\,dz}{z^2 + 4z + 1}$$

The poles are given by putting the denominator equal to zero.

$$z^2+4z+1=0 \text{ or } z=\frac{-4\pm\sqrt{16-4}}{2}=\frac{-4\pm 2\sqrt{3}}{2}=-2\pm\sqrt{3}$$

The pole within the unit circle C is a simple pole at $z=-2+\sqrt{3}$. Now we calculate the residue at this pole.

$$\text{Residue at } (z=-2+\sqrt{3}) = \lim_{z\to(-2+\sqrt{3})} \frac{1}{i}\frac{(z+2-\sqrt{3})2}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$$

$$=\lim_{z\to(-2+\sqrt{3})} \frac{2}{i(z+2+\sqrt{3})}=\frac{2}{i(-2+\sqrt{3}+2+\sqrt{3})}=\frac{1}{\sqrt{3}i}$$

Hence by Cauchy's Residue Theorem, we have

$$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}=2\pi i \text{ (sum of the residues within the contour)}$$

$$=2\pi i\frac{1}{i\sqrt{3}}=\frac{2\pi}{\sqrt{3}}$$

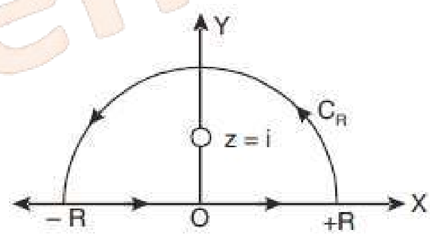
Q55:

Evaluate $\int_0^\infty \frac{\cos mx}{(x^2+1)}dx$

Solution. $\int_0^\infty \frac{\cos mx}{x^2+1}dx$

Consider the integral $\int_C f(z)\,dz$, where

$f(z)=\frac{e^{imz}}{z^2+1}$, taken round the closed contour c consisting of the upper half of a large circle $|z|=R$ and the real axis from $-R$ to R.



Poles of $f(z)$ are given by

$z^2+1=0$ *i.e.* $z^2=-1$ *i.e.* $z=\pm i$

The only pole which lies within the contour is at $z=i$.

The residue of $f(z)$ at $z=i$

$$=\lim_{z\to i}\frac{(z-i)e^{imz}}{(z^2+1)}=\lim_{z\to i}\frac{e^{imz}}{z+i}=\frac{e^{-m}}{2i}$$

Hence by Cauchy's residue theorem, we have

$$\int_C f(z)\,dz=2\pi i\times \text{ sum of the residues}$$

$$\Rightarrow \int_C \frac{e^{imz}}{z^2+1}dz=2\pi i\times\frac{e^{-m}}{2i} \qquad \Rightarrow \int_{-R}^{R}\frac{e^{imx}}{x^2+1}dx=\pi e^{-m}$$

Equating real parts, we have

$$\int_{-\infty}^{\infty}\frac{\cos mx}{x^2+1}dx=\pi e^{-m} \qquad \Rightarrow \int_0^\infty\frac{\cos mx}{x^2+1}dx=\frac{\pi e^{-m}}{2}$$

Q56: Using the complex variable techniques, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx$$

Solution. $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx$

Consider $\int_C f(z)\, dz$, where $f(z) = \frac{1}{z^4+1}$

taken around the closed contour consisting of real axis and upper half C_R, *i.e.* $|z| = R$.

Poles of $f(z)$ are given by

$$z^4+1 = 0 \text{ i.e. } z^4 = -1 = (\cos\pi + i\sin\pi)$$

$$\Rightarrow \quad z^4 = [\cos(2n+1)\pi + i\sin(2n+1)\pi]$$

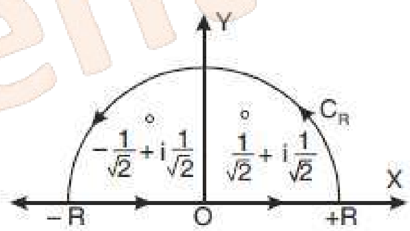
$$z = [\cos(2n+1)\pi + i\sin(2n+1)\pi]^{\frac{1}{4}} = \left[\cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4}\right]$$

If $n = 0$, $z_1 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = e^{i\frac{\pi}{4}}$

$n = 1$, $z_2 = \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = e^{i\frac{3\pi}{4}}$

$n = 2$, $z_3 = \left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$n = 3$, $z_4 = \left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$



There are four poles, but only two poles at z_1 and z_2 lie within the contour.

$$\text{Residue}\left(\text{at } z = e^{\frac{i\pi}{4}}\right) = \left[\frac{1}{\frac{d}{dz}(z^4+1)}\right]_{z=e^{\frac{i\pi}{4}}} = \left[\frac{1}{4z^3}\right]_{z=e^{\frac{i\pi}{4}}} = \frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3} = \frac{1}{4e^{i\frac{3\pi}{4}}}$$

$$= \frac{1}{4}e^{-i\frac{3\pi}{4}} = \frac{1}{4}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right] = \frac{1}{4}\left[-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]$$

$$\text{Residue}\left(\text{at } z = e^{\frac{3i\pi}{4}}\right) = \left[\frac{1}{\frac{d}{dz}(z^4+1)}\right]_{z=e^{\frac{3i\pi}{4}}} = \frac{1}{[4z^3]_{z=\frac{3i\pi}{4}}} = \frac{1}{4\left(e^{i\frac{3\pi}{4}}\right)^3} = \frac{1}{4e^{i\frac{9\pi}{4}}}$$

$$= \frac{1}{4}e^{-i\frac{9\pi}{4}} = \frac{1}{4}\left(\cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right) = \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues at poles within c)}$$

$$\int_{-R}^{R} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i \text{ (sum of the residues)}$$

$$\int_{-R}^{R} \frac{1}{x^4+1}\,dx + \int_{C_R} \frac{1}{z^4+1}\,dz = 2\pi i \text{ (sum of the residues)}$$

Now, $\left|\int_{C_R} \frac{1}{z^4+1}dz\right| \le \int_{C_R} \frac{1}{|z^4+1|}|dz|$

$$\le \int_{C_R} \frac{1}{(|z^4|-1)}|dz| \qquad \left[\text{Since } z = Re^{i\theta}, |dz| = |Re^{i\theta}\, i\, d\theta| = R\,d\theta\right]$$

$$\le \int_0^{\pi} \frac{1}{R^4-1}R\,d\theta \le \frac{R}{R^4-1}\int_0^{\pi} d\theta$$

$$\le \frac{R\pi}{R^4-1} = \frac{\pi/R^3}{1-1/R^4} \quad \text{which} \to 0$$

as $R \to \infty$.

Hence, $\int_{-R}^{R} \frac{1}{x^4+1}\,dx = 2\pi i$ (Sum of the residues within contour)

As $R \to \infty$

Hence, $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx = 2\pi i$ (Sum of the residues within contour)

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx = 2\pi i \left[\frac{1}{4}\left(-\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right) + \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi}{2}\, i\left(-\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right) = \frac{\pi i}{2}\left(-i\,\frac{2}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$$

Q57: Use the residue theorem to show that

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

Solution. $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \int_0^{2\pi} \frac{d\theta}{\left(a+b\cdot\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2}$

Put $e^{i\theta} = z$, so that $e^{i\theta}(i\,d\theta) = dz \Rightarrow iz\,d\theta = dz \Rightarrow d\theta = \frac{dz}{iz}$

$$= \int_c \frac{1}{\left\{a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right\}^2}\frac{dz}{iz}$$ where c is the unit circle $|z| = 1$.

$$\int_c \frac{1}{\left(a+\frac{bz}{2}+\frac{b}{2z}\right)^2}\frac{dz}{iz} = \int_c \frac{-4iz}{\left(a+\frac{bz}{2}+\frac{b}{2z}\right)^2}\frac{dz}{(2z)^2}$$

$$= \int_c \frac{-4iz\,dz}{(bz^2+2az+b)^2} = \frac{-4i}{b^2}\int_c \frac{z\,dz}{\left(z^2+\frac{2az}{b}+1\right)^2}$$

The poles are given by putting the denominator equal to zero.

i.e., $\left(z^2+\frac{2a}{b}z+1\right)^2 = 0$

$\Rightarrow$ $(z-\alpha)^2\,(z-\beta)^2 = 0$

where $$\alpha = \frac{-\frac{2a}{b} + \sqrt{\frac{4a^2}{b^2} - 4}}{2} = \frac{-a + \sqrt{a^2 - b^2}}{b}$$

$$\beta = \frac{-\frac{2a}{b} - \sqrt{\frac{4a^2}{b^2} - 4}}{2} = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

There are two poles, at $z = \alpha$ and at $z = \beta$, each of order 2.

Since $|\alpha\beta| = 1$ or $|\alpha|\,|\beta| = 1$ if $|\alpha| < 1$ then $|\beta| > 1$.

There is only one pole $[|\alpha| < 1]$ of order 2 within the unit circle c.

Residue (at the double pole $z = \alpha$) $= \lim_{z\to\alpha} \frac{d}{dz} (z-\alpha)^2 \frac{(-4iz)}{b^2 (z-\alpha)^2 (z-\beta)^2}$

$$= \lim_{z\to\alpha} \frac{d}{dz} \frac{-4iz}{b^2 (z-\beta)^2}$$

$$= -\frac{4i}{b^2} \lim_{z\to\alpha} \frac{(z-\beta)^2 .1 - 2(z-\beta) z}{(z-\beta)^4} = \frac{-4i}{b^2} \lim_{z\to\alpha} \frac{z-\beta-2z}{(z-\beta)^3} = \frac{-4i}{b^2} \lim_{z\to\alpha} \frac{-(z+\beta)}{(z-\beta)^3}$$

$$= \frac{4i}{b^2} \frac{(\alpha+\beta)}{(\alpha-\beta)^3} = \frac{4i}{b^2} \frac{\alpha+\beta}{[(\alpha+\beta)^2 - 4\alpha\beta]^{\frac{3}{2}}} = \frac{4i}{b^2} \frac{\frac{-2a}{b}}{\left[\left(-\frac{2a}{b}\right)^2 - 4(1)\right]^{\frac{3}{2}}}$$

$$= \frac{-8ai}{(4a^2 - 4b^2)^{\frac{3}{2}}} = -\frac{ai}{(a^2 - b^2)^{\frac{3}{2}}}$$

Hence, $$\int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2} = 2\pi i \times \frac{-ai}{(a^2 - b^2)^{3/2}} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

Q58: Using complex variable techniques evaluate the real integral

$$\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\, d\theta$$

Solution. If we write $z = e^{i\theta}$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right), \quad d\theta = \frac{dz}{iz}$$

and so $I=\int_0^{2\pi}\frac{\sin^2\theta}{5-4\cos\theta}\,d\theta=\frac{1}{2}\int_0^{2\pi}\frac{1-\cos 2\theta}{5-4\cos\theta}\,d\theta$

$$\text{I} = \text{Real part of } \frac{1}{2}\int_0^{2\pi}\frac{1-\cos 2\theta - i\sin 2\theta}{5-4\cos\theta}\,d\theta$$

[where c is a circle of unit radius with centre $z=0$]

$$= \text{Real part of } \frac{1}{2}\int_0^{2\pi}\frac{1-e^{2i\theta}}{5-4\cos\theta}\,d\theta$$

$$= \text{Real part of } \frac{1}{2}\int_c\frac{1-z^2}{5-2(z+\frac{1}{z})}\left(\frac{dz}{iz}\right) = \text{Real part of } \frac{1}{2i}\int_c\frac{1-z^2}{5z-2z^2-2}\,dz$$

$$= \text{Real part of } \frac{1}{2i}\int_c\frac{z^2-1}{2z^2-5z+2}\,dz$$

Poles are determined by $2z^2-5z+2=0$ or $(2z-1)(z-2)=0$ or $z=\frac{1}{2},\ 2$

So inside the contour c there is a simple pole at $z=\frac{1}{2}$

Residue at the simple pole $\left(z=\frac{1}{2}\right)=\lim\limits_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^2-1}{(2z-1)(z-2)}$

$$= \lim_{z\to\frac{1}{2}}\frac{z^2-1}{2(z-2)} = \frac{\frac{1}{4}-1}{2\left(\frac{1}{2}-2\right)} = \frac{1}{4}$$

$I=$ Real part of $\frac{1}{2i}\int_c\frac{(z^2-1)}{2z^2-5z+2}\,dz=\frac{1}{2i}2\pi i$ (sum of the residues)

$$\Rightarrow \int_0^{2\pi}\frac{\sin^2\theta}{5-4\cos\theta}\,d\theta=\pi\left(\frac{1}{4}\right)=\frac{\pi}{4}$$